CHAPTER 6

Restricted Three-Body Problem

The *restricted three-body problem* is the three-body problem with two positive masses and one zero mass. Although the same phrase may also refer to the three-body problem with two zero masses, we only consider the case with one zero mass, which exhibits intriguing and complex phenomena, while the other case is simply the coupling of two independent Kepler problems.

Throughout this chapter we assume masses m_1 and m_2 are positive, denote their respective positions by $x_1,\ x_2 \in \mathbb{R}^d$, and $x_2 - x_1$ is an elliptical Keplerian orbit. Denote the position of the zero mass m_3 by q. Then equation of motion for the zero mass is

$$\ddot{q} \;=\; \frac{m_1(x_1 - q)}{|x_1 - q|^3} + \frac{m_1(x_2 - q)}{|x_2 - q|^3}. \tag{6.0.1}$$

This chapter is merely an introduction to the restricted three-body problem, we shall revisit this model in several later chapters.

6.1. The planar circular problem

Let $x_2 - x_1$ be a circular Keplerian orbit. By suitable choice of mass, time, and length units, we may assume $m_1 = 1 - \mu$, $m_2 = \mu$ for some $\mu \in (0, 1)$, angular velocity and the gravitational constant are both equal to 1. Then, by suitable choice of space coordinates, we may assume x_1 and x_2 are given by

$$x_1(t) \;=\; -\mu e^{it}, \quad x_2(t) \;=\; (1 - \mu)e^{it}.$$

The equation of motion for the *planar circular restricted 3-body problem*, abbreviated as (PCR3BD), becomes an non-autonomous Newtonian mechanical system:

$$\ddot{q} \;=\; \frac{\partial}{\partial q}U(q,t), \quad \text{where } U(q,t) \;=\; \frac{1-\mu}{|x_1(t) - q|} + \frac{\mu}{|x_2(t) - q|}.$$

Introduce a rotating coordinate system on which x_1 and x_2 are stationary: $z = x + iy = e^{-it}q$. Then the equation above can be easily seen to be equivalent to

$$\begin{aligned}
\ddot{x} - 2\dot{y} - x \;&=\; -\frac{(1-\mu)(x+\mu)}{\big((x+\mu)^2 + y^2\big)^{3/2}} - \frac{\mu(x-1-\mu)}{\big((x-1+\mu)^2 + y^2\big)^{3/2}},\\
\ddot{y} + 2\dot{x} - y \;&=\; -\frac{(1-\mu)y}{\big((x+\mu)^2 + y^2\big)^{3/2}} - \frac{\mu y}{\big((x-1+\mu)^2 + y^2\big)^{3/2}}.
\end{aligned}$$

A briefer formulation is

$$(\ddot{x} - 2\dot{y},\, \ddot{y} + 2\dot{x}) \;=\; \nabla V, \tag{6.1.1}$$

where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2},$$
$$\rho_1 = \sqrt{(x+\mu)^2+y^2}, \quad \rho_2 = \sqrt{(x-1+\mu)^2+y^2}.$$

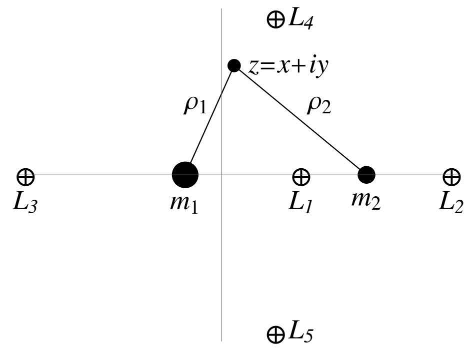

FIGURE 1. The planar circular restricted 3-body problem in rotating coordinate system, and the five Lagrange points. Here $\mu = 0.2$, $L_1 \approx 0.438$, $L_2 \approx 1.271$, $L_3 \approx -1.083$.

The system (PCR3BD) is a Lagrangian system, with Lagrangian

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + x\dot{y} - y\dot{x} + V(x,y).$$

By setting $u = \dot{x} - y, \ v = \dot{y} + x$,

$$H(x,y,u,v) = \frac{1}{2}(x^2+y^2+u^2+v^2) + yu - xv - V(x,y),$$

(PCR3BD) can be alternatively expressed as a Hamiltonian system:

$$\begin{cases} \dot{x} = y + u = H_u \\ \dot{y} = -x + v = H_v \\ \dot{u} = -x + v + V_x = -H_x \\ \dot{v} = -y - u + V_y = -H_y. \end{cases} \tag{6.1.2}$$

Observe that the system (6.1.2) has exactly five equilibrium points. To see this, note that equilibrium points are given by equations $\dot{x} = \dot{y} = 0$ and

$$0 = V_x = x\left[1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3}\right] - \mu(1-\mu)\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right),$$
$$0 = V_y = y\left[1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3}\right].$$

If $y \neq 0$, then the square brackets $[\cdots]$ in identities above must be zero. From $V_x = 0$ we conclude $\rho_1 = \rho_2 = 1$. This shows that there are exactly two non-collinear equilibrium points, located $(1-2\mu)/2 \pm i\sqrt{3}/2$, forming equilateral triangles with m_1 and m_2.

If $y = 0$, then $V_y = 0$. The function

$$g(x) := V(x, 0) = \frac{x^2}{2} + \frac{1-\mu}{|x+\mu|} + \frac{\mu}{|x-1+\mu|}$$

has strictly positive second derivative on intervals $(-\infty, -\mu)$, $(-\mu, 1-\mu)$, and $(1-\mu, \infty)$. It approaches infinity as x approaches boundaries of these intervals, so each interval contains exactly one critical point. See figure 2. This shows that there are exactly three collinear equilibrium points.

The five equilibrium points are called *Lagrange points* or *libration points*. Assuming $m_1 \gg m_2$, conventional notations for these Lagrange points are $L_1 \sim L_5$, where $L_1 \sim L_3$ are collinear, L_4 is the one in the upper half plane, L_5 is the one in the lower half plane. The collinear one between m_1 and m_2 is L_1, the one on m_2's side is L_2, the remaining one is L_3 (see figure 1).

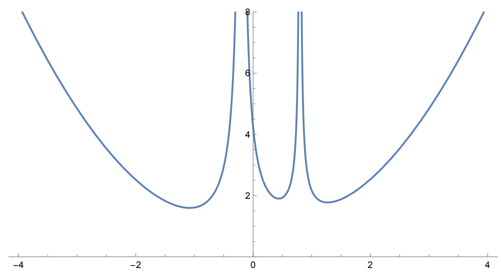

FIGURE 2. The graph of g with $\mu = 0.2$.

Let $\zeta = (x, y, u, v)^T$, and write the Hamiltonian system (6.1.2) as $\dot{\zeta} = F(\zeta)$. The linearized system at ζ_0 is

$$\dot{\zeta} = F(\zeta_0) + DF(\zeta_0)(\zeta - \zeta_0).$$

The total derivative of F is given by

$$DF(\zeta) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 + V_{xx} & V_{xy} & 0 & 1 \\ V_{xy} & -1 + V_{yy} & -1 & 0 \end{pmatrix}.$$

We say an equilibrium solution ζ_0 is *spectrally stable* if the linearized map $DF(\zeta_0)$ is diagonalizable and has only pure imaginary eigenvalues. We leave it as an exercise to show that an equilibrium is unstable if it is not spectrally stable.

LEMMA 6.1.1. *Eigenvalues of the matrix $DF(\zeta)$ has pure imaginary eigenvalues if and only if the following three conditions hold:*

(1) $4 - V_{xx} - V_{yy} > 0$,
(2) $V_{xx}V_{yy} - V_{xy}^2 > 0$,
(3) $(4 - V_{xx} - V_{yy})^2 > 4(V_{xx}V_{yy} - V_{xy}^2)$.

PROOF. The characteristic equation of the matrix is

$$0 = \det(DF(z) - \lambda I) = \lambda^4 + (4 - V_{xx} - V_{yy})\lambda^2 + (V_{xx}V_{yy} - V_{xy}^2). \tag{6.1.3}$$

Let $\nu = \lambda^2$ and rewrite it as

$$\nu^2 + (4 - V_{xx} - V_{yy})\nu + (V_{xx}V_{yy} - V_{xy}^2) \;=\; 0. \tag{6.1.4}$$

From this identity we see that eigenvalues of $DF(\zeta)$ are pure imaginary if and only if roots ν_1, ν_2 of (6.1.4) are both negative, which is equivalent to (i) ν_1, ν_2 are distinct and real, (ii) $\nu_1 + \nu_2 < 0$, and (iii) $\nu_1\nu_2 > 0$. These three conditions are exactly the three asserted conditions in the Lemma. $\square$

THEOREM 6.1.2. *Collinear Lagrange points L_1, L_2, L_3 are unstable; equilateral Lagrange points L_4, L_5 are spectrally stable if and only if $27\mu(1-\mu) < 1$.*

PROOF. Observe that

$$\begin{aligned}
V_{xx} &= 1 + \frac{1-\mu}{\rho_1^5}\left(3(x+\mu)^2 - \rho_1^2\right) + \frac{\mu}{\rho_2^5}\left(3(x-1+\mu)^2 - \rho_2^2\right)\\
V_{yy} &= 1 + \frac{1-\mu}{\rho_1^5}\left(3y^2 - \rho_1^2\right) + \frac{\mu}{\rho_2^5}\left(3y^2 - \rho_2^2\right)\\
V_{xy} &= 3y\left(\frac{(1-\mu)(x+\mu)}{\rho_1^5} + \frac{\mu(x-1+\mu)}{\rho_2^5}\right).
\end{aligned}$$

Denote the xy-coordinates of L_i by $(\xi_i, 0)$, $i = 1,2,3$. Then $\xi_3 < -\mu < \xi_1 < 1-\mu < \xi_2$. At L_1, $\rho_1 = \xi_1 + \mu$, $\rho_2 = 1 - \mu - \xi_1 = 1 - \rho_1$. From $V_x(L_1) = 0$ we obtain

$$\xi_1 - \frac{1-\mu}{\rho_1^2} + \frac{\mu}{\rho_2^2} = 0; \quad \text{i.e.} \quad \frac{1-\mu}{\rho_1^2} = \rho_1 - \mu + \frac{\mu}{\rho_2^2}$$

Therefore

$$\begin{aligned}
\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} &= 1 + \frac{\mu}{\rho_1}\left(-1 + \frac{1}{\rho_2^2}\right) + \frac{\mu}{\rho_2^3} \;=\; 1 + \frac{\mu}{\rho_1}\left(\frac{1}{\rho_2^3} - 1\right),\\
V_{xx} &= 1 + 2\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) = 3 + \frac{2\mu}{\rho_1}\left(\frac{1}{\rho_2^3} - 1\right) > 0,\\
V_{yy} &= 1 - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) = -\frac{\mu}{\rho_1}\left(\frac{1}{\rho_2^3} - 1\right) < 0,\\
V_{xy} &= 0.
\end{aligned}$$

The second condition in Lemma 6.1.1 fails to hold, so L_1 is unstable. Discussions for L_2, L_3 are similar.

At $L_4 = \left(\frac{1-2\mu}{2}, \frac{\sqrt{3}}{2}\right)$, $\rho_1 = \rho_2 = 1$, and it can be easily verified that

$$V_{xx} = \frac{3}{4}, \quad V_{yy} = \frac{9}{4}, \quad V_{xy} = \frac{3\sqrt{3}}{4}(1-2\mu).$$

Therefore the first and second conditions in Lemma 6.1.1 holds. The third condition in Lemma 6.1.1 is equivalent to $27\mu(1-\mu) < 1$. This proves that at L_4 eigenvalues of DF are pure imaginary. To prove spectral stability it remains to show that $DF(L_4)$ is diagonalizable when $27\mu(1-\mu) < 1$.

Note that roots of $27\mu(1-\mu) - 1$ are $\frac{1}{2}\left(1 \pm \sqrt{\frac{23}{27}}\right)$, Assuming $0 < \mu < 1$, then

$$27\mu(1-\mu) < 1 \Leftrightarrow 0 < \mu < \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) (\approx 0.03852\cdots)$$

At L_4, equation (6.1.4) becomes

$$\nu^2 + \nu + \frac{27}{4}\mu(1-\mu) = 0.$$

Its roots are $\nu_1 = -\frac{1}{2}\left(1 + \sqrt{1 - 27\mu(1-\mu)}\right)$, $\nu_2 = -\frac{1}{2}\left(1 - \sqrt{1 - 27\mu(1-\mu)}\right)$. Assuming $27\mu(1-\mu) < 1$, then $\nu_1 \neq \nu_2$, $\nu_1, \nu_2 < 0$. This implies (6.1.3) has 4 distinct roots, so DF at L_4 is diagonalizable. The proof for L_5 is the same. $\square$

This system has an integral of motion

$$h = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y),$$

called the *Jacobi integral*.

Given $h \in \mathbb{R}$, the *Hill region* for (PCR3BD) is defined by

$$\mathcal{H}(h) = \left\{(x, y) \in \mathbb{R}^2 : V(x, y) + h \geq 0\right\}.$$

It is the region of possible positions for orbits with Jacobi integral h. The topology of Hill's region $\mathcal{H}(h)$ varies as h changes, and the bifurcation points are exactly Lagrange points. See figure 3.

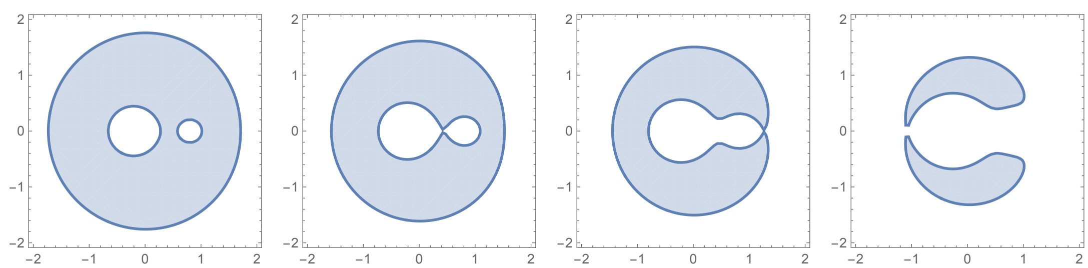

FIGURE 3. Hill's regions of (PCR3BD) with $\mu = 0.2$.