LECTURES ON THE TWO-BODY PROBLEM (Extract)

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The Radial Potential Problem and Newton's Method

Here we treat a problem more general than Kepler's problem, defined instead by the system

$$\ddot{x} = -\varphi(I)x, \qquad \ddot{y} = -\varphi(I)y, \tag{6}$$

where $I = x^2 + y^2$ and φ is an arbitrary real function. The Kepler problem corresponds to the particular case $\varphi(I) = I^{-3/2}$.

We shall reduce this problem to quadrature. The specific features of Kepler's problem will appear only in the process of quadrature.

In order to justify our use of the term "radial potential problem," we note that the "force field" which defines the system (6) may be characterized by any two of the following three properties, which in turn imply the third property:

i) The force field is invariant under rotations of the plane that fix the origin

ii) The force is central

iii) The force field is derived from a potential

Property (ii) signifies that the force is directed toward (or away from) the origin. Property (iii) signifies that there exists a function whose field of gradient vectors is the force field. When the force field is that of system (6), this function may be written $-\frac{1}{2}\Phi(I)$, where Φ denotes a primitive of the function φ. In the Kepler problem, we choose the homogeneous primitive $\Phi(I) = -2I^{-1/2}$ of $\varphi(I) = I^{-3/2}$.

In order to solve (6), we note the following three properties.

i) The isometries of the plane fixing $(0,0)$ form a symmetry group of (6). That is, the transformation

$$\begin{pmatrix} x & \dot{x} \\ y & \dot{y} \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \begin{pmatrix} x & \dot{x} \\ y & \dot{y} \end{pmatrix}$$

sends any solution of (6) onto a solution of (6) provided $a^2 + b^2 = a'^2 + b'^2 = 1$ and $aa' + bb' = 0$.

ii) The quantity $C = x\dot{y} - y\dot{x}$ is a first integral of system (6).

iii) The quantity $H = \frac{1}{2}\big(\dot{x}^2 + \dot{y}^2 + \Phi(I)\big)$, the energy, is a first integral of system (6).

We also note that the first integrals H and C^2 are invariant under the isometries (i), and are functionally independent. We can already foresee that the reduction process we are about to use will lead us to an autonomous system of order one.

The reduction takes place in three stages, one for each of the ingredients (i), (ii), and (iii). We may choose the order of the stages. We begin by using (i) to reduce the order by one. We choose three independent functions of $(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4$ invariant under the action of the group. The most natural choices are $I = x^2 + y^2$, $J = x\dot{x} + y\dot{y}$ and $K = \dot{x}^2 + \dot{y}^2$. Upon calculating the derivative and replacing $\ddot{x}$ and $\ddot{y}$ by the right-hand sides of (6), we obtain:

$$\dot{I} = 2J, \quad \dot{J} = K - I\varphi(I), \quad \dot{K} = -2J\varphi(I). \tag{7}$$

This is the system of order three that we sought. Some remarks will serve to clarify its status.

The classical treatises on Lagrangian mechanics describe the preceding reduction in terms of *ignorable variables*. Consider the angle θ such that $x = \sqrt{I}\cos\theta$ and $y = \sqrt{I}\sin\theta$. On suitable open sets, the transformation $(x, y, \dot{x}, \dot{y}) \mapsto (I, J, K, \theta)$ is a smooth change of variables. System (6) is transformed into the three equations (7) and the equation

$$\dot{\theta} = \frac{C}{I}. \tag{8}$$

The angle θ is "ignorable" because it does not appear in the right-hand sides above; we may study system (7) while ignoring θ. Once an explicit solution of (7) is found, we may deduce a family of solutions of (6) from it by integrating $d\theta = C dt/I$, thereby introducing a constant of integration.

To this classical presentation, we prefer an explanation in terms of the *quotient space*, which has the advantage of not introducing an angle θ which is subsequently ignored. In $(x, y, \dot{x}, \dot{y})$-space, let us visualize the *orbits of the action of the group of isometries* (i), not to be confused with the orbits of the system which we ultimately seek to describe. Except for degeneracies, these are topologically pairs of circles in the space $\mathbb{R}^4$, because the group of plane isometries is topologically a pair of circles. We may check that C is positive on one circle and negative on the other, and that the degeneracies are characterized precisely by the equation $C = 0$. To give meaning to system (7), it is enough to note that a triple (I, J, K) characterizes one of these orbits. The system thus describes motion in a new, three-dimensional space whose "points" are the pairs of circles above. This is the quotient space.

We continue to reduce the order. The quantities H and C^2 are first integrals of system (7). We have

$$2H = K + \Phi(I) \qquad \text{and} \qquad C^2 = IK - J^2. \tag{9}$$

From the geometric standpoint, this remark achieves the reduction to order one: these two equations define curves in (I, J, K)-space which are invariant under system (7) and on which this system induces an autonomous differential equation of order one.

In preparation for quadrature, one ordinarily writes this differential equation explicitly by eliminating J and K. By eliminating K from (9), we obtain

$$C^2 + J^2 = -I\Phi(I) + 2HI\,, \tag{10}$$

which may be solved for J. Then system (7) becomes the autonomous differential equation

$$\dot{I} = \pm 2\sqrt{-I\Phi(I) + 2HI - C^2}. \tag{11}$$

This is the form of the equation we expected. However, we shall put (11) aside to avoid entering into considerations of the appropriate sign in front of the square root. We prefer to work with (10), since it more faithfully represents the geometry and topology of curves in (I, J, K)-space: a closed curve does not lend itself to parametrization by one of the three coordinates. Having made this remark, we are ready to present a good algorithm for integrating system (6).

i) Choose a quadruple $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ of initial conditions and an initial time t_0. Calculate H and C. Calculate the initial values I_0, J_0, and θ_0 of the variables I, J, and the angle θ.

ii) Consider the curve (10) in the (I, J)-plane. It contains the initial point (I_0, J_0). Now *move* along this curve, in other words choose a parameter w and a path (I_w, J_w) which describes the curve. We take $w = 0$ at the initial point.

iii) Integrate along this path the following two differential forms (which are smooth on the curves (10) thanks to the symmetry $J \mapsto -J$ of the curves):

$$dt = \frac{dr}{\dot{r}} = \frac{rdr}{r\dot{r}} = \frac{dI}{2J} \qquad \text{and} \qquad d\theta = \dot{\theta}dt = \frac{Cdt}{r^2} = \frac{CdI}{2IJ}. \tag{12}$$

In this way we obtain the functions t_w and θ_w of the parameter w.

iv) Now return to the original variables. The quintuple $(x_w, y_w, \dot{x}_w, \dot{y}_w, t_w)$ is a function of the parameter w. We may consider w as a function of t since dt does not vanish.

Bernoulli's Method for the Particular Case of $1/r^2$-Attraction

When $\varphi(I) = I^{-3/2}$, equation (10) becomes

$$C^2 + J^2 = 2\sqrt{I} + 2HI \tag{13}$$

We arrive to a rational equation by setting $r = \sqrt{I}$ or $\rho = 1/\sqrt{I}$.

i) *The variable r and the eccentric anomaly u.* We call the curve described by $J^2 = 2Hr^2 + 2r - C^2$ the *first auxiliary conic*. We set

$$k^2 = 1 + 2HC^2. \tag{14}$$

The condition $k^2 \geq 0$ is equivalent to the existence of a real point on this conic section, or to the existence of a real point in the intersection of the conic section and the half-plane $r > 0$. We distinguish three possibilities: $H < 0$, $H = 0$, or $H > 0$. The particular case $C = 0$ is notable only for the following reason:

the conic section contains a point such that $r = 0$. We restrict ourselves to the case $H < 0$, where the auxiliary conic is an ellipse entirely contained in the half-plane $r \geq 0$. We set

$$a = -\frac{1}{2H} \tag{15}$$

and we introduce the trigonometric parametrization of the auxiliary ellipse by the angle u, called the *eccentric anomaly*:

$$r = a(1 - k\cos u), \qquad J = k\sqrt{a}\sin u. \tag{16}$$

We obtain

$$dt = r\sqrt{a}\,du = a^{3/2}(1 - k\cos u)du \qquad \text{and} \qquad d\theta = \frac{C\,du}{\sqrt{a}(1 - k\cos u)}. \tag{17}$$

The first equation immediately gives *Kepler's equation*

$$a^{-3/2}(t - t_0) = u - k\sin u. \tag{18}$$

To integrate the second equation, it is enough to use a rational parametrization of the ellipse, in other words, to take as the new variable $\tan(u/2)$, which leads us to introduce a new angle v, depending on u through the formula

$$\tan\frac{u}{2} = \sqrt{\frac{1-k}{1+k}}\tan\frac{v}{2}.$$

We shall be led to this angle in the following section (ii), where we will prefer the variable $\rho = 1/r$ to the variable r at the outset. The hypothesis $H < 0$ will not be necessary.

ii) *The variable ρ and the true anomaly v.* We multiply both sides of (13) by ρ^2 and we note that it is convenient to use the variable $\dot{r}$ as the ordinate rather than $J = r\dot{r}$. There remains $\dot{r}^2 = 2H + 2\rho - C^2\rho^2$. This *second auxiliary conic* possesses a real point if and only if $k^2 = 1 + 2HC^2 \geq 0$. It is an ellipse provided C is nonzero, with trigonometric parametrization:

$$\rho = \frac{1 + k\cos v}{C^2}, \qquad \dot{r} = \frac{k\sin v}{C}. \tag{19}$$

We obtain

$$dt = \frac{C^3}{(1 + k\cos v)^2}dv \qquad \text{and} \qquad d\theta = dv.$$

The first quadrature is difficult this time, but the second is miraculously simple:

$$\theta - \theta_0 = v. \tag{20}$$

iii) *Synthesis.* Equation (20) makes v into a polar angle. The first equation (19) is thus the equation of a conic section with eccentricity k. The angle θ_0 indicates the direction of the pericenter: r is minimal for $\theta = \theta_0$.

Classically, one introduces three *anomalies*, defined as angles starting from the pericenter, and distinguished from the *longitudes*, which are angles starting from a fixed direction. Thus v is the *true anomaly* and θ the *true longitude*. The angle $l = a^{-3/2}(t - t_0)$, appearing on the left-hand side of Kepler's equation (18), is called the *mean anomaly*. Lastly, we have introduced the *eccentric anomaly* u, the trigonometric parametrization of the first auxiliary ellipse. It remains to see that it is also a trigonometric parametrization of the trajectory ellipse. This is easily deduced from the following proposition.

9. *Proposition.* Suppose the eccentricity k is nonzero. Let $\xi = r\cos v$ and $\eta = r\sin v$ be the coordinates of the body in a frame having the fixed center as its origin and the direction of the pericenter as the ξ-axis. The affine transformation $(r, J) \mapsto (\xi, \eta) = k^{-1}(C^2 - r, CJ)$ takes the first auxiliary conic onto the trajectory while preserving the time parametrization.

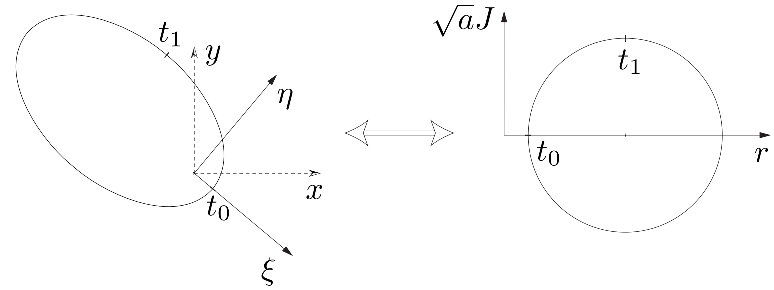

Figure. An elliptic trajectory and its first auxiliary conic

Proof. We multiply by r both equations (19). The first gives $r = -k\xi + C^2$, the second $J = k\eta/C$.

To give the trigonometric parametrization explicitly in the case of an ellipse of semi-major axis a, we use the system (16), taking into account the relation $C^2 = a(1 - k^2)$ which we deduce from (14) and (15). We obtain

$$\xi = a(\cos u - k), \qquad \eta = a\sqrt{1-k^2}\,\sin u. \tag{21}$$

Choosing the pericenter as the origin, rather than the apocenter, is justified—for example in Gauss's *Theoria Motus*—by the absence of an apocenter in the case of hyperbolic motion. The choice of the pericenter is thus required for the true anomaly, and is extended to the other anomalies. The eccentric anomaly is specific to the ellipse, and does not continue through $H = 0$. However it has an exact analog for $H > 0$. In the formulas it suffices to replace the sin and cos by sinh and cosh, the $\sqrt{a}$ by $\sqrt{-a}$, and finally the $a^{3/2}$ by $a\sqrt{-a}$.