A Few Exercises About Abstract Measures

1. Suppose μ_n is a monotone sequence of measures on the measurable space $(X, \mathscr{B})$. Define $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ for any $E \in \mathscr{B}$.

   (a) Show that μ is a measure if μ_n is increasing; i.e. $\mu_n(E) \leq \mu_{n+1}(E)$ for any n and $E \in \mathscr{B}$.

   (b) Show that μ may not be a measure if μ_n is decreasing.

2. Let μ_1, μ_2 be finite signed measures on the measurable space $(X, \mathscr{B})$. Show that μ_1 and μ_2 has a greatest lower bound μ_*. That is, μ_* is a signed measure such that $\mu_* \leq \mu_1,\ \mu_2$, and if σ is a sigend measure satisfying $\sigma \leq \mu_1,\ \mu_2$, then $\sigma \leq \mu_*$. Similarly, show that μ_1 and μ_2 has a least upper bound μ^*.

3. Suppose $f \geq 0$ is a measurable function on a σ-finite measure space $(X, \mathscr{B}, \mu)$. Show that f is the pointwise limit of an increasing sequence of nonnegative simple functions ϕ_n with each ϕ_n vanishes outside a set of finite μ-measure.

4. Suppose $f \geq 0$ is an integrable function on a measure space $(X, \mathscr{B}, \mu)$. Show that the statement in the previous exercise holds even if μ is not σ-finite.

5. Suppose f is measurable on a complete measure space $(X, \mathscr{B}, \mu)$ and $f = g$ μ-a.e. Show that g is measurable. Find a counterexample if μ is not complete.

6. Suppose f_n are measurable on a complete measure space $(X, \mathscr{B}, \mu)$ and f_n converges to f μ-a.e. Show that f is measurable. Find a counterexample if μ is not complete.