Suppose A has Jordan form J.

$\Rightarrow \exists$ invertible Q s.t. $Q^{-1}AQ = J$.

Q: How are sol. of $\dot{x} = Ax$ and $\dot{y} = Jy$ related?

Consider $\dot{x} = Ax$.

General sol. is $e^{At}\xi = Qe^{Jt}\underbrace{(Q^{-1}\xi)}_{= \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}}$

$= \sum_i c_i (i^{th}$ column of $Qe^{Jt})$

— linear combination of columns of Qe^{Jt}.

Set $y = Q^{-1}x$. $\dot{y} = Q^{-1}\dot{x} = Q^{-1}Ax = Q^{-1}AQy = Jy$.

$\therefore$ $x = Qy$, where y is a sol. of $\dot{y} = Jy$.

Remark. The D.E. $\dot{x} = Ax$ can be generalized to $A \in \mathbb{C}^{d\times d}$. $x(t) \in \mathbb{C}^d$.

$x = x_0 + i x_1$. $A = A_0 + iA_1$. — real & imag. parts.

$\dot{x} = Ax \Leftrightarrow \dot{x}_0 = A_0 x_0$, $\dot{x}_1 = A_1 x_1$

Theorems for existence, uniqueness, matrix exp. stability are still valid in the complex setting.

Theorem 3. (Stability of $\dot{x} = Ax$)

(a) If $\max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 0$, then $\exists\, \alpha > 0, K \geq 1$ s.t. $\|e^{At}\| \leq K e^{-\alpha t}$ $\forall t \geq 0$. ($\therefore x \equiv 0$ is asy. stable)

In particular, $e^{At} \to 0$ as $t \to \infty$.

(b) If $\max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \leq 0$ and eigenvalues w/ zero real part has algebraic multiplicity = geometric multiplicity, then $\exists\, M > 0$ s.t.
$$\|e^{At}\| \leq M \quad \forall t \geq 0.$$ ($\therefore$ Sol. are bdd.)

(c) If $\exists\, \lambda \in \sigma(A)$ s.t. $\mathrm{Re}\,\lambda > 0$, then
$$\|e^{At}\| \to \infty \text{ as } t \to \infty.$$ (unstable)

pf. (a) Let $-\alpha \in (\max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}, 0)$.

Each Jordan block of A is of the form
$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \text{ or } (\lambda). \quad \lambda = a + ib,\ a < -\alpha < 0.$$

$$e^{Jt} = e^{at} \cdot e^{ibt} \begin{pmatrix} 1 & t & t^2/2 & \cdots \\ & 1 & t & \cdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix} \text{ or } e^{at} \cdot e^{ibt}$$

$\Rightarrow \|e^{Jt}\| \leq e^{-\alpha t}$ for $t \geq 0$ large.

$\therefore \|e^{Jt}\| \leq K e^{-\alpha t}$ $\forall t \geq 0$, for some $K \geq 1$.

(b) When $\lambda = a + ib$, $a = 0$, the Jordan block is (λ). Then $e^{Jt} = e^{ibt} I$

$\therefore \|e^{Jt}\|$ is bounded for every Jordan block of A.

(c) For some Jordan block J, $J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$.

$\lambda = a + ib$. $a > 0$.

May assume it is the first Jordan block J_1.

$e^{J_1 t} = e^{at} \cdot e^{ibt} \begin{pmatrix} 1 & t & \cdots & \\ & \ddots & \ddots & \\ & & \ddots & t \\ & & & 1 \end{pmatrix}$ or $e^{at} \cdot e^{ibt} I$.

Pick $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. $\Rightarrow |e^{J_1 t} e_1| = e^{at} \longrightarrow \infty$ as $t \to \infty$.

$\therefore \|e^{Jt}\| \to \infty$ as $t \to \infty$. $\underline{\text{QED}}$.

Suppose $\det(A - tI) = a_0 t^d + a_1 t^{d-1} + \cdots + a_{d-1} t + a_d$.

$a_i \in \mathbb{R}$. $a_0 \neq 0$

Simple criteria for the # of roots w/ positive/negative real parts.

Descarte's Rule of Signs

Let p = # of sign changes in $(a_0, a_1, \ldots, a_d)$

g = # " positive real roots.

$\Rightarrow g \leq p$ and $p - g$ is even.

In particular, if $p = 1$, then $g = 1$.

if $p = 0$, then $g = 0$.

Routh-Hurwitz Criterion

$d=2.$ $a_1, a_2 > 0$

$d=3.$ $a_1, a_3, a_1 a_2 - a_0 a_3 > 0$

$d=4.$ $a_1, a_2, a_4, a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4 > 0$

$\Rightarrow$ real parts of roots < 0

eg. $3t^4 + 2t^3 - t - 4$. $\exists !$ positive real root

$t^4 + 7t^3 + 17t^2 + 17t + 6$ $(= (t+1)^2(t+2)(t+3))$

Coeff. > 0, no positive real root.

$17(7\cdot 17 - 1\cdot 17) - 7^2\cdot 6 = (17^2 - 7^2)6 > 0$

$\Rightarrow$ Real parts of roots < 0.

<u>Example</u> Consider $\dot{x} = Ax$ w/ $A = \begin{pmatrix} -6 & -2 & 2 & 2 \\ 4 & 0 & -2 & -2 \\ 6 & 3 & -5 & -1 \\ -2 & -1 & 1 & -3 \end{pmatrix}$

$\det(A - tI) = t^4 + 14t^3 + 68t^2 + 136t + 96$

$(= (t+2)^2(t+4)(t+6))$

$(a_0, a_1, a_2, a_3, a_4) = (1, 14, 68, 136, 96)$

$a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4 = 136(14\cdot 68 - 136) - 14^2\cdot 96$

$= 8^2\cdot 6\cdot 17^2 - 8^2\cdot 6\cdot 7^2 > 0.$

$\therefore$ By H-R criterion, it is stable.

Descarte's rule tells us that $\nexists$ positive root.

but does <u>not</u> imply $\nexists$ root w/ positive real part.

§3. Two-dimensional Linear Autonomous System

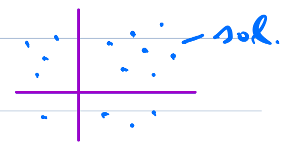
Consider the special case $\dot{x} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A} x$, $A \in \mathbb{R}^{2\times 2}$.
$\sigma(A) = \{\lambda_1, \lambda_2\}$.

Case 1. A has zero eigenvalue

Subcase 1.1. 2 zero eigenvalues $\Rightarrow A \sim O$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

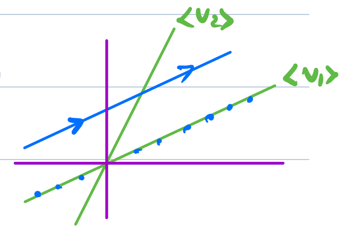
If $A \sim O$, then $A = O$

$\dot{x} = 0 \Rightarrow x = \text{const.} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$



If $A \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then A has eigenvector v_1 & generalized eigenvector v_2. $Av_2 = v_1$. $Av_1 = 0$

Sol. are
$$(v_1, v_2) \exp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t\right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= (v_1, v_2) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= (c_1 + t c_2) v_1 + c_2 v_2 = c_1 v_1 + c_2 (t v_1 + v_2)$$



eg. $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. $A = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$

$\det(A - \lambda I) = (-2-\lambda)(2-\lambda) + 4 = \lambda^2$. e.values: 0, 0.

$Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. $Q^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. $Q^{-1} A Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = J$

$e^{Jt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. $Qe^{Jt} = \begin{pmatrix} 2 & 2t+1 \\ 1 & t+1 \end{pmatrix}$

$\therefore$ Sol. are linear combinations of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

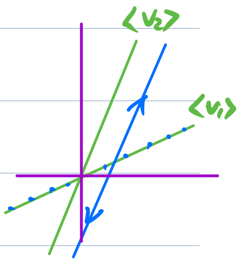
Subcase 1.2. 1 zero eigenvalue

$A \sim \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \Rightarrow \exists$ linearly indep. eigenvectors v_1, v_2

v_1 corresp. to 0, v_2 corr. to λ.

$Q = (v_1, v_2) \Rightarrow Q^{-1} A Q = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} = D$

$$Q e^{Dt} = (v_1, v_2) \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = (v_1, e^{\lambda t} v_2)$$



$\therefore$ Sol. are linear comb. of $v_1, e^{\lambda t} v_2$

Case 2. $\lambda_1 \neq \lambda_2$. $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$.

Subcase 2.1 $\lambda_2 < \lambda_1 < 0$ (stable node)

Corresponding eigenvectors: v_1, v_2. — lin. indep.

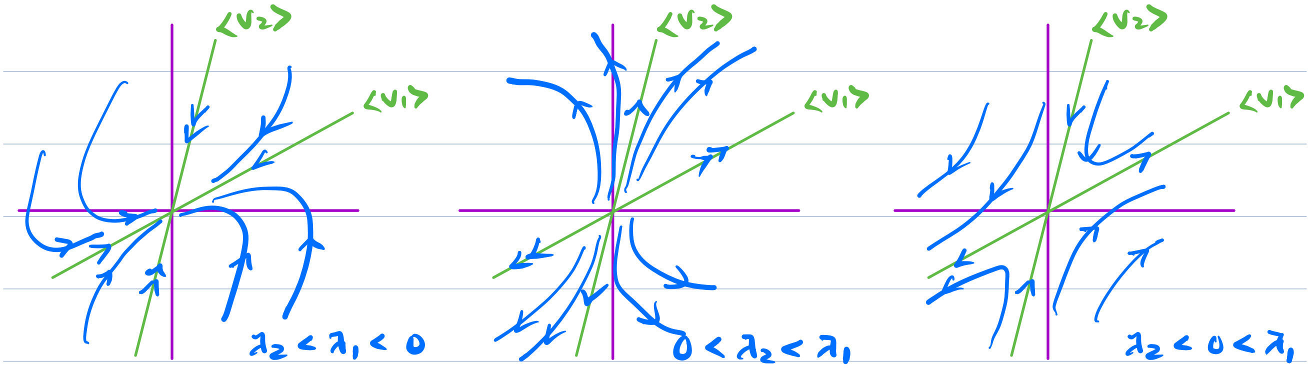
$Q = (v_1, v_2)$. $Q^{-1} A Q = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} = D$. $e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{pmatrix}$

$$Q e^{Dt} = (e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2)$$

$\therefore$ Sol. are linear comb. of $e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2$

Subcase 2.2. $0 < \lambda_2 < \lambda_1$ similar (unstable node)

Subcase 2.3. $\lambda_2 < 0 < \lambda_1$. Similar. (saddle)



Case 3. $\lambda_1, \lambda_2 \notin \mathbb{R}$. $\lambda_1 = \lambda = \alpha + i\beta$, $\lambda_2 = \bar\lambda$, $\beta \neq 0$.

$(\Rightarrow \lambda_1 \neq \lambda_2)$

Eigenvector corresp. to λ is v

" " " $\bar\lambda$ is $\bar v$

Complex sol. are linear comb. of $e^{\lambda t} v$, $e^{\bar\lambda t}\bar v$

i.e. $c_1 e^{\lambda t} v + c_2 e^{\bar\lambda t} \bar v$. $c_1, c_2 \in \mathbb{C}$.

Let $v = u + iw$, $u = \mathrm{Re}\, v$, $w = \mathrm{Im}\, v$.

$$e^{\lambda t} v = e^{\alpha t}(\cos\beta t + i \sin\beta t)(u + iw)$$

$$= e^{\alpha t}(u\cos\beta t - w \sin\beta t + i(w\cos\beta t + u \sin\beta t))$$

$$e^{\bar\lambda t}\bar v$$

$$= e^{\alpha t}(u\cos\beta t - w\sin\beta t - i(w\cos\beta t + u\sin\beta t))$$

$$\therefore \frac{1}{2}(e^{\lambda t} v + e^{\bar\lambda t}\bar v) = e^{\alpha t}(u\cos\beta t - w\sin\beta t)$$

$$\frac{1}{2i}(e^{\lambda t} v - e^{\bar\lambda t}\bar v) = e^{\alpha t}(w\cos\beta t + u\sin\beta t)$$

They are columns of $e^{\alpha t}(u, w)\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t\end{pmatrix}$

$$= (u, w)\exp\underbrace{\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}_{\text{Real Jordan form of } A}.$$

Take arbitrary consts c_1, c_2.

Rewrite them by $c_1 = a\cos\delta$, $c_2 = a\sin\delta$

(i.e. let $a^2 = c_1^2 + c_2^2$, $\cos\delta = \frac{c}{a}$, ...)

$$e^{\alpha t}(u, w)\begin{pmatrix}\cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t\end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

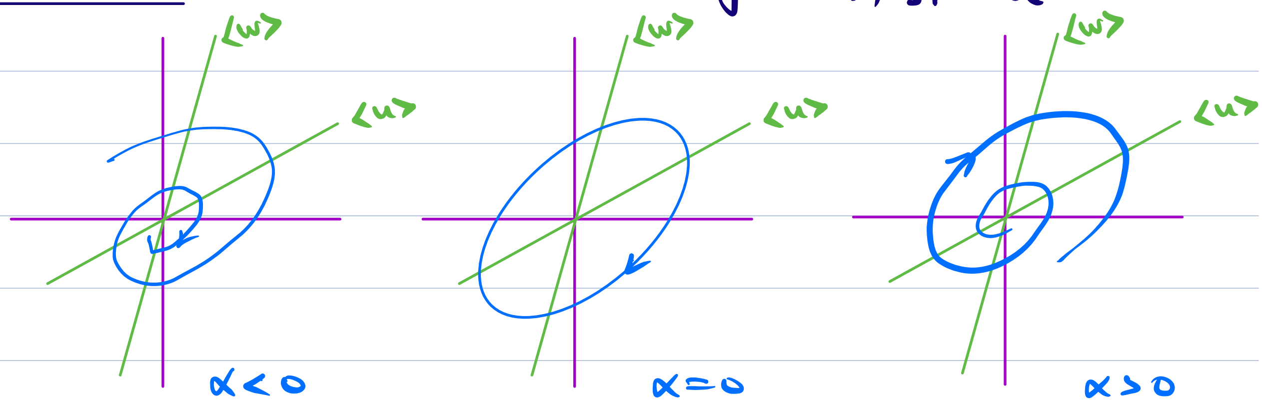
$$= a e^{\alpha t}(u, w)\begin{pmatrix}\cos(\beta t - \delta) \\ -\sin(\beta t - \delta)\end{pmatrix}$$

$$= a e^{\alpha t}\left(u\cos(\beta t-\delta) - w\sin(\beta t-\delta)\right)$$

Subcase 3.1. $\alpha < 0$ (stable focus, spiral in)

Subcase 3.2. $\alpha = 0$ (center)

Subcase 3.3. $\alpha > 0$ (unstable focus, spiral out)



Case 4. $\lambda_1 = \lambda_2 = \lambda \neq 0$. $\lambda_1, \lambda_2 \in \mathbb{R}$.

Subcase 4.1 $A \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\Rightarrow \exists$ lin. indep. eigenvectors v_1, v_2

$\therefore$ sol. are $(c_1 v_1 + c_2 v_2) e^{\lambda t}$

Subcase 4.2 $A \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

Let v_1 be an eigenvector, v_2 be the generalized eigenvector s.t. $A v_2 = v_1$.

$$Q = (v_1, v_2) \Rightarrow Q^{-1} A Q = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = J$$

$$Qe^{Jt} = e^{\lambda t}(v_1, v_2)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= e^{\lambda t}(v_1,\ t v_1 + v_2).$$

$\therefore$ Sol. are lin. combinations of

$e^{\lambda t} v_1,\ e^{\lambda t}(t v_1 + v_2).$