數理思維

第九講：正義與權利義務

John Rawls (1921-2002)

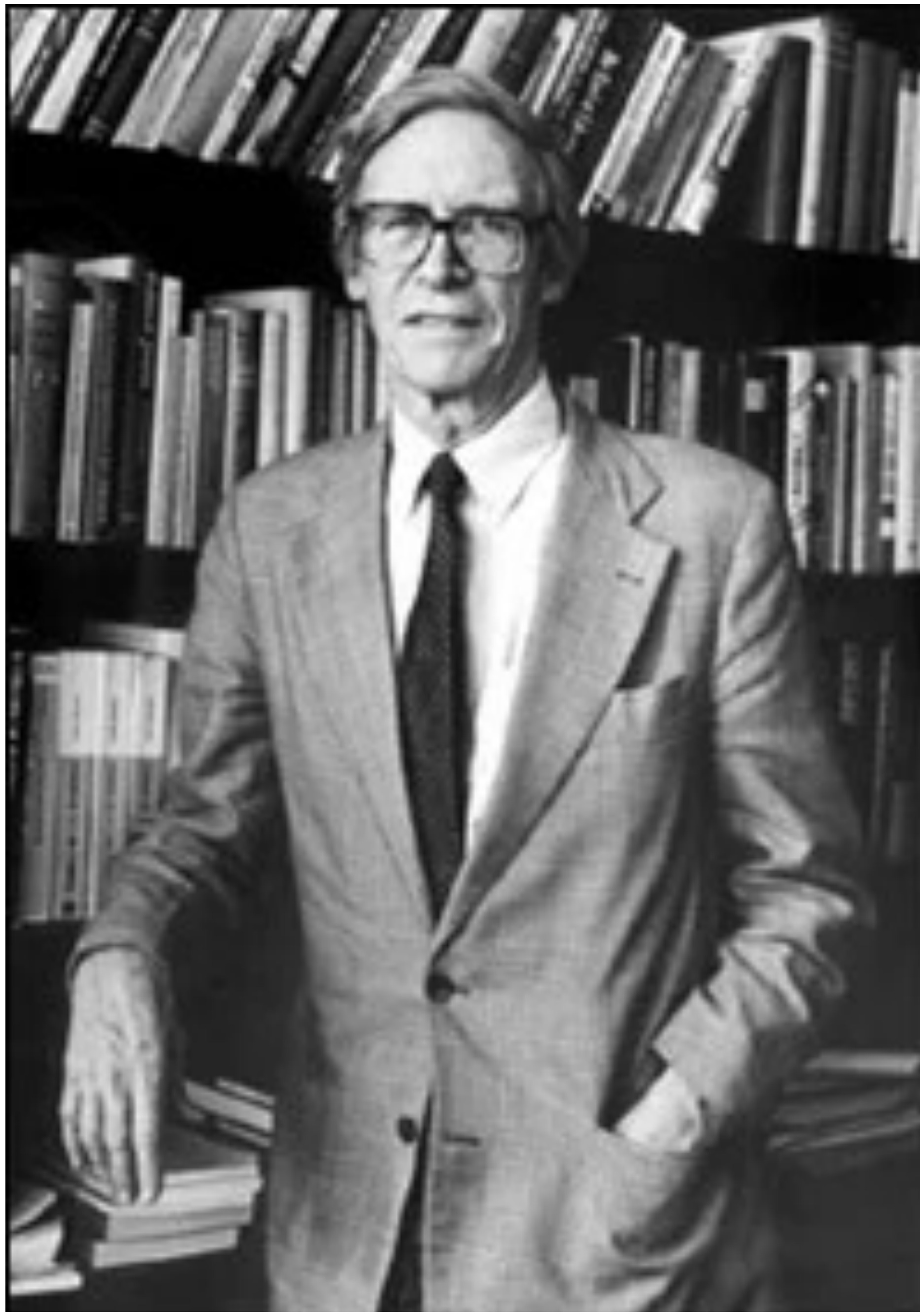
"Justice is the first virtue of social institutions, as truth is of systems of thought."

"While the distribution of wealth and income need not be equal, it must be to everyone's advantage, and at the same time, positions of authority and responsibility must be accessible to all."

"Injustice, then, is simply inequalities that are not to the benefit of all."

"The natural distribution is neither just nor unjust; nor is it unjust that persons are born into society at some particular position. These are simply natural facts. What is just and unjust is the way that institutions deal with these facts."

— John Rawls, *A Theory of Justice*, 1971



Source: The Harvard Gazette

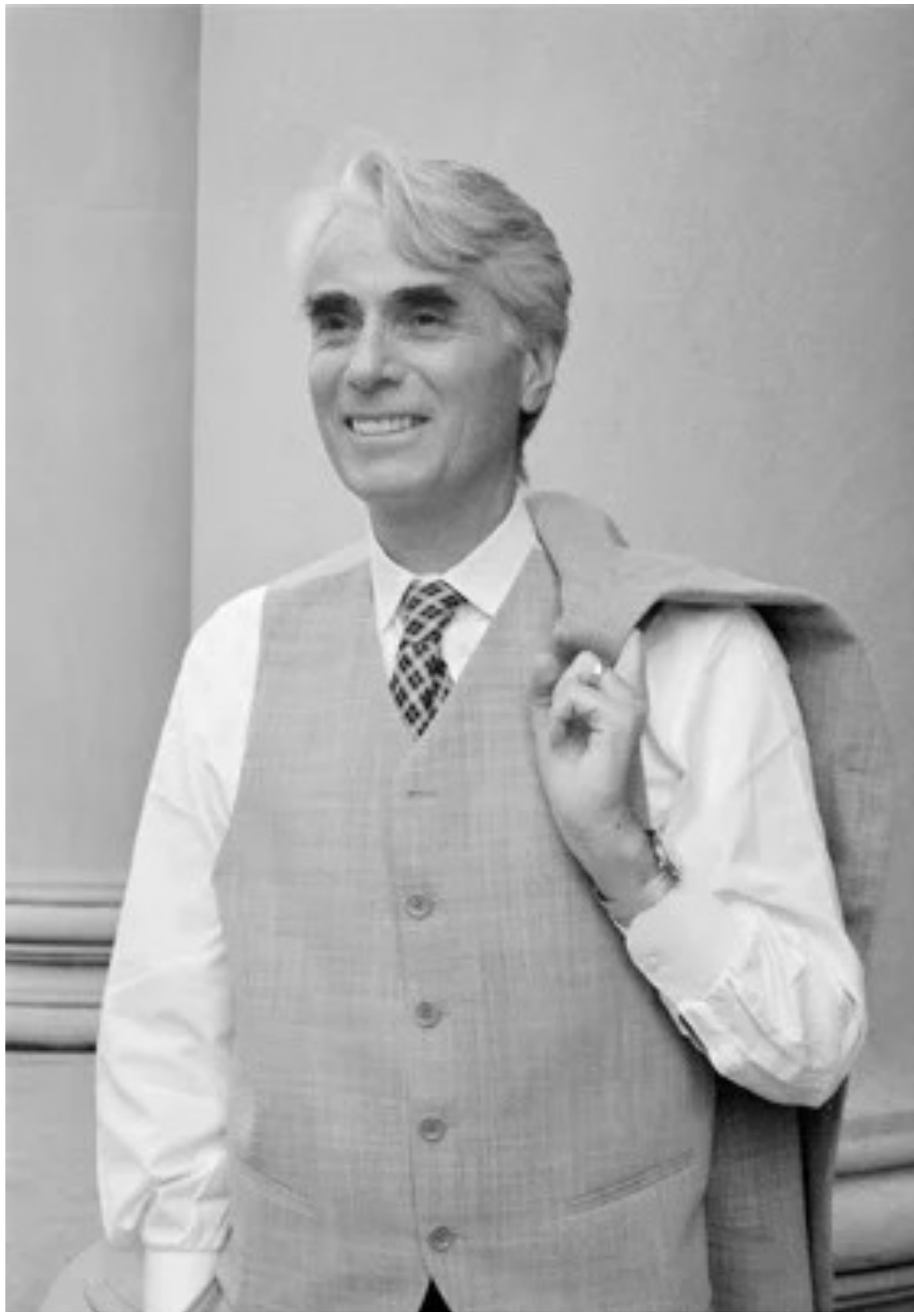
Robert Nozick (1938-2002)

"A distribution is just if it arises from another just distribution by legitimate means."

"Whatever arises from a just situation by just steps is itself just."

"Some people steal from others, or defraud them, or enslave them, seizing their product and preventing them from living as they choose, or forcibly exclude others from competing in exchanges. None of these are permissible modes of transition from one situation to another."

— Robert Nozick, *Anarchy, State, and Utopia*, 1974



Source: The Harvard Gazette

Michael Sandel (1953-)

"The mere fact that a group of people in the past agreed to a constitution is not enough to make that constitution just."

"Justice is not only about the right way to distribute things. It is also about the right way to value things."

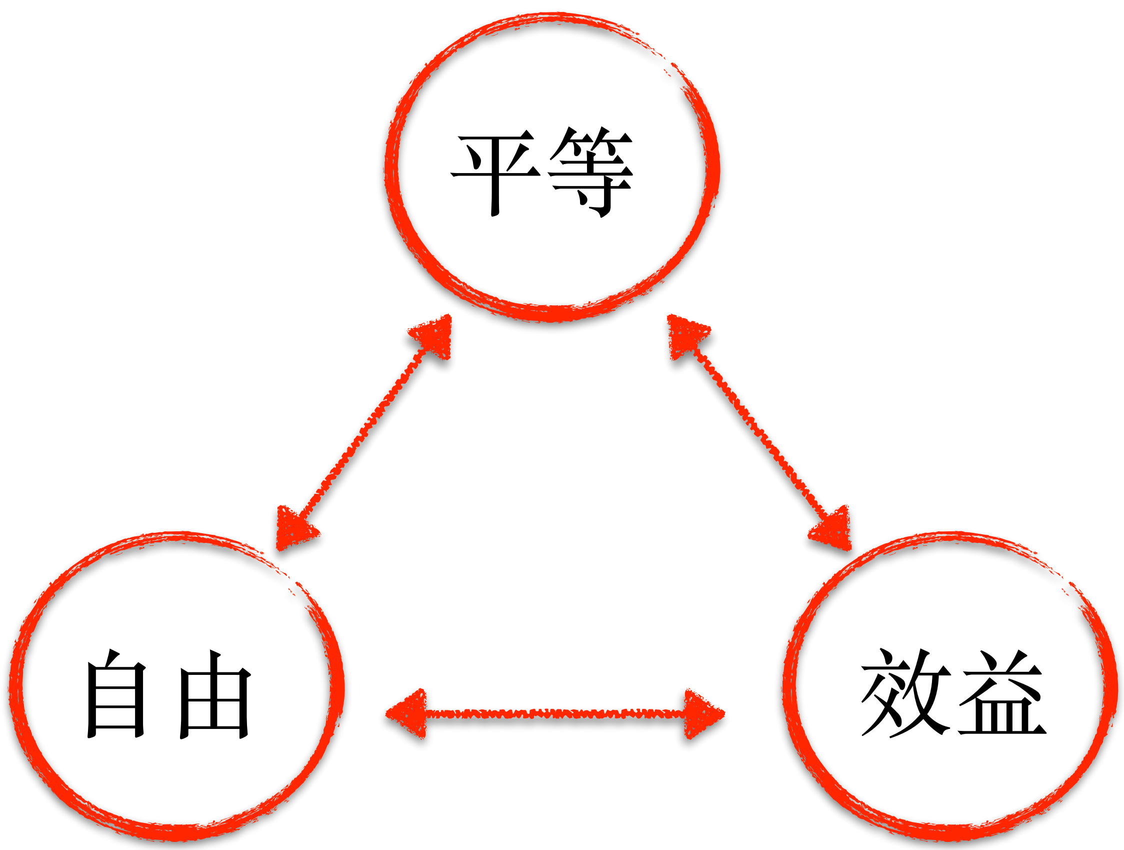
"To achieve a just society we have to reason together about the meaning of the good life, and to create a public culture hospitable to the disagreements that will inevitably arise."

"Debates about justice and rights are often, unavoidably, debates about the purpose of social institutions, the goods they allocate, and the virtues they honor and reward. Despite our best attempts to make law neutral on such questions, it may not be possible to say what's just without arguing about the nature of the good life."

— Michael Sandel, *Justice: What's the Right Thing to Do?* 2009

Source: The Harvard Gazette

分配正義的原則



如何兼顧/實踐？

Friedrich A. von Hayek (1899-1992)

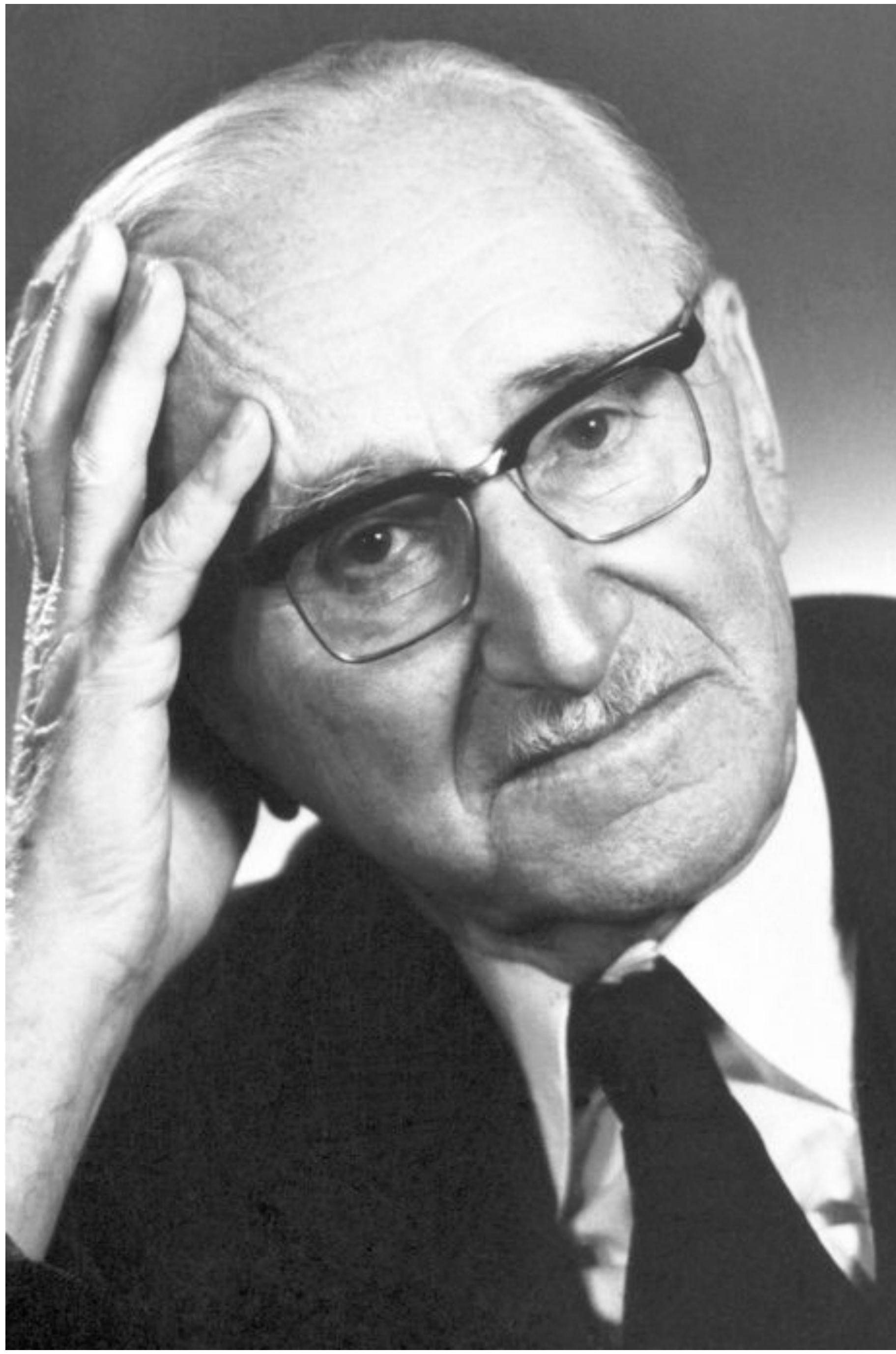
"There is all the difference in the world between treating people equally and attempting to make them equal. While the first is the condition of a free society, the second means as De Tocqueville describes it, a new form of servitude."

"We must face the act that the preservation of individual freedom is incompatible with a full satisfaction of our views of distributive justice."

— *Individualism and Economic Order*, 1948

"While an equality of rights under a limited government is possible and an essential condition of individual freedom, a claim for equality of material position can be met only by a government with totalitarian powers."

— *The Mirage of Social Justice*, 1976



Source: nobelprize.org

切蛋糕問題 — 隱喻分配正義的數學遊戲

n 個孩童分享一塊蛋糕, 如何做到公平分配？

隱喻：土地分割（繼承）、公共資源分配、時間配置等

課堂活動：考慮 n=2, n=3

公平原則

考慮 n 個人參與分配, 各人對蛋糕 C 的價值評估用加法函數

$V_1, V_2, \ldots, V_n : 2^C \rightarrow [0,1]$ 表示, $V_i(C) = 1 \ \forall i$ 。

2^C 表示C之子集合構成之集合 (power set of C)

令第 i 個人獲得分配的部分為X_i

1. 比例原則 (Proportionality): $V_i(X_i) \geq 1/n$

2. 無嫉妒原則 (Envy-freeness): $V_i(X_i) \geq V_i(X_j) \quad \forall i, j$

觀察：無嫉妒原則 $\Rightarrow$ 比例原則

證明：$\forall i, 1 = \Sigma_j V_i(X_j) \leq n V_i(X_i) \Rightarrow 1/n \leq V_i(X_i)$

抽象化的公平原則

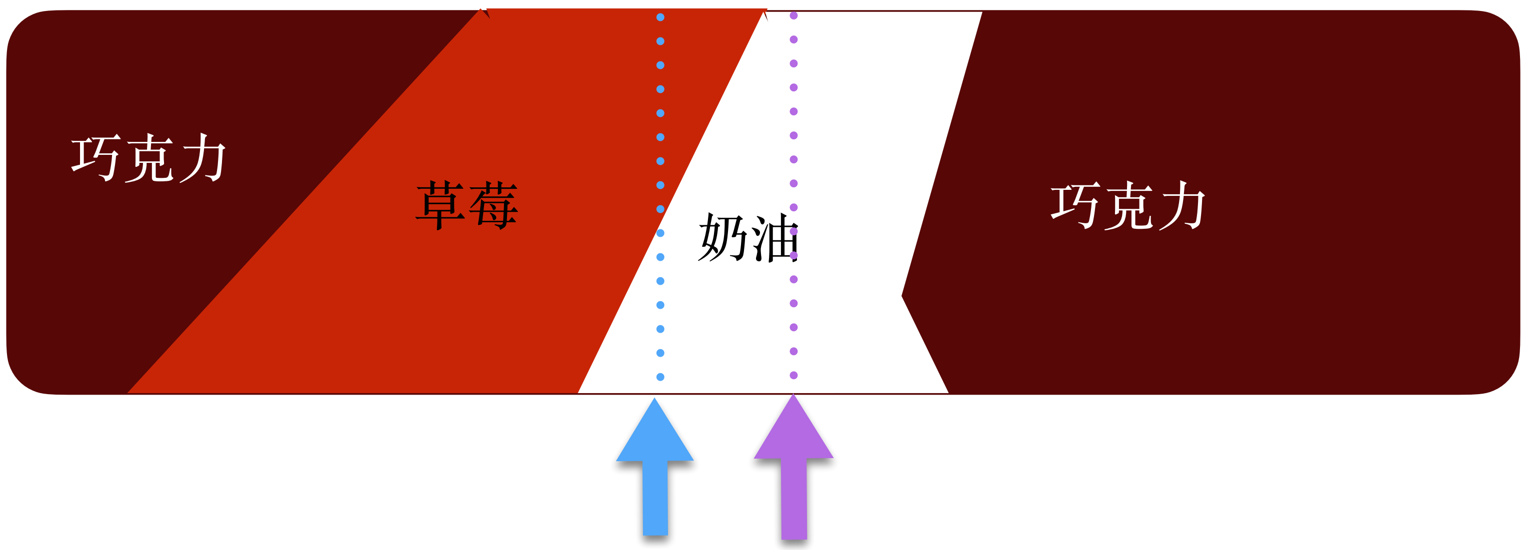
考慮集合(拓撲空間) C 與其(Borel)可測子集 M(C)，
給定 n 個概率測度 $V_1, V_2, \ldots, V_n : M(C) \to [0,1]$。
考慮 C 的一個可測分割 $\{X_i : 1 \le i \le n\}$
其中 X_i 表示第 i 個人獲得分配的部分

1. 比例原則 (Proportionality): $V_i(X_i) \ge 1/n$

2. 無嫉妒原則 (Envy-freeness): $V_i(X_i) \ge V_i(X_j) \quad \forall\, i, j$

無嫉妒原則 ⇒ 比例原則 （相同證明）

切蛋糕問題：n = 2



n = 2 的“自然”分配法：先分後取

優點：每人都自認為獲得至少一半價值的蛋糕，符合公平原則

分配法二：各人標註左右價值均等點，偏左者取該點左邊蛋糕

優點：同上

課堂活動：考慮 n=3

切蛋糕問題：n ≥ 3

切蛋糕問題 (n≥3) 由數學家Steinhaus提出，他的學生Banach、Knaster提出“公平”的最後削減者法 (Last-Diminiser Protocol, 1948)。



Hugo Steinhaus, Stefan Banach, Bronisław Knaster (left to right), Wikipedia

最後削減者法

考慮 n 個人$A_1, A_2, \ldots , A_n$ 參與分配，A_1將蛋糕切割出一塊，此時A_2得到對被切出的這塊蛋糕進一步削減的權利，無進一步削減的義務，然後下一位分配者A_3得到對同樣一塊蛋糕進一步削減的權利，依次輪至A_n行使權利為止。最後一位進行削減者，取得削減後的蛋糕，然後排除此人，剩餘 n-1 人重複前面步驟。

觀察：(1) 剩餘2人即先分後取; (2) 符比例原則; (3) 需步驟$O(n^2)$

缺點：不符合無嫉妒原則，例如

$n=3, V_1 (X_1)=1/3, V_1 (X_2)=0, V_1 (X_3)=2/3$

改進的最後削減者法

Brahms-Taylor方法 (1996)

前面步驟同最後削減者法，差別是允許最後削減者進行下一輪競爭，若再次獲勝（i.e.仍然為最後削減者），則需放棄原先取得的蛋糕，改持有新的一塊。為保證不陷入無窮迴圈，允許最後削減者重新競爭 n^k 次。

觀察：(1) 接近於無嫉妒原則; (2) 需步驟$O(n^{2+k})$

說明：最後削減者可選擇每次重新競爭都確保被切出的蛋糕價值 $\leq$ 目前持有蛋糕價值$+1/n^k$

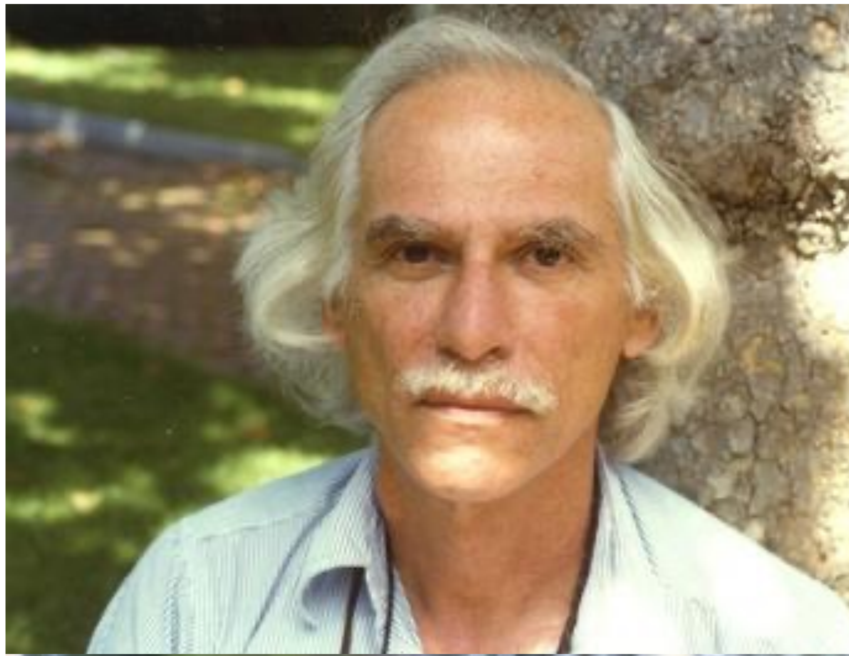
移動刀法

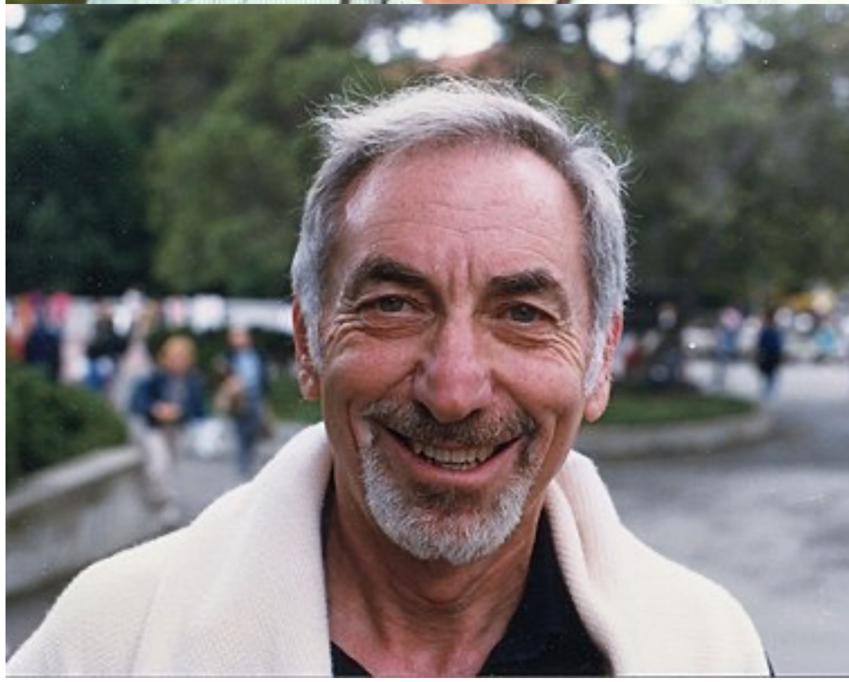
Dubins-Spanier 移動刀法 (1961):

1. 一個人將刀由左往右移動, 有n個分配者, 任何人認為左邊部分達到1/n 即喊停, 並取得左邊部分蛋糕, 此人退出分配

2. 將刀繼續往右移, 重複步驟1, 直到僅剩一人

觀察：(1) 剩二人時，等同先分後取
(2) 可視為時間連續的最後削減者法
(3) 滿足比例原則





Lester Dubins (up) and Edwin Spanier(down)
Wikipedia

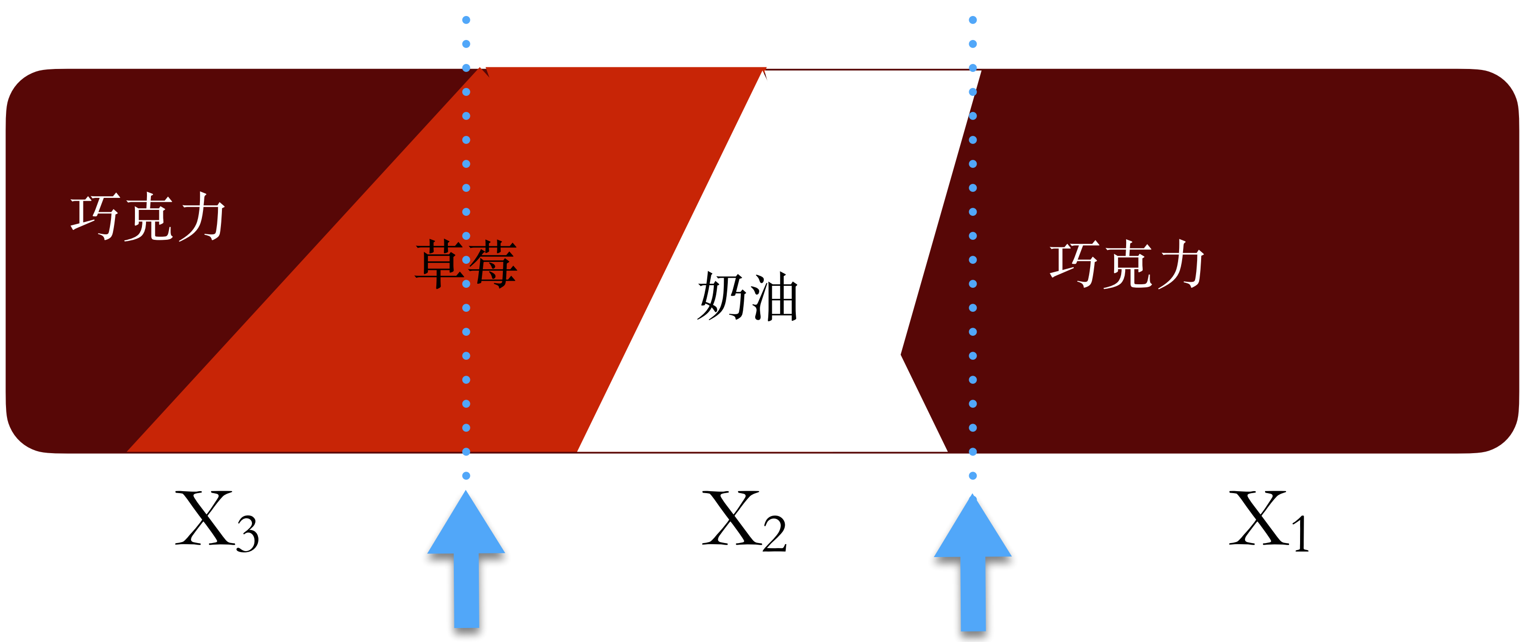
移動刀法

n ≥ 3: 不滿足無嫉妒原則

e.g. 考慮Player 1移動刀, 以及矩陣 $\{V_i (X_j)\}$

1/2	1/4	1/4
5/12	5/12	1/6
1/2	1/6	1/3

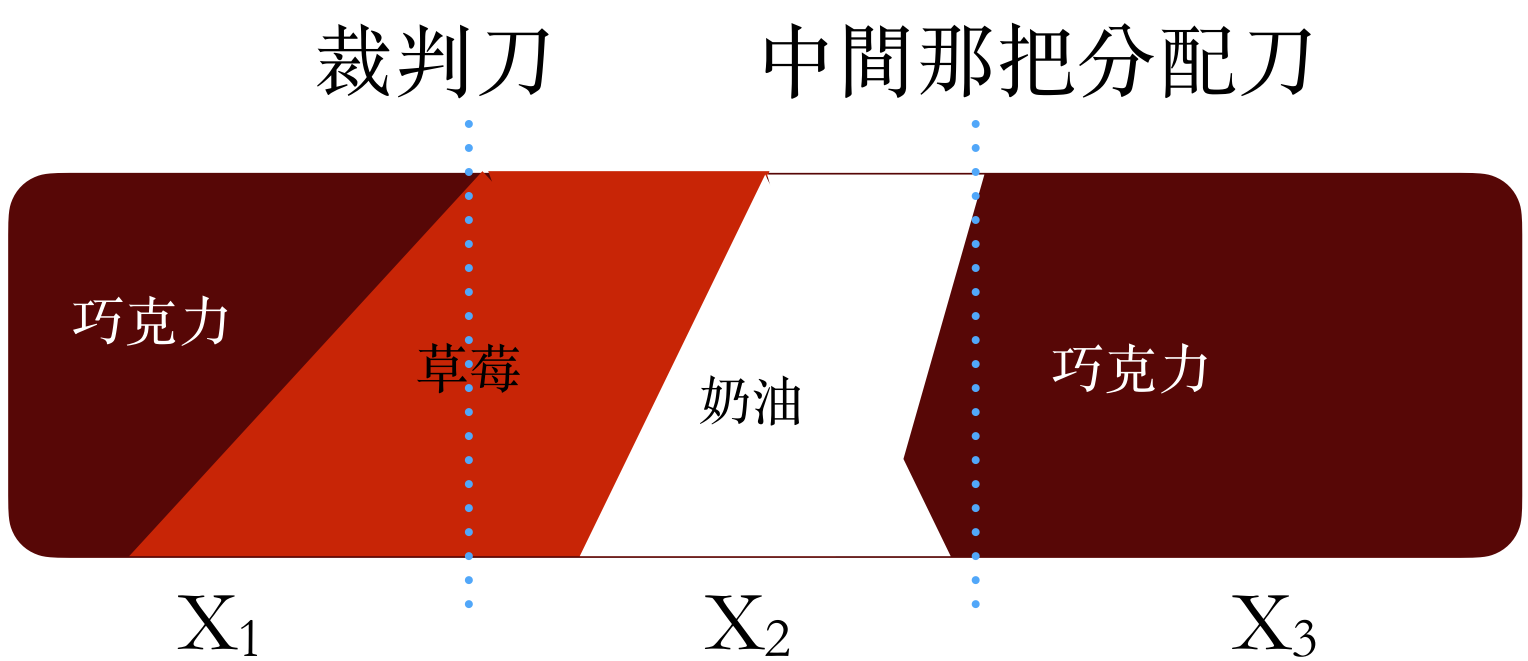
則 $V_3 (X_3) = 1/3 < 1/2 = V_3 (X_1)$



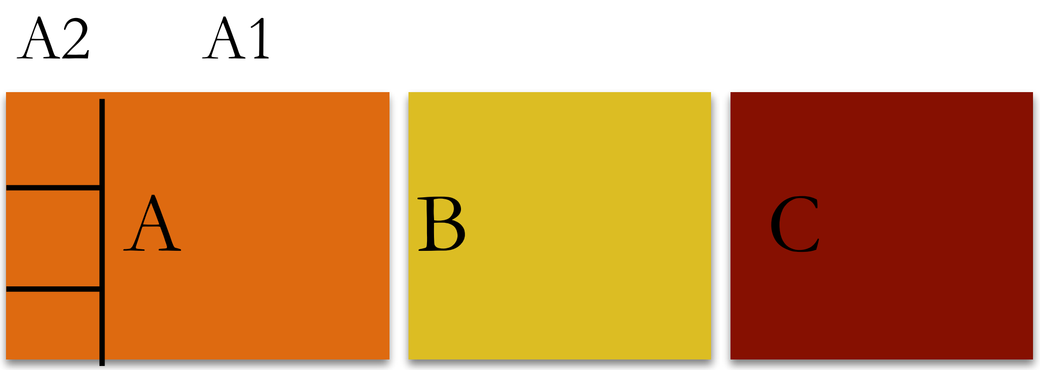
無嫉妒的移動刀法

n=3: Stromquist 移動刀法 (1980)

增加一裁判，每人一把刀，裁判由左至右，分配者由右至左，連續並平行移動刀，任何分配者(eg. Player 1)喊停即中止，喊停者取得左邊的那塊，另兩個分配者中，刀最接近裁判刀者(eg. Player 2)，取得裁判刀與第二接近裁判刀者（中間那把分配刀）之間的蛋糕，剩餘那塊給第三人。



Selfridge-Conway方法 (1960)



令分配者為P1, P2, P3

1. P1將蛋糕分為他認為公平的三份A、B、C

2. 若P2認為最好的兩塊A、B等值, 則按照P3、P2、P1順序選蛋糕, 並結束分配;
   若P2認為有一塊A最好, 則將A切下一小塊A2, 使A1=A\A2與第二好的B等值

3. P3就A1、B、C擇一

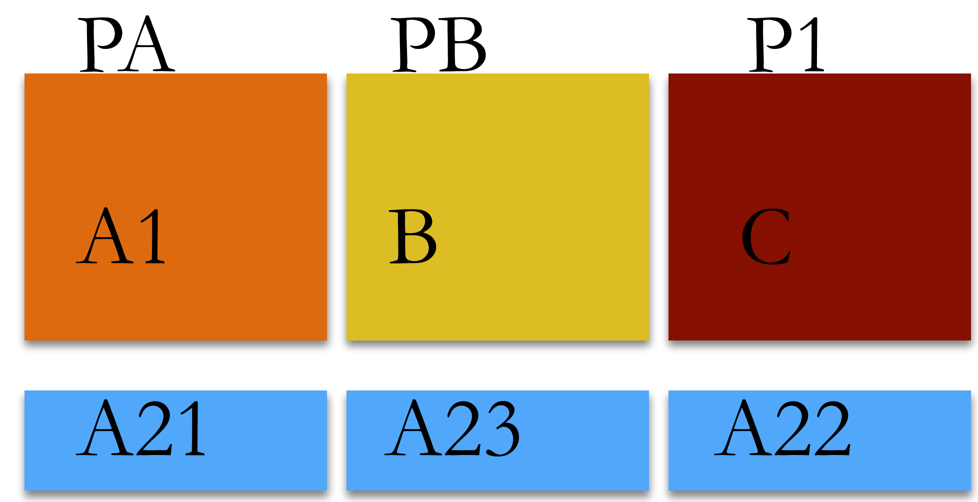
4. 若P3選A1, 則P2就B、C擇一, 最後一塊給P1;
   若P3未選A1, 則P2需選A1, 最後一塊給P1

5. 選擇A1者是P2或P3, 令其為PA, 另一為PB.
   PB將A2分為他認為公平的三份, 按PA、P1、PB
   順序選取分別為A21、A22、A23

John Conway(1937-2020), Wikipedia

Selfridge-Conway方法 (1960)

以PA=P3為例 (PA=P2的討論類似)



1. P1將蛋糕分為公平三份A、B、C
2. P2認為A最好, 將A切下一小塊A2, 使A1=A\A2與第二大塊B等值
3. P3選A1, P2就B、C擇一, 不妨設為B, 最後一塊給P1

$V_3(B), V_3(C) \leq V_3(A1)$

$V_2(C) \leq V_2(A1) = V_2(B)$

$V_1(A1) \leq V_1(B) = V_1(C)$

4. P2將A2分為他認為公平的三份, 按P3、P1、P2順序選取, 分別為A21、A22、A23

$V_3(A23), V_3(A22) \leq V_3(A21)$ (P3不嫉妒)

$V_2(A23) = V_2(A21) = V_2(A22)$ (P2不嫉妒)

$V_1(A23) \leq V_1(A22)$ (P1不嫉妒P2)

$V_1(A1)+V_1(A21) \leq V_1(C)$ (P1不嫉妒P3)

Cutting Piece of Cake is not Piece of Cake!

事實：

1.存在符合無嫉妒原則的切割法
(Brahms-Taylor 1995, Robertson-Webb 1998, Simmons, Su 1999)

2.計算步驟至少是（通常遠大於）$O(n^2)$

3.幾何限制（例如聯通性）增加計算複雜度

4.允許「接近於無嫉妒原則」可大幅降低計算複雜度

對實現分配正義的啟示

所有公平分配法均具備「先切後選」的原則

拿刀者 —「切」是權利,「後選」是義務
不拿刀者 — 失去「切」的權利, 獲得「先選」的權利

1.公平分配法的內涵是權利與義務的平衡

2.完美的分配正義不易實現，需要包容的社會

Recommended Readings (☞ Required)

☞ 1. Crash Course Philosophy #40: What is justice? https://www.youtube.com/watch?v=H0CTHVCkm90&t=479s
2. Ariel D. Procaccia, “Cake Cutting Algorithms”, Handbook of Computational Social Choice (Brandt, Conitzer, Endriss, Lang, and Procaccia, eds.), chapter 13, 2016
3. Fair Division Problems and Fair Division Schemes, Discrete Mathematics Project at the University of Colorado at Boulder https://web.archive.org/web/20091022113055/http://www.colorado.edu/education/DMP/fair_division.html
4. Harvard University Online Course by Michael Sandel. Justice: What's The Right Thing To Do? Episode 08: "WHATS A FAIR START?” https://www.youtube.com/watch?v=VcL66zx_6No&t=1354s
5. Francis Su, Fair Division Calculator, https://math.hmc.edu/su/fairdivision/