CHAPTER 7

Introduction to Central Configurations

Recall that equations of motion for the the restricted and non-restricted Newtonian N-body problem can be written

$$\ddot{x}_k = \sum_{i \neq k} \frac{m_i(x_i - x_k)}{|x_i - x_k|^3}, \quad k = 1, 2, \cdots, N,$$

where the right-hand side of the equation is the instantaneous acceleration of mass m_k. If the system starts from rest and for each k the right-hand side is pointing toward or away from the mass center, then the system has zero torque and zero angular momentum. If, additionally, the right-hand side is in a fixed proportion to the position relative to the mass center, it is easy to see that the system would collapse in a self-similar way. Configurations and solutions of these types, called central configurations, are interesting and important for several reasons. We will begin with equivalent mathematical formulations for such configurations, discuss their significance, and then show some simple classes of central configurations.

7.1. Definition and equivalent formulations

The position vector $x = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$ is often referred to as the *configuration* of the system, and vectors $\{x_k\}$ are *vertices* of the configuration x. We assume the total mass M is strictly positive, for otherwise we would have a trivial system consisting of zero masses. For convenience, some notations are introduced for some special positions and vectors:

NOTATIONS.

$$\begin{aligned}
\Delta &= \{x \in (\mathbb{R}^d)^N : x_i = x_j \text{ for some } i \neq j\}, \quad \text{(collision set)} \\
c_x &= \frac{1}{M}(m_1 x_1 + \cdots + m_N x_N), \quad \text{(mass center of } x\text{)} \\
\mathscr{A}_k &= \sum_{i \neq k} \frac{m_i(x_i - x_k)}{|x_i - x_k|^3}, \quad \text{(gravitational acceleration of mass } m_k\text{)} \\
\mathcal{I}(x; c) &= \sum_{k=1}^{N} m_k |x_k - c|^2. \quad \text{(moment of inertia about position vector } c \in \mathbb{R}^d\text{)} \\
\mathcal{I}(x) &= \sum_{k=1}^{N} m_k |x_k - c_x|^2. \quad \text{(moment of inertia about the mass center)}
\end{aligned}$$

DEFINITION 7.1.1. Given a system of masses $m = (m_1, \cdots, m_N) \in \mathbb{R}_+^N \setminus \{0\}$. A non-collision configuration $x = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N \setminus \Delta$ is called a *central configuration* for masses m if there exists some constant λ, called the *multiplier*, such that

$$-\lambda(x_k - c_x) = \mathscr{A}_k, \quad k = 1, 2, \cdots, N. \tag{7.1.1}$$

To put in words, central configurations are special configurations characterized by the property that the acceleration vector $\mathscr{A}_k$ of each mass m_k is a constant multiple of the vector from x_k to the mass center c_x. The set of central configurations are invariant under similarity transformations; i.e. compositions of translations, rotations, and scalings. Translations and rotations do not affect the multiplier, while the scaling $x \mapsto cx$ changes the multiplier from λ to λ/c^3. Conventionally, central configurations within the same similarity class are considered equivalent, so the term "central configurations" is often referred to similarity classes of central configurations.

Now we discuss some alternative formulations for central configurations.

LEMMA 7.1.1. *The moment of inertia about the mass center can be written*

$$\mathcal{I}(x) \;=\; \frac{1}{M}\sum_{i<j} m_i m_j |x_i - x_j|^2.$$

PROOF. For any $c \in \mathbb{R}^d$,

$$\mathcal{I}(x;c) \;=\; \sum_{k=1}^N m_k|x_k|^2 + M|c|^2 - 2\sum_{k=1}^N m_k\langle x_k, c\rangle \;=\; \sum_{k=1}^N m_k|x_k|^2 + M\langle c, c - 2c_x\rangle.$$

In particular, when $c = c_x$ we have

$$\mathcal{I}(x) \;=\; \sum_{k=1}^N m_k|x_k|^2 - M|c_x|^2.$$

Therefore

$$\begin{aligned}
\sum_{i<j} m_i m_j |x_i - x_j|^2 \;&=\; \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N m_i m_j |x_i - x_j|^2 \\
&=\; M\sum_{k=1}^N m_k|x_k|^2 - \sum_{i=1}^N\sum_{j=1}^N m_i m_j\langle x_i, x_j\rangle \\
&=\; M(\mathcal{I}(x) + M|c_x|^2) - \langle Mc_x, Mc_x\rangle \;=\; M\mathcal{I}(x).
\end{aligned}$$

□

THEOREM 7.1.2. *Given a system of masses $m = (m_1, \cdots, m_N)$ with $m_k \geq 0$ for each k and with total mass $M > 0$, and given a non-collision configuration $x = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N \setminus \Delta$. The followings are equivalent:*

(a) x is a central configuration for m with multiplier λ;
(b) $-\lambda(x_i - x_j) \;=\; \mathscr{A}_i - \mathscr{A}_j$ for any $i \neq j$;
(c) $\frac{\lambda}{2}\nabla\mathcal{I}(x) \;=\; -\nabla U(x)$;
(d) x is a critical point of $\sqrt{\mathcal{I}}\,U$ and $\lambda = U(x)/\mathcal{I}(x)$;
(e)

$$\sum_{i\neq k} m_i\left(\frac{1}{|x_i - x_k|^3} - \frac{\lambda}{M}\right)(x_i - x_k) = 0, \quad k = 1, 2, \cdots, N.$$

PROOF. Clearly (a) implies (b). That (b) implies (a) follows from

$$M\mathscr{A}_i \;=\; \sum_{j=1}^{N} m_j(\mathscr{A}_i - \mathscr{A}_j) \;=\; -\lambda \sum_{j=1}^{N} m_j(x_i - x_j) \;=\; -\lambda M(x_i - c_x).$$

Equivalence of (a) and (c) is obvious. Observe that

$$\nabla\big(\sqrt{\mathcal{I}}\,U\big) \;=\; \sqrt{\mathcal{I}}\,\nabla U + \frac{U}{2\sqrt{\mathcal{I}}}\nabla I\,.$$

Equivalence of (c) and (d) follows from this observation and Euler's theorem for homogeneous functions. Equivalence of (c) and (e) follows immediately from Lemma 7.1.1. □

Part (c) of the theorem above justifies the terminology "multiplier" for λ; x is a central configuration if and only if it is a constraint extremum of $-U$ restricted to constant levels of $\frac{1}{2}\mathcal{I}$ with Lagrange multiplier λ. Coefficients -1 and $\frac{1}{2}$ reflect degrees of homogeneity of U and $\mathcal{I}$. Existence of central configuration is clear from formulation (c) since constant levels of $\mathcal{I}$ are ellipsoids and U is positive and smooth except on Δ, near which U approaches infinity.

Another equivalent formulation is as follows. The term $\mathcal{I}/U$ depends not only on mutual distances $r_{ij} = |x_i - x_j|$ but also on masses $m = (m_1, \cdots, m_N)$. It is homogeneous in $r = (r_{ij})_{i<j}$ of degree -3 and homogeneous in m of degree 1, so the set of dimension-less "shape" variables $u = (u_{ij})_{i<j}$ given by

$$u_{ij} \;=\; r_{ij}\left(\frac{\lambda}{M}\right)^{\frac{1}{3}}$$

reveals the shape of the central configuration, and is independent of the total mass and size. With this in mind, it is natural to introduce a variable r_0 in place of λ/M and consider their ratios with r_{ij}'s:

$$r_0 = \left(\frac{\lambda}{M}\right)^{-\frac{1}{3}}, \quad u_{ij} = \frac{r_{ij}}{r_0}. \tag{7.1.2}$$

In terms of mutual distances r_{ij} and r_0, the system Theorem 7.1.2 (e) can be written

$$\sum_{i=1}^{N} m_i\left(\frac{1}{r_{ik}^3} - \frac{1}{r_0^3}\right)(x_i - x_k) \;=\; 0, \quad k = 1, 2, \cdots, N. \tag{7.1.3}$$

In terms of dimension-less variables u_{ij}, the above system can be written

$$\sum_{i=1}^{N} m_i\left(\frac{1}{u_{ik}^3} - 1\right)(x_i - x_k) \;=\; 0, \quad k = 1, 2, \cdots, N. \tag{7.1.4}$$

Let Δ_{ijk} be the oriented area of the triangle with vertices (x_i, x_j, x_k); i.e.

$$\Delta_{ijk} \;=\; \frac{1}{2}(x_j - x_i) \wedge (x_k - x_i).$$

By taking wedge product of the identity in Theorem 7.1.2 (e) with $x_j - x_k$, we find

$$\sum_{i=1}^{N} m_i\left(\frac{1}{r_{ik}} - \frac{\lambda}{M}\right)\Delta_{ijk} \;=\; 0.$$

Interchange x_j and x_k and then subtract the two resulting identities, then

$$\sum_{i\neq j,\,k} m_i\left(\frac{1}{r_{ij}}-\frac{1}{r_{ik}}\right)\Delta_{ijk} \;=\; 0,\quad \forall j\neq k. \tag{7.1.5}$$

Equations (7.1.5) are called *Laura-Andoyer equations* for the central configuration x.

THEOREM 7.1.3. *A non-collinear configuration $x=(x_1,\cdots,x_N)\in(\mathbb{R}^d)^N$ is a central configuration for masses $(m_1,\cdots,m_N)$ if and only if (7.1.5) holds.*

PROOF. It is sufficient to show that (7.1.5) implies Theorem 7.1.2 (b) holds for some constant λ. Assume (7.1.5) holds. Then $\mathscr{A}_j-\mathscr{A}_k$ and x_j-x_k are parallel for any $j\neq k$ since

$$(\mathscr{A}_j-\mathscr{A}_k)\wedge(x_j-x_k) \;=\;=\; -\sum_{i\neq j,\,k} m_i\left(\frac{1}{r_{ij}^3}-\frac{1}{r_{ik}^3}\right)\Delta_{ijk} \;=\; 0.$$

Let $\lambda_{jk}\in\mathbb{R}$ be chosen so that $\mathscr{A}_j-\mathscr{A}_k=\lambda_{jk}(x_j-x_k)$. Clearly $\lambda_{jk}=\lambda_{kj}$ for any $j\neq k$. What we need to show is that every λ_{ij} is the same.

Given distinct i, j, $k\in\{1,2,\cdots,N\}$. If x_i, x_j, x_k are not collinear, then $\lambda_{ij}=\lambda_{jk}=\lambda_{ik}$ since

$$\begin{aligned}
0 &= (\mathscr{A}_i-\mathscr{A}_j)+(\mathscr{A}_j-\mathscr{A}_k)+(\mathscr{A}_k-\mathscr{A}_i)\\
&= \lambda_{ij}(x_i-x_j)+\lambda_{jk}(x_j-x_k)+\lambda_{ki}(x_k-x_i)\\
&= \lambda_{ij}(x_i-x_j)+\lambda_{jk}(x_j-x_k)+\lambda_{ik}((x_k-x_j)+(x_j-x_i))\\
&= (\lambda_{ij}-\lambda_{ik})(x_i-x_j)+(\lambda_{jk}-\lambda_{ik})(x_j-x_k).
\end{aligned}$$

Suppose x_i, x_j, x_k are collinear, then there exists some $\ell\notin\{i,j,k\}$ such that x_ℓ is not collinear with them. Then

$$\lambda_{ij} \;=\; \lambda_{i\ell} \;=\; \lambda_{ik}.$$

The first identity holds because x_i, x_j, x_ℓ are not collinear, the second identity holds because x_i, x_k, x_ℓ are not collinear. In any case, we have $\lambda_{ij}=\lambda_{ik}$, so every λ_{ij} is the same. □

We finish this section by outlining the significance of central configurations without showing technical details:

(1) Observed by Laplace, each planar central configuration gives rise to a family of periodic solutions. To see this, given any elliptical Keplerian orbit $k(t)\in\mathbb{C}$ and any planar central configuration $(a_1,\cdots,a_N)\in\mathbb{C}^N$ for (1.1.1), the orbit $x(t)=(x_1(t),\cdots,x_N(t))$ defined by $x_i(t)=k(t)a_i$, $\forall i$, is a periodic solution of (1.1.1). Any relative equilibrium is of the this form with circular $k(t)$.

(2) If the initial configuration of the N bodies is a central configuration and if it starts with zero velocity, then the solution has the form $x(t)=k(t)x(0)$, where the scalar function $k(t)$ is a collinear Keplerian orbit starting with zero velocity. The solution ends in *total collapse* (i.e. collision of all N bodies). If a solution ends in total collapse, it is not necessarily a central configuration but it is asymptotically a central configuration. Therefore, knowledge of central configurations is important in the study of total collapse.

(3) The topology of the integral manifold $\mathcal{M}(h,\omega)\subset TV$ may change as the total energy h and angular momentum ω vary. Let $\pi: TV\to V$ denote the natural projection. In the planar case, if the topology of the integral manifold bifurcates at the level (h,ω), then the projection $\pi(\partial M(h,\omega))$ of the boundary of $\mathcal{M}(h,\omega)$ into the configuration space contains

a central configuration. Let $C(h,\omega) := \pi(\mathcal{M}(h,\omega))$, called the *Hill's region.* Bifurcation points in V at which the topology of Hill's region changes are also central configurations.

7.2. Examples of central configurations

In this section we introduce some simple classes of central configurations.

(Details to be added for Lagrange's equilateral triangles (1772), generation by Lehmann-Filhés (1891), regular n-gon by Perko-Walter (1985).)

Collinear central configurations are also known as *Euler-Moulton central configurations* in honor of Euler's complete classification for the case $N = 3$ (1762, 1764) and Moulton's generalization to general N (1910).

For each $1 \leq i < j \leq N$ the set

$$\Delta_{ij} = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_i = x_j\}$$

is an $(N-1)$-dimensional subspace with a normal vector $e_i - e_j$. It separates $\mathbb{R}^N$ into two half spaces: one characterized by $x_i < x_j$ and the other by $x_j < x_i$. Clearly

$$\Delta = \bigcup_{i<j} \Delta_{ij}.$$

Each component of $\mathbb{R}^N \setminus \Delta$ corresponds a specific ordering of collinear configurations, so there is a total of $N!$ components.

Theorem 7.2.1. (Moulton's theorem, 1910)
Given N positive masses $m_1, \cdots, m_N$. For any ordering of masses on the line, there is exactly one similarity class of collinear central configurations.

Proof. Given an ordering, say $x_1 < x_2 < \cdots < x_N$, the component Ω of $\mathbb{R}^N \setminus \Delta$ consisting of configurations with this ordering is convex since it is the intersection of half spaces. More generally, the intersection of convex sets is convex, so the intersection $\Omega \cap V$ is also convex. It is sufficient to prove that, with a fixed $\lambda > 0$, there exists unique central configuration in $\Omega \cap V$ with multiplier λ. That is, we need to show the function

$$F(x) = \frac{\lambda}{2}\mathcal{I}(x) + U(x)$$

has precisely one critical point in $\Omega \cap V$.

On V the function F can be expressed as

$$F(\mathbf{r}) = \sum_{i<j} m_i m_j \left(\frac{\lambda}{2} r_{ij}^2 + \frac{1}{r_{ij}}\right), \quad \mathbf{r} = (r_{ij})_{1\leq i<j\leq N}.$$

Then

$$\frac{\partial^2}{\partial r_{ij}^2} F(\mathbf{r}) = m_i m_j \left(\lambda + \frac{1}{r_{ij}^3}\right) > 0.$$

Given $x,\ y \in \Omega \cap V$. By convexity the line segment $\ell(t) := (1-t)x + ty$, $t \in [0,1]$, is contained entirely in $\Omega \cap V$. The mutual distance $r_{ij}(t) = |\ell_i(t) - \ell_j(t)|$ is actually a polynomial in t of degree ≤ 1 when $t \in [0,1]$. This is because x and y have the same ordering, so for each $i \neq j$ the differences

$x_i - x_j$ and $y_i - y_j$ have the same sign. If $x \neq y$, then for some $i < j$ the mutual distance $r_{ij}(t)$ has degree 1. Now, restricting F to this line segment yields

$$\frac{d^2}{dt^2} F(\mathbf{r}(t)) = \sum_{i<j} m_i m_j \left(\lambda + \frac{1}{r_{ij}(t)^3} \right) r'_{ij}(t)^2 > 0.$$

Therefore x and y can not be both critical points of F. This shows that F has at most one critical point in $\Omega \cap V$. But since $F > 0$ and it approaches ∞ as $x \to \partial\Omega$ or $|x| \to \infty$, there must be a minimum point of F in the interior of $\Omega \cap V$. This finishes the proof. □

For the case $N = 3$ here is an alternative approach. Given an ordering, say $x_1 < x_2 < x_3$ or $x_3 < x_2 < x_1$, both of which correspond the constraint $r_{12} + r_{23} = r_{13}$ on mutual distances. Setting $r_{12} = r$, $r_{23} = 1 - r$, $r_{13} = 1$, then $x = (x_1, x_2, x_3)$ is a central configuration if and only if $r \in (0, 1)$ is a critical point of

$$F(r) := \mathcal{I} U^2 = \frac{1}{M} \left(m_1 m_2 r^2 + m_2 m_3 (1-r)^2 + m_1 m_3 \right) \left(\frac{m_1 m_2}{r} + \frac{m_2 m_3}{1-r} + m_1 m_3 \right)^2.$$

Note that

$$F'(r) = \frac{2 m_1 m_2 m_3}{M r^3 (1-r)^3} f(r) g(r),$$

where

$$\begin{aligned} f(r) &= m_1 m_2 (1-r) + m_2 m_3 r + m_1 m_3 r (1-r), \\ g(r) &= (m_1 + m_3) r^5 - (2m_1 + 3m_3) r^4 + (m_1 + 2m_2 + 3m_3) r^3 \\ &\qquad - (m_1 + 3m_2) r^2 + (2m_1 + 3m_2) r - (m_1 + m_2). \end{aligned}$$

Polynomial $f(r)$ is strictly positive on $(0, 1)$, and g has unique root in $(0, 1)$ since $g(0) = -(m_1 + m_2)$, $g(1) = m_2 + m_3$, and

$$g'(r) = m_1 (1-r)^2 (5r^2 + 2r + 2) + m_2 (6r^2 - 6r + 3) + m_3 r^2 (5r^2 - 12r + 9) > 0 \quad \text{for } r \in (0, 1).$$

This proves the existence of unique collinear central configuration with the prescribed ordering.