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One application of convexity <Toric manifold>

Recall: effective $\bigcap_{m\in M} G_m = \{e\}$

Thm: Let (M,ω,Φ) be a Ham T^m-space. If the T^m action is effective, then $\dim M \geq 2m$.

Pf: Consider $\mathcal{O}_x = \{g.x \mid g\in G\} \subset M$ an orbit of dim m

Then $\Phi(\mathcal{O}_x) = g.\Phi(x) = \Phi(x)$ ($\because$ T^m is abelian so conjugation action is trivial)

Let $p\in\mathcal{O}_x$ be any pt, then

$$\begin{array}{ccc} d\Phi_p : T_pM & \longrightarrow & \mathfrak{t}^* \ (\cong \mathbb{R}^n) \\ \cup & & \\ T_p\mathcal{O} & \longrightarrow & 0 \quad (\because \Phi \text{ is const on } \mathcal{O}) \end{array}$$

So $T_p\mathcal{O} \subset \ker d\Phi_p$

proved in symplectic reduction

O.T.H. $\ker d\Phi_p = (T_p\mathcal{O})^\omega := \{W\in T_pM \mid \omega(V,W)=0, \forall V\in T_p\mathcal{O}\}$

pf: $T_p\mathcal{O} = \{\xi_M(p) \mid \xi\in\mathfrak{t}\}$

and $\omega_p(\xi_M, W) = \langle d\Phi_p(W), \xi\rangle$

$\therefore$ if $W\in\ker d\Phi_p$, then $\omega(\xi_M, W)=0$ $\forall\xi$ $\Rightarrow$ $W\in(T_p\mathcal{O})^\omega$

if $W\in(T_p\mathcal{O})^\omega$, then $\langle d\Phi_p(W),\xi\rangle=0$ $\forall\xi$ $\Rightarrow$ $d\Phi_p(W)=0$

$\Rightarrow T_p\mathcal{O}\subset(T_p\mathcal{O})^\omega$ i.e. $\mathcal{O}$ is an isotropic submfd of M

so in particular, $\dim\mathcal{O} \leq \frac{1}{2}\dim M$.

Thus $\dim\mathcal{O} = m$ and $\dim M \geq 2m$

※

(The reason for the existence of such orbit $\mathcal{O}$ is the following theorem of G-S)

Prop: (Guilleman-Sternberg)

If T^m-action is effective, then it has orbits of dim m. In particular, there is a T^m-inv. open dense subset on which the action is free.

Def: A (symplectic) toric mfd is a cpt connected symplectic mfd (M, ω) equipped with an effective Hamiltonian action of a torus T of dim equal to half the dim of the mfd $\dim T = \frac{1}{2} \dim M$ and with a choice of a corresponding moment map Φ.

Def: A convex polytope Δ in $(\mathbb{R}^n)^*$ is Delzant if:

(1) There are n edges meeting in each vertex p.

(2) The edges meeting in the vertex p are rational, i.e. each edge is of the form $p + t v_i$, $0 \le t \le \infty$, where $v_i \in (\mathbb{Z}^n)^*$.
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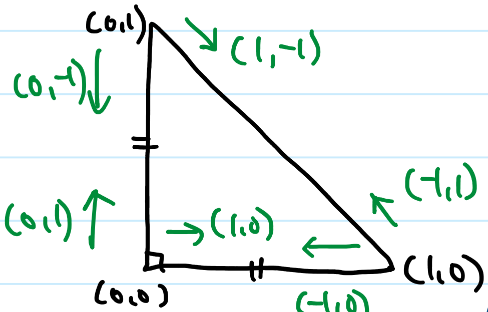
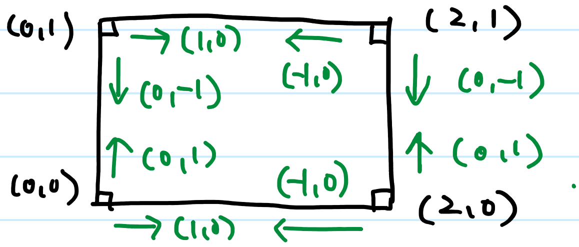
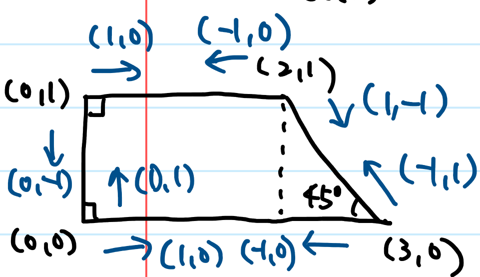
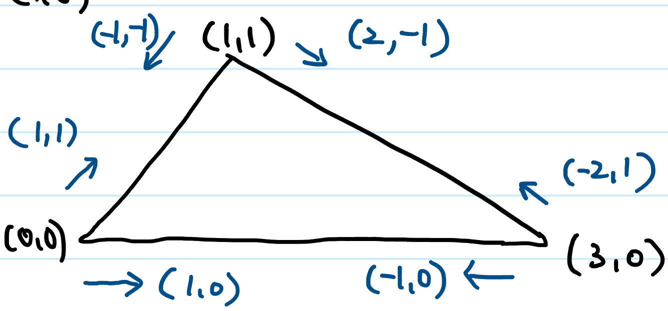
(3) The $v_1, \cdots, v_n$ in (2) can be chosen to be a basis of $(\mathbb{Z}^n)^*$.

Examples:









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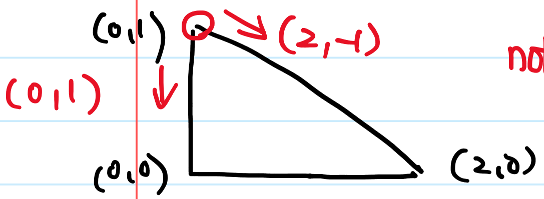
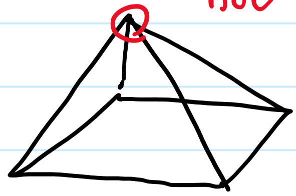
Examples that are not Delzant:



not (3)



not (1)

Thm (Delzant)

Toric manifolds are classified by Delzant polytopes. More specifically

(1) Given a Delzant polytope Δ^n, $\exists\ (M_\Delta, \omega_\Delta, T^n, \Phi)$ with $\Phi(M_\Delta) = \Delta$

(2) If (X, ω, T^n, Φ) is a (symplectic) toric mfd, then the image $\Phi(X)$ is a Delzant polytope, and X is isomorphic to M_Δ as a Hamiltonian T^n-space.

we will look at (1), e.g. Guillemin "Moment map & combinatorial invariants of Ham T^n-spaces", chap 1.

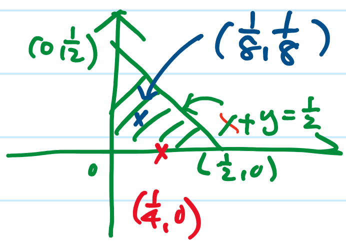
One important property coming from the Delzant's construction is $\forall x \in \Delta$, the level set $\Phi^{-1}(x)$ is a single T^n-orbit, which is isomorphic to $T^n/\text{stabilizer}$

Example: $T^2 \curvearrowright \mathbb{CP}^2 \xrightarrow{\Phi} \mathbb{R}^2 \simeq \mathfrak{t}^*$

$[1:0:0] \longmapsto (0,0)$

$[0:1:0] \longmapsto (\frac{1}{2},0)$

$[0:0:1] \longmapsto (0,\frac{1}{2})$



$\Phi^{-1}((0,0)) = [1:0:0] \simeq T^2/T^2 \longleftarrow$ fixed by the whole T^2

$\Phi^{-1}((\frac{1}{4},0)) \simeq T^2/T^1 \simeq T^1 \longrightarrow$ fixed by $T^1 \simeq S^1$

$\Phi^{-1}((\frac{1}{8},\frac{1}{8})) \simeq T^2/\mathrm{id} \simeq T^2 \longrightarrow$ free action

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Example: $T^3 \curvearrowright \mathbb{CP}^3 \xrightarrow{\Phi} \mathbb{R}^3 \simeq \mathfrak{t}^*$

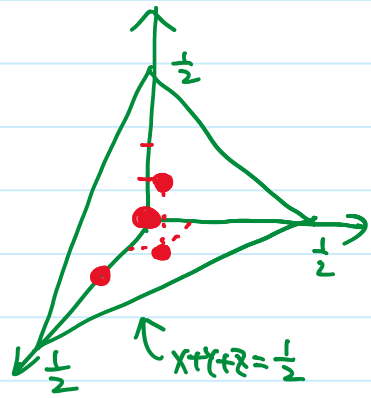
$(t_1, t_2, t_3) . [z_0 : z_1 : z_2 : z_3] = [z_0 : t_1 z_1 , t_2 z_2 , t_3 z_3]$

where $(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}$, $[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \simeq \mathbb{C}^4 \setminus \{0\} / \mathbb{C}^*$

the moment map is $\Phi([z_0 : \cdots : z_3]) = \frac{1}{2} \frac{(|z_1|^2, |z_2|^2, |z_3|^2)}{\|z\|^2}$

and fixed pts are

$[1:0:0:0] \longmapsto (0,0,0)$
$[0:1:0:0] \longmapsto (\frac{1}{2},0,0)$
$[0:0:1:0] \longmapsto (0,\frac{1}{2},0)$
$[0:0:0:1] \longmapsto (0,0,\frac{1}{2})$



and the fibers (i.e. preimages) are

$\Phi^{-1}((0,0,0)) = [1:0:0:0] \cong T^3/T^3$, fixed by the whole T^3

vertex

$\Phi^{-1}((\frac{1}{4},0,0)) \cong T^3/T^2 \cong T^1$, fixed by T^2

on edge

$\Phi^{-1}((\frac{1}{8},\frac{1}{8},0)) \cong T^3/T^1 \cong T^2$, fixed by T^1

on surface

$\Phi^{-1}((\frac{1}{8},\frac{1}{8},\frac{1}{8})) \cong T^3/id \cong T^3$, free action

in the interior

To prove Delzant's theorem, it's convenient to use a different way to describe the Delzant polytope:

< Algebraic description of Delzant polytope >:

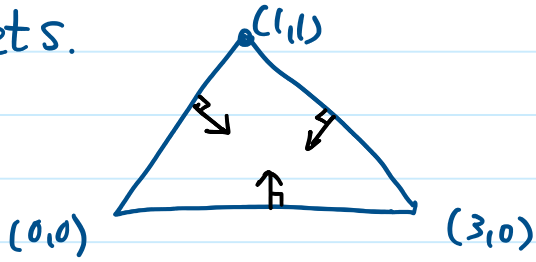
A facet of an n-polytope is a $(n-1)$-dim'l face. Let Δ be a Delzant polytope with $n = \dim \Delta$ and $d =$ number of facets (clearly $d \geq n$)

A lattice vector $u \in \mathbb{Z}^n$ is primitive if it can't be written as $u = k\nu$, with $\nu \in \mathbb{Z}^n$, $k \in \mathbb{Z}$, and $|k| > 1$.

e.g. $(1,1)$, $(2,3)$, $(4,3)$ are primitive, but $(2,2)$, $(3,6)$ are not.

Let $u_i \in \mathbb{Z}^n$, $i = 1, \dots, d$ be the primitive inward pointing normal vectors to the facets.

e.g.



$n = 2$

$d = 3$

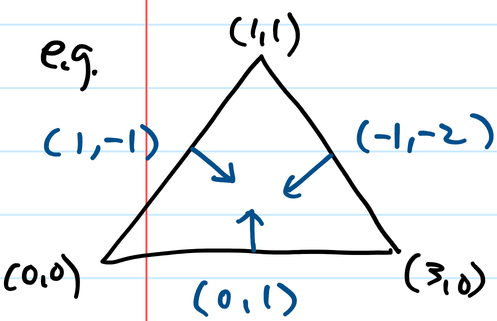
Then we can describe Δ as an intersection of half spaces:

$$\Delta = \{ x \in (\mathbb{R}^n)^* \mid \langle x, u_i \rangle \geq \lambda_i,\ i = 1, \dots, d \} \text{ for some } \lambda_i \in \mathbb{R}$$

and the facets are defined by "equality" in the above def.

inspired by root system

e.g.



$$\Delta = \{ x = (x_1, x_2) \in (\mathbb{R}^2)^* \mid \begin{array}{l} x_1 - x_2 \geq 0 \\ x_1 + 2x_2 \leq 3 \\ x_2 \geq 0 \end{array} \}$$

$$= \{ x \in (\mathbb{R}^2)^* \mid \begin{array}{l} \langle x, (1,-1) \rangle \geq 0 \\ \langle x, (-1,-2) \rangle \geq -3 \\ \langle x, (0,1) \rangle \geq 0 \end{array} \}$$

What we'll show is that given a Δ Delzant n-polytope, we can construct a T^n- Ham system $(M_\Delta, \omega_\Delta, T^n, \Phi)$ s.t. $\Phi(M_\Delta) = \Delta$ (i.e. item (1) of Delzant's theorem)

pf: Start with a given Delzant polytope in $\mathbb{R}^n$ with d facets. Let $u_i \in \mathbb{Z}^n$ be the primitive outward pointing normal vectors to the facets. Then $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, u_i \rangle \geq \lambda_i,\ i = 1, \cdots, d\}$

for some $\lambda_i \in \mathbb{R}$.

Let $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider the map $\pi: \mathbb{Z}^d \to \mathbb{Z}^n$, $e_i \mapsto u_i$, and its extension $\pi: \mathbb{R}^d \longrightarrow \mathbb{R}^n$, claim: $\pi: \mathbb{Z}^d \to \mathbb{Z}^n$ & $\mathbb{R}^d \to \mathbb{R}^n$ are onto.

Since $\{e_1, \cdots, e_d\}$ is a basis of $\mathbb{Z}^d$, $d \geq n$, the set $\{u_1, \cdots, u_d\}$ spans $\mathbb{Z}^n$: because at any vertex p, the edge vector $v_{p(1)}, \cdots, v_{p(n)} \in (\mathbb{R}^n)^*$ form a basis for $(\mathbb{Z}^n)^*$

(this is condition (3) of Delzant polytope: The v_i in (2) can be chosen to be a basis of $(\mathbb{Z}^n)^*$ $p + tv_i$)

so the corresponding primitive normal vectors $u_{p(i)}$ to the facets meeting at p also forms a basis of $\mathbb{Z}^n$.

So $\pi: \mathbb{Z}^d \longrightarrow \mathbb{Z}^n$ is onto and so the extension $\mathbb{R}^d \to \mathbb{R}^n$ is also onto.

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So π induces a surjective map on the quotients, still called π:

$$\begin{array}{ccccc} \pi: & \mathbb{R}^d/\mathbb{Z}^d & \longrightarrow & \mathbb{R}^n/\mathbb{Z}^n & \\ & \| & & \| & \\ & T^d & \longrightarrow & T^n & \longrightarrow 0 \end{array}$$

$\because$ surjective.

Let $N := \ker(\pi: T^d \to T^n)$ (then N is a Lie subgp of T^d)

$\mathfrak{n} := \mathrm{Lie}(N)$, $\mathbb{R}^d = \mathrm{Lie}(T^d)$, $\mathbb{R}^n = \mathrm{Lie}(T^n)$

we have the exact sequence of tori:

$$0 \to N \overset{i}{\hookrightarrow} T^d \xrightarrow{\pi} T^n \longrightarrow 0$$

inducing an exact sequence of Lie algebras:

$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0$$

With dual exact sequence:

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0$$

notice that exactness implies π^* is injective

Next, consider a standard Ham space: $(\mathbb{R}^{2d} \cong \mathbb{C}^d, \omega_0, T^d)$:

$\omega_0 = \sum dx_i \wedge dy_i = \frac{\sqrt{-1}}{2} \sum dz_k \wedge d\bar{z}_k$,

$(e^{2\pi\sqrt{-1}t_1}, \cdots, e^{2\pi\sqrt{-1}t_d}).(z_1, \cdots, z_d) = (e^{2\pi\sqrt{-1}t_1} z_1, \cdots, e^{2\pi\sqrt{-1}t_d} z_d)$

Then the moment map is

$$\Psi: \mathbb{C}^d \to (\mathbb{R}^d)^*, \quad (z_1, \cdots, z_d) \longmapsto \pi(|z_1|^2, \cdots, |z_d|^2) + \text{const.}$$

for abelian action, moment maps are unique up to a constant.

Choose the constant to be the $(\lambda_1, \cdots, \lambda_d)$ in the def of given Δ.

$$\therefore \Psi(z_1, \cdots, z_d) = \pi(|z_1|^2, \cdots, |z_d|^2) + (\lambda_1, \cdots, \lambda_d) \geq (\lambda_1, \cdots, \lambda_d)$$

The subtorus N acts on $\mathbb{C}^d$ in a Ham way with the moment map $i^* \circ \Phi$ where $i^* : (\mathbb{R}^d)^* \to \mathfrak{n}^*$ is dual to $i : \mathfrak{n} \to \mathbb{R}^d$

($\because$ the action is $N \overset{i}{\hookrightarrow} T^d \curvearrowright \mathbb{C}^d$)

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{R}^n)^* \arrow[r, "\pi^*"] & (\mathbb{R}^d)^* \arrow[r, "i^*"] & \mathfrak{n}^* \\ & \mathbb{C}^d \arrow[ur, "\Phi"] \arrow[urr, dashed, "i^* \circ \Phi"'] & & \end{tikzcd}$$

Thm A : $(i^* \circ \Phi)^{-1}(0)$ is a cpt subset of $\mathbb{C}^d$ and N acts freely on it.

In other words, it's legitimate to reduce $\mathbb{C}^d$ with respect to the action of N, and if we do, we'll get cpt symplectic mfd on which the quotient gp $T^n = T^d / N$ acts.

Denote $(i^* \circ \Phi)^{-1}(0)$ by Z and assume Thm A is true. Then N acts freely on Z implies that $0 \in \mathfrak{n}^*$ is a regular value of $i^* \circ \Phi$, Hence Z is a submfd of $\mathbb{C}^d$, and is cpt by Thm A with $\dim_{\mathbb{R}} = 2d - \underbrace{(d-n)}_{\text{dim of } \mathfrak{n}^*} = d+n$ by regular value thm.

Now since N is cpt and the action is free, the quotient space $M_\Delta := (i^* \circ \Phi)^{-1}(0) / N$ is a cpt mfd of $\dim_{\mathbb{R}} M_\Delta = (d+n) - \underbrace{(d-n)}_{\text{dim of } N} = 2n$

Moreover, also b/c the action is free & (Z/N) cpt $\Rightarrow$ $Z \xrightarrow{/N} M_\Delta$ is a principal N-bundle over M_Δ.

Consider the diagram:

$$\begin{tikzcd} Z \arrow[r, hook, "j"] \arrow[d, "p"', "/N"] & \mathbb{C}^d \arrow[d, "i^* \circ \Phi"] \arrow[loop right, "N"] \\ M_\Delta & \mathfrak{n}^* \end{tikzcd}$$

, then Marsden-Weinstein-Meyer thm on this setting $\Rightarrow$ $\exists\ \omega_\Delta$ symplectic form on M_Δ s.t. $p^* \omega_\Delta = j^* \omega_0$

of $\mathbb{C}^d$

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claim: this $(M_\Delta, \omega_\Delta)$ is a Ham T^n-space with a moment map Φ s.t. $\Phi(M_\Delta) = \Delta \subset (\mathbb{R}^n)^*$

In the proof of Thm A, case (3) of the free action claim, we showed that if $z \in \Phi^{-1}(\pi^*(p))$, where p is a vertex of Δ, then $(T^d)_z \cap N = \{1\}$ in T^d. This group is mapped bijectively by π onto T^n, (see the proof), hence by identifying this gp with T^n, we get an embedding $T^n \longrightarrow T^d$ (of couse this depends on the choice of this vertex p)

Thus we can think of T^n as a subgp of T^d.

The action of T^n on $\mathbb{C}^d$ commutes with the action of N so there is an induced Ham action of T^n on $(i^* \circ \Phi)^{-1}(0)/N$ whose moment map is given by the following Lemma

Lemma: (M, ω, Φ) is a Ham G-space & G acts freely on $\Phi^{-1}(0)$. If a Lie gp H acts Ham on M and commute with G-action then the induced action on $\Phi^{-1}(0)/G$ is again Ham, and the moment maps are related by: $\Phi_H \circ j = \underline{\Phi}_H \circ q$, where

$\underline{\Phi}_H : \Phi^{-1}(0)/G \to \mathfrak{h}^*$, $\Phi_H : M \to \mathfrak{g}^* \to \mathfrak{h}^*$, $j : \Phi^{-1}(0) \to M$, $q : \Phi^{-1}(0) \to \Phi^{-1}(0)/G$

stands for quotient

$$\begin{tikzcd} \Phi_H : M \arrow[r, "\Phi"] & \mathfrak{g}^* \arrow[r, "Pr"] & \mathfrak{h}^* \\ \Phi^{-1}(0) \arrow[u, "j"] \arrow[r, "q"] & \Phi^{-1}(0)/G \arrow[ur, "\underline{\Phi}_H"'] & \end{tikzcd}$$

so denote this moment map of the T^n-action on $(i^* \circ \Phi)^{-1}(0)/N$ by Φ

Claim: $\Phi(M_\Delta) = \Delta$ (will prove it after proving Thm A) ※

pf of Thm A:

1. Consider the exact sequence

$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0$$

and its dual

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{R}^n)^* \arrow[r, "\pi^*"] & (\mathbb{R}^d)^* \arrow[r, "i^*"] & \mathfrak{n}^* \arrow[r] & 0 \\ & \Delta \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \Delta' \arrow[u, phantom, "\cup"] & & \end{tikzcd}$$

so $\ker i^* = \operatorname{Im} \pi^*$. Let $\Delta' = \pi^*(\Delta)$

Since π^* is injective $\Delta' \simeq \Delta$.

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{R}^n)^* \arrow[r] & (\mathbb{R}^d)^* \arrow[r, "i^*"] & \mathfrak{n}^* \\ & & \mathbb{C}^d \arrow[u, "\Phi"] \arrow[ur, "i^*\circ\Phi"'] & \end{tikzcd}$$

claim: $(i^* \circ \Phi)^{-1}(0) = \Phi^{-1}(\Delta')$

pf: since $\Phi(z) = \pi(|z|^2) + \lambda \geq \lambda$

$\therefore$ image of Φ are $\{ x \in (\mathbb{R}^d)^* \mid (x_1, \cdots, x_d) \geq (\lambda_1, \cdots, \lambda_d) \}$

$\therefore$ $\Phi(\mathbb{C}^d) = \{ x \in (\mathbb{R}^d)^* \mid \langle x, e_i \rangle \geq \lambda_i, \ \forall i = 1, \cdots, d \}$

If $x \in (\mathbb{R}^d)^*$ is in the ker of i^* i.e. $i^*(x) = 0$

then by exactness, $x = \pi^*(y)$ for some $y \in (\mathbb{R}^n)^*$

$\therefore$ if $i^*(x) = 0$ & $x \in \operatorname{Im}(\Phi)$ then

$$\langle e_i, \pi^*(y) \rangle = \langle e_i, x \rangle \geq \lambda_i \quad \forall i = 1, \cdots, d$$

$$\parallel$$

$\therefore$ $\langle \pi(e_i), y \rangle = \langle u_i, y \rangle$

$\therefore$ $y \in \Delta$, and since $x = \pi^* y \Rightarrow x \in \Delta'$

$\therefore$ $i^*(x) = 0$ & $x \in \operatorname{Im}(\Phi) \Leftrightarrow x \in \Delta'$ $\therefore$ $(i^* \circ \Phi)^{-1}(0) = \Phi^{-1}(\Delta')$ #

Now Δ' cpt and Φ proper $\Rightarrow (i^* \circ \Phi)^{-1}(0) \subset \mathbb{C}^d$ is cpt.

2. Let $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$, and let $I(z) = \{ i, \ z_i = 0 \}$

Let $\mathbb{R}^{I(z)} = \{ (x_1, \cdots, x_d) \in \mathbb{R}^d, \ x_i = 0 \text{ for } i \notin I(z) \} \subset \mathbb{R}^d$

Let $T^{I(z)}$ be the image of $\mathbb{R}^{I(z)} \subset \mathbb{R}^d$ in T^d

claim: The stabilizer of z in T^d is $T^{I(z)}$

Pf: $(T^d)_z = \{ t \in T^d \mid t.z = z \}$

$= \{ (e^{2\pi\sqrt{-1}t_1}, \cdots, e^{2\pi\sqrt{-1}t_d}) \in T^d \mid$

$(z_1, \cdots, z_d) = (e^{2\pi\sqrt{-1}t_1} z_1, \cdots, e^{2\pi\sqrt{-1}t_d} z_d) \}$

$\therefore$ if $z_j \neq 0$, then "$=$" iff $e^{2\pi\sqrt{-1}t_j} = 1$

$\therefore (T^d)_z = \{ (e^{2\pi\sqrt{-1}t_1}, \cdots, e^{2\pi\sqrt{-1}t_d}) \in T^d \mid e^{2\pi\sqrt{-1}t_j} = 1, \forall j \notin I(z) \}$

$= T^{I(z)}$ #

3. Since $(i^* \circ \Phi)^{-1}(0) = \Phi^{-1}(\Delta') = \Phi^{-1}(\pi^*(\Delta))$

and $\ker i^* = \operatorname{Im} \pi^*$,

Thus $z \in (i^* \circ \Phi)^{-1}(0)$ iff $\Phi(z) = \pi^*(p)$ for some $p \in \Delta$

Case 1: If $p \in \mathring{\Delta}$, the interior of Δ, then T^d acts freely on $\Phi^{-1}(\pi^*(p))$

$p \ \Delta \longrightarrow \Delta$

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} n^* \to 0$$

$\Delta \qquad \Delta'$

$$\mathbb{C}^d \xrightarrow{\Phi} (\mathbb{R}^d)^*$$

denote $x = \pi^*(p) = \Psi(z)$ (i.e. z is any element in $\Psi^{-1}(\pi^*(p))$)

now since $p \in \mathring{\Delta} \iff \langle u_i, p \rangle > \lambda_i \ \forall i$ ("=" are the facets)

$$\langle u_i, p\rangle = \langle \pi(e_i), p \rangle = \langle e_i, \pi^*(p) \rangle = \langle e_i, x \rangle = x_i$$

O.T.H., since $x = \Psi(z) = \pi(|z_1|^2 + \cdots) + (\lambda_1, \cdots, \lambda_d) \iff x_i = \pi |z_i|^2 + \lambda_i$

$\therefore x_i > \lambda_i \iff |z_i|^2 > 0 \ \forall i \quad \therefore I(z) =$ empty set

$\therefore (T^d)_z = T^{I(z)} = \{(1, 1, \cdots, 1)\}$ i.e. free action

case 2: p is on the boundary & can be described as

$\langle u_i, p \rangle = \lambda_i, \quad i \in I$ for some I index set

then $(T^d)_z = T^I$ (where $z \in \Psi^{-1}(\pi^*(p))$)

pf: exercise (slight modification of case 1)

Case 3: worse case: p is one of the vertex.

W.L.O.G, assume $p = 0$, and reindex them if necessary so that the hyperplanes meeting at p are $\langle u_i, p \rangle = 0$, $i = 1, \cdots, n$

So if $z \in \Psi^{-1}(\pi^*(p))$, then $I(z) = \{1, \cdots, n\}$

Delzant polytope means $u_1, \cdots, u_n$ is the standard basis of $\mathbb{R}^n$ $\therefore$ The map $\pi: e_i \to u_i$, $i = 1, \cdots, d$ maps the subspace

$$\mathbb{R}^{I(z)} = \{(x_1, \cdots, x_n, 0, \cdots, 0) \in \mathbb{R}^d\} \cong \mathbb{R}^n \cong \{\sum_{i=1}^n c_i e_i, \ c_i \in \mathbb{R}\}$$

bijectively onto $\mathbb{R}^n$ and maps the corresponding torus $T^{I(z)}$ bijectively onto T^n. Since $N = \ker \pi$, this means

$T^{I(z)} \cap N = \{1\}$ trivial subgp of T^d $\therefore (T^d)_z \cap N = \{1\}$

i.e. N acts freely at $z \in \Phi^{-1}(\pi^*(p))$

Combine all $\Rightarrow$ N acts freely on $\Phi^{-1}(\pi^*(\Delta)) = \Phi^{-1}(\Delta') = (i^* \circ \Phi)^{-1}(0)$

$\divideontimes$

The claim for $\Phi(M_\Delta) = \Delta$:

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{R}^n)^* \arrow[r, "\pi^*"] & (\mathbb{R}^d)^* \arrow[l, bend left, "s"] \arrow[rr, "i^*"] & & \mathfrak{n}^* \\
 & & \mathbb{C}^d \arrow[u, "\Phi"] \arrow[urr, "i^* \circ \Phi"'] & & \\
 & M_\Delta = Z/N \arrow[uu, "\Phi"] & Z = (i^* \circ \Phi)^{-1}(0) \arrow[l, "Pr"'] \arrow[u, hook, "j"] & &
\end{tikzcd}$$

$$\begin{aligned}
\Phi(M_\Delta) &= \Phi(Pr(Z)) \\
&\overset{\text{by Lemma on 3 pages before}}{=} S \circ \Phi \circ j(Z) \\
&= S \circ \Phi(Z) \\
&= S(\Delta') = \Delta
\end{aligned}$$

$S \circ \pi^* = id$ & $S(\underbrace{\pi^*(\Delta)}_{= \Delta'}) = \Delta$

because

$Z = i^* \circ \Phi^{-1}(0)$

$= \Phi^{-1}(\Delta')$

$v = u_1 \quad u_2 = -v$

$\overset{\rightarrow}{[}\ \overset{\leftarrow}{]}$ two facets

$0 \quad 1$

Example: let $\Delta = [0,1] \subset \mathbb{R}^*$ $(n=1, d=2)$

let $v=1$ be the standard vector in $\mathbb{R}$

then $\Delta = \{ x \in \mathbb{R}^* \mid \langle x, u_1 \rangle \geq \underbrace{0}_{\lambda_1}, \langle x, u_2 \rangle \geq \underbrace{-1}_{\lambda_2} \}$

the projection $\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}$, $e_1 \longmapsto u_1$, $e_2 \longmapsto u_2$

$\therefore \ker \pi = \{ a e_1 + b e_2 \mid a - b = 0 \} = \{ a(e_1+e_2) \mid a \in \mathbb{R} \}$

So $N \cong S^1 = \{ (g,g) \in S^1 \times S^1 \}$ is the diagonal subgp of $S^1 \times S^1$

The exact seq: $0 \to N \xrightarrow{i} T^2 \xrightarrow{\pi} S^1 \to 0$

$$0 \to \mathbb{R}^* \xrightarrow{\pi^*} (\mathbb{R}^2)^* \xrightarrow{i^*} n^* \to 0$$

$$(x_1, x_2) \overset{i^*}{\longmapsto} x_1 + x_2$$

the induced action of N on $\mathbb{C}^2$ is

$$(e^{2\pi\sqrt{-1}t}, e^{2\pi\sqrt{-1}t}).(z_1, z_2) = (e^{2\pi\sqrt{-1}t} z_1, e^{2\pi\sqrt{-1}t} z_2)$$

(λ_1, λ_2)

and moment map $(i^* \circ \Phi)(z_1, z_2) = \pi(|z_1|^2 + |z_2|^2) + (0, -1)$

$\therefore$ the zero level set is $\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = \frac{1}{\pi} \}$

Hence $M_\Delta = (i^* \circ \Phi)^{-1}(0) / N \cong S^3 / S^1 \underset{\underset{(\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^*}{\uparrow}}{\cong} \mathbb{C}P^1$