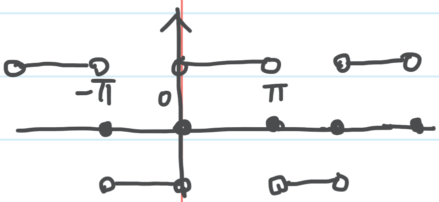
P27

2024年3月11日

Fourier series : An example

$$f(x) = \operatorname{sgn} x = \begin{cases} -1 & -\pi < x < 0 \\ 0 & x = 0, \pi, -\pi \\ 1 & 0 < x < \pi \end{cases}$$

period 2π $[-\pi \sim \pi]$



Fourier series is $f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$

Check at $x = \frac{\pi}{2} \Rightarrow \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\right) = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1$!

A thought about the coefficient :

similar to before, we can try to guess the coefficients.

Suppose the fun $f(x)$ can be written as a linear combination of $\cos kx$ and $\sin kx$, ie. $f(x) = \left(\sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx\right) + \frac{a_0}{2}$

What should a_n be ?

multiply by $\cos nx$ on both side and take integration :

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi} \tfrac{1}{2}a_0 \cos nx\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} a_k \cos kx \cos nx\,dx$$
$$+ \int_{-\pi}^{\pi} \sum_k b_k \sin kx \cos nx\,dx$$

assume the infinite sum conv. so the integration becomes $\sum_k \int_{-\pi}^{\pi} a_k \cos kx \cos nx\,dx$ and $\sum_k \int_{-\pi}^{\pi} b_k \sin kx \cos nx\,dx$

合角 $\Rightarrow \int_{-\pi}^{\pi} a_k \cos kx \cos nx\,dx$

$$= \frac{a_k}{2}\int_{-\pi}^{\pi} \cos(k+n)x + \cos(k-n)x\,dx$$

$$= \begin{cases} \frac{a_k}{2}\left[\frac{+1}{k+n}\sin(k+n)x\Big|_{-\pi}^{\pi} + \frac{+1}{k-n}\sin(k-n)x\Big|_{-\pi}^{\pi}\right] & \forall k \neq n \\ \frac{a_k}{2}\left[\frac{+1}{k+n}\sin(k+n)x\Big|_{-\pi}^{\pi} + x\Big|_{-\pi}^{\pi}\right] & \text{when } k = n \end{cases}$$

b/c $\int_{-\pi}^{\pi} \cos(n-n)x\,dx = \int_{-\pi}^{\pi} \cos 0\,dx = \int_{-\pi}^{\pi} 1\,dx = x\Big|_{-\pi}^{\pi}$

Note: here we are only talking about what the coefficient should be like, we didn't say it conv, nor it conv. to f(x) !!

and $\int_{-\pi}^{\pi} b_k \sin k\pi \cos n\pi \, dx = \frac{b_k}{2} \int_{-\pi}^{\pi} \sin(k+n)x + \sin(k-n)x \, dx$

$$= \begin{cases} -\frac{b_k}{2}\left[\frac{1}{k+n}\cos(k+n)x \Big|_{-\pi}^{\pi} + \frac{1}{k-n}\cos(k-n)x \Big|_{-\pi}^{\pi}\right] & \text{if } k \neq n \\ -\frac{b_k}{2}\left[\frac{1}{k+n}\cos(k+n)x \Big|_{-\pi}^{\pi} + 0\right] & \text{if } k = n \end{cases}$$

b/c $\int_{-\pi}^{\pi} \sin(n-n)x \, dx = \int_{-\pi}^{\pi} \sin 0 \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0$

also $\int_{-\pi}^{\pi} \frac{a_0}{2} \cos nx \, dx = \frac{-a_0}{2n} \sin nx \Big|_{-\pi}^{\pi} = 0$

and $\sin(k+n)x \Big|_{-\pi}^{\pi} = 0 - 0$, $\sin(k-n)x \Big|_{-\pi}^{\pi} = 0 - 0$

$\cos(k+n)x \Big|_{-\pi}^{\pi} = * - * = 0$, $\cos(k-n)x \Big|_{-\pi}^{\pi} = * - * = 0$

Thus, we have

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 + 0 + \frac{a_n}{2} \cdot 2\pi + 0 + 0$$

$$\Rightarrow \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

similarly, we will have

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 + 0 + 0 + \frac{b_n}{2} \cdot 2\pi + 0$$

$$\begin{cases} 2\cos kx \sin nx = \sin(k+n)x - \sin(k-n)x \\ 2\sin kx \sin nx = \cos(k-n)x - \cos(k+n)x \end{cases}$$

$$\Rightarrow \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

What about a_0 ? $\int_{-\pi}^{\pi} a_k \cos kx \, dx = \frac{+a_k}{k} \sin kx \Big|_{-\pi}^{\pi} = 0$

$\int^{\pi} b_k \sin kx \, dx = 0 \quad \therefore \int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \, dx = a_0 \cdot \pi$

$\int_{-\pi}^{\pi} b_k \sin kx \, dx = 0$ $\therefore \int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \, dx = a_0 \cdot \pi$