CHAPTER 5

Lambert's Problem

5.1. Rectilinear Lambert's Problem

Lambert's problem is boundary value problem for the Kepler problem. It concerns the determination of the Keplerian orbit from one point is space to another using a predetermined transfer time. The problem has important applications in space mission designs and targeting problems.

In this section we deal with the rectilinear case. The rectilinear Kepler problem is the one-dimensional Kepler problem. This is a simple special case which deserves special attention, for its behavior exhibits several fundamental features of general N-body problems, and its regularization provides a handy tool for analysis of double collisions.

Let us begin with a special solution – the *parabolic ejection orbit* x_{pe} given by

$$x_{\mathrm{pe}}(t) \;=\; c_{\mathrm{pe}}\, t^{\frac{2}{3}}, \quad \text{where } c_{\mathrm{pe}} = \left(\frac{9\alpha}{2\mu}\right)^{\frac{1}{3}}. \tag{5.1.1}$$

It is easy to check that this is the only solution for (3.1.1) of the form $c\,t^{\lambda}$, and it is indeed parabolic.

Consider the rectilinear Kepler problem along the positive real axis with boundary conditions:

$$\mu\ddot{x} \;=\; -\frac{\alpha}{x^2}, \quad x(0) = x_0, \;\; x(T) = x_1, \;\; 0 \le x_0 < x_1. \tag{5.1.2}$$

By conservation of energy, solution curves are given by

$$\frac{\mu}{2}\dot{x}^2 - \frac{\alpha}{x} \;=:\; h \;=\; \text{Constants}. \tag{5.1.3}$$

Figure 1 shows typical phase curves of the differential equation (5.1.2). Along the positive real axis $\dot{x}$ is decreasing, so the flow along these phase curves are downward.

When $h < 0$, x is bounded above by $-\alpha/h$; when $h = 0$, x goes to infinity as $|\dot{x}|$ approaches 0; when $h > 0$, $|\dot{x}| > \sqrt{\frac{2h}{\mu}}$ and x goes to infinity as $|\dot{x}|$ approaches $\sqrt{\frac{2h}{\mu}}+$. Let

$$x_{\max} \;=\; \begin{cases} -\alpha/h, & \text{when } h < 0, \\ \infty & \text{when } h \ge 0. \end{cases}$$

Then $x_{\max}$ is the least upper bound of x.

Consider orbits with negative energy. For elliptical orbits the energy H equals $-\frac{\alpha}{2\mathfrak{a}}$, while in rectilinear case the energy h equals $-\frac{\alpha}{x_{\max}}$. We shall therefore call $\mathfrak{a} := \frac{1}{2}x_{\max}$ the semi-major axis for this case, so that the total energy equals $-\frac{\alpha}{2\mathfrak{a}}$ in both planar and rectilinear cases.

If we let $T(x_0, x_1, h) > 0$ be the transfer time from x_0 to x_1 following an orbit with energy h, then there are cases for which T is undefined. When $h \ge 0$, $T(x_0, x_1, h)$ is well-defined since there exists unique solution for (5.1.2). When $h < 0$, $T(x_0, x_1, h)$ is well-defined if $x_1 = x_{\max}$, undefined if $x_1 > x_{\max}$, and is double-valued if $x_1 < x_{\max}$. These observations are evident from figure 1.

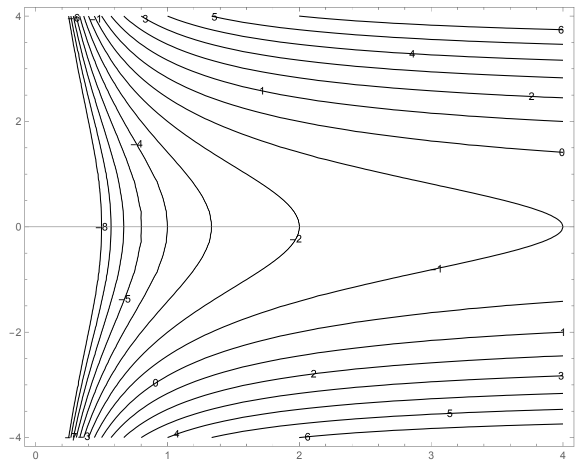

FIGURE 1. Integral curves of the rectilinear Kepler problem. Phase curve crosses the x-axis if and only if $h < 0$.

For $x_1 \leq x_{\max}$ we let $T_1(x_0, x_1, h)$ be the first time the rectilinear orbit passes x_1. For elliptic orbits (i.e. $h < 0$) we let $T_2(x_0, x_1, h)$ be the second time it passes x_1. Then $T_1(x_0, x_{\max}, h) < T_2(x_0, x_{\max}, h)$ and

$$\begin{aligned} T_1(x_0, x_1, h) &= T_1(0, x_1, h) - T_1(0, x_0, h), \\ T_2(x_0, x_1, h) &= T_2(0, x_1, h) - T_1(0, x_0, h), \\ T_2(0, x_1, h) &= T_1(0, x_{\max}, h) + T_1(x_1, x_{\max}, h). \end{aligned}$$

Therefore, to determine the relation between transfer time and energy, we only need a general formula for $T_1(0, x, h)$.

Given $\xi > 0$, let x be the solution curve which goes from 0 to ξ using transfer time $T_1(0, \xi, h)$. Then for $t \in (0, T_1(0, \xi, h))$ we have $\dot{x} > 0$ and from (5.1.3) we have

$$\dot{x} = \sqrt{\frac{2}{\mu}\left(h + \frac{\alpha}{x}\right)}.$$

Thus, for $h < 0$,

$$T_1(0, \xi, h) = \int_0^\xi \frac{1}{\dot{x}} dx = \sqrt{\frac{\mu}{2}} \int_0^\xi \frac{x}{\sqrt{hx^2 + \alpha x}} dx = \sqrt{\frac{-\mu}{2h}} \int_0^\xi \frac{x}{\sqrt{-x^2 - \frac{\alpha}{h} x}} dx.$$

The righthand side can be easily evaluated using the integral formula:

$$c > 0, \quad \int \frac{s\, ds}{\sqrt{-s^2 + cs}} = -\sqrt{-s^2 + cs} + c \sin^{-1}\left(\sqrt{\frac{s}{c}}\right).$$

The case $h > 0$ is similar but we shall use another integral formula:

$$c > 0, \quad \int \frac{s\, ds}{\sqrt{s^2 + cs}} = \sqrt{s^2 + cs} - c \sinh^{-1}\left(\sqrt{\frac{s}{c}}\right).$$

When $h=0$, we have

$$\dot{x} = \sqrt{\frac{2\alpha}{\mu x}} \quad \Rightarrow \quad T_1(0,\xi,0) = \int_0^\xi \frac{1}{\dot{x}}dx = \sqrt{\frac{\mu}{2\alpha}}\int_0^\xi \sqrt{x}dx = \frac{1}{3}\sqrt{\frac{2\mu}{\alpha}}\xi^{\frac{3}{2}}.$$

This can be also obtained from (5.1.1) by observing the solution x_{pe} is parabolic.

To summarize, we find

$$\begin{aligned}
T_1(0,x,h) &= \alpha\sqrt{\frac{\mu}{2(-h)^3}}\left[\sin^{-1}\left(\sqrt{\frac{-hx}{\alpha}}\right) - \sqrt{\frac{-h}{\alpha}x\left(1+\frac{h}{\alpha}x\right)}\right], \quad h<0,\\
T_2(0,x,h) &= \alpha\sqrt{\frac{\mu}{2(-h)^3}}\left[\pi - \sin^{-1}\left(\sqrt{\frac{-hx}{\alpha}}\right) + \sqrt{\frac{-h}{\alpha}x\left(1+\frac{h}{\alpha}x\right)}\right], \quad h<0,\\
T_1(0,x,0) &= \frac{1}{3}\sqrt{\frac{2\mu}{\alpha}}x^{\frac{3}{2}},\\
T_1(0,x,h) &= \alpha\sqrt{\frac{\mu}{2h^3}}\left[-\sinh^{-1}\left(\sqrt{\frac{hx}{\alpha}}\right) + \sqrt{\frac{h}{\alpha}x\left(1+\frac{h}{\alpha}x\right)}\right], \quad h>0.
\end{aligned} \tag{5.1.4}$$

See figure 2 for graphs of these transfer times.

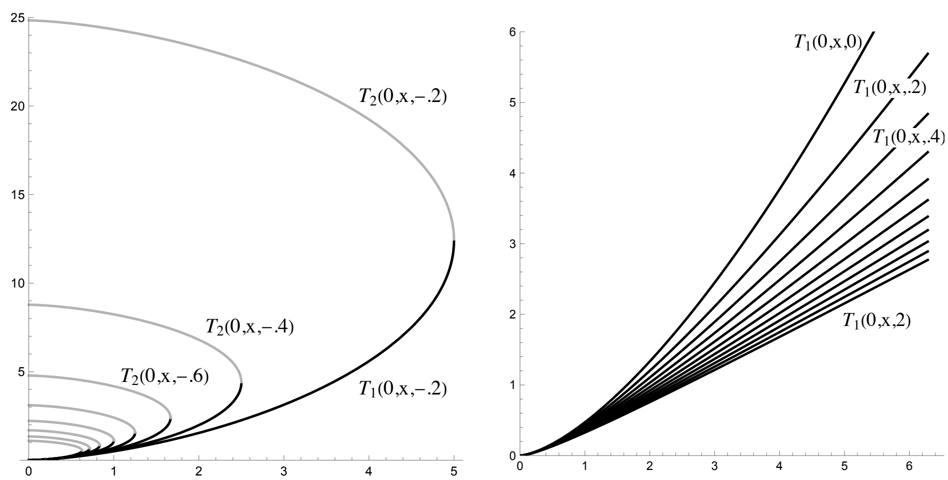

FIGURE 2. Left: graphs of transfer time $T_1(0,x,h)$ (black) and $T_2(0,x,h)$ (gray) as functions of x. Energies ranges from -1.6 to -0.2. Right: graphs of transfer time $T_1(0,x,h)$ as functions of x. Energies ranges from 0 to 2.

5.2. Planar Lambert's problem