CHAPTER 4

Kepler's Equation

4.1. Derivation of Kepler's equation

As soon as the trajectory (semi-major axis and eccentricity) of a Keplerian orbit is determined, the next question is to describe the dynamics on it. Namely, we wish to determine the position of the celestial body as a function of time. To this end we need the concept of apsis and anomalies.

For convenience we set the phase angle $\theta_0 = 0$ throughout this chapter, so that the Keplerian orbit $x = re^{i\theta}$ are given by

$$r(\theta) \;=\; \frac{\mathfrak{p}}{1+\mathfrak{e}\cos\theta} \;=\; \frac{\mathfrak{a}(1-\mathfrak{e}^2)}{1+\mathfrak{e}\cos\theta}\,.$$

We also assume the true anomaly θ satisfies $\dot{\theta} > 0$, so that the scalar angular momentum is strictly positive, and the eccentric or hyperbolic anomaly ψ satisfies $\dot{\psi} > 0$.

Definition 4.1.1. An *apsis* is the point of greatest or least distance from the attractive focus F. The point of least distance is called the *periapsis* or *pericenter*. For elliptic orbits, the point of greatest distance is called the *apoapsis* or *apocenter*.[1]

Denote the minimum value of t by $r_{\min}$ and the its maximum value for elliptic orbits by $r_{\max}$. Then clearly

$$r_{\min} \;=\; \frac{p}{1+\mathfrak{e}}; \quad r_{\max} \;=\; \frac{p}{1-\mathfrak{e}}.$$

By conservation of energy, the maximum speed $v_{\max}$ occurs at periapsis and, for elliptic orbits, the minimum speed $v_{\min}$ occurs at apoapsis. Following from (3.3.2) and (3.3.8), they are given by

$$v_{\max} = \sqrt{\frac{\alpha}{\mathfrak{p}\mu}}(1+\mathfrak{e}) = \sqrt{\left(\frac{1+\mathfrak{e}}{1-\mathfrak{e}}\right)\frac{\alpha}{\mathfrak{a}\mu}}; \quad v_{\min} = \sqrt{\frac{\alpha}{\mathfrak{p}\mu}}(1-\mathfrak{e}) = \sqrt{\left(\frac{1-\mathfrak{e}}{1+\mathfrak{e}}\right)\frac{\alpha}{\mathfrak{a}\mu}}.$$

Definition 4.1.2. Let ω be the *mean motion*; i.e. the average angular velocity $\omega = 2\pi/T$, where T is the period. According to Kepler's third law, we may define the mean motion using the semi-major axis:

$$\omega \;=\; \sqrt{\frac{\alpha}{\mu|\mathfrak{a}|^3}}. \tag{4.1.1}$$

Let t be the time since the last periapsis passage. The *mean anomaly* is defined by

$$M \;=\; \omega\, t. \tag{4.1.2}$$

[1] There are several synonyms. When the attractive focus F is Sun, the periapsis is often called the *perihelion* and the apoapsis is often called the *aphelion*. If the attractive focus F is Earth, then the periapsis if often called the *perigee* and the apoapsis is often called the *apogee*.

Equations (3.2.6) and (3.2.7) determine relations between the true anomaly θ and the eccentric anomaly ψ. The Kepler equation determines their relationships with the mean anomaly:

THEOREM 4.1.1. (THE KEPLER EQUATION)
For elliptic Keplerian orbits, the mean anomaly M and eccentric anomaly ψ are related by

$$M = \psi - \mathfrak{e} \sin \psi.$$

PROOF. Let $x = re^{i\theta}$ be as in (3.3.5). For convenience, and without loss of generality, we assume $\theta(0) = \theta_0 = 0$. Let T be the period of the elliptical orbit. Given $0 \le \tau < T$. By (3.2.7), the area of the sector swept by the position vector x for $t \in [0, \tau]$ is

$$\frac{1}{2} \int_0^{\theta(\tau)} r^2 \, d\theta = \frac{\mathfrak{a}^2}{2} \int_0^{\psi(\tau)} (1 - \mathfrak{e} \cos \psi)^2 \frac{d\theta}{d\psi} \, d\psi = \frac{\mathfrak{ab}}{2} (\psi(\tau) - \mathfrak{e} \sin \psi(\tau)).$$

By Kepler's second law, the areal velocity is constant, so the area of the above mentioned sector also equals

$$\frac{(\text{Area of ellipse}) \, t}{T} = \frac{(\pi \mathfrak{ab}) \, \omega \tau}{2\pi} = \frac{\mathfrak{ab}}{2} M(\tau).$$

This finishes the proof. □

The proof above is essentially contained in the derivation of Kepler's third law. Expressing semi-major axis $\mathfrak{a}$ and semi-minor axis $\mathfrak{b}$ in terms of $\mathfrak{p}$ and $\mathfrak{e}$ has the advantage of including non-elliptical cases; using the formula (3.3.6) for areal velocity, calculations above can be written

$$\begin{aligned} \frac{\mathfrak{p}^2}{2(1 - \mathfrak{e}^2)^{3/2}} (\psi(\tau) - \mathfrak{e} \sin \psi(\tau)) &= \frac{1}{2} \int_0^{\theta(\tau)} r^2 \, d\theta = \frac{1}{2} \int_0^{\tau} r^2 \dot{\theta} \, dt = \frac{1}{2} \sqrt{\frac{\mathfrak{p} \alpha}{\mu}} \tau = \frac{1}{2} \sqrt{\mathfrak{p} \mathfrak{a}^3} M(\tau) \\ &= \frac{\mathfrak{p}^2}{2(1 - \mathfrak{e}^2)^{3/2}} M(\tau). \end{aligned}$$

Since the mean anomaly M is a linear function in time t, Kepler's equation determines the eccentric anomaly ψ as a function of t, and so r and θ can be expressed in terms of t. This gives us precise location of the planet at any given time. A natural question arises: How do we express ψ in terms of t? This will be the subject of the next two sections.

Now let us turn to hyperbolic orbits. In this case we do not use "period" to define mean motion, but the definition of mean motion in (4.1.1) still make sense, and so is (4.1.2). When $h > 0$ the mean anomaly is also known as the *hyperbolic mean anomaly.*

THEOREM 4.1.2. (THE HYPERBOLIC KEPLER EQUATION)
For hyperbolic Keplerian orbits, the mean anomaly M_H and hyperbolic eccentric anomaly ψ_H are related by

$$M_H = \mathfrak{e} \sinh \psi_H - \psi_H.$$

The proof is similar to the elliptic case and is left to the reader.

Lastly, we turn to parabolic orbits. In this case the semi-major axis is infinity and so the mean motion in (4.1.1) is simply zero. However, using

$$r(\theta) = \frac{\mathfrak{p}}{1 + \cos \theta} = \frac{\mathfrak{p}}{2} \left(1 + \tan^2 \frac{\theta}{2} \right),$$

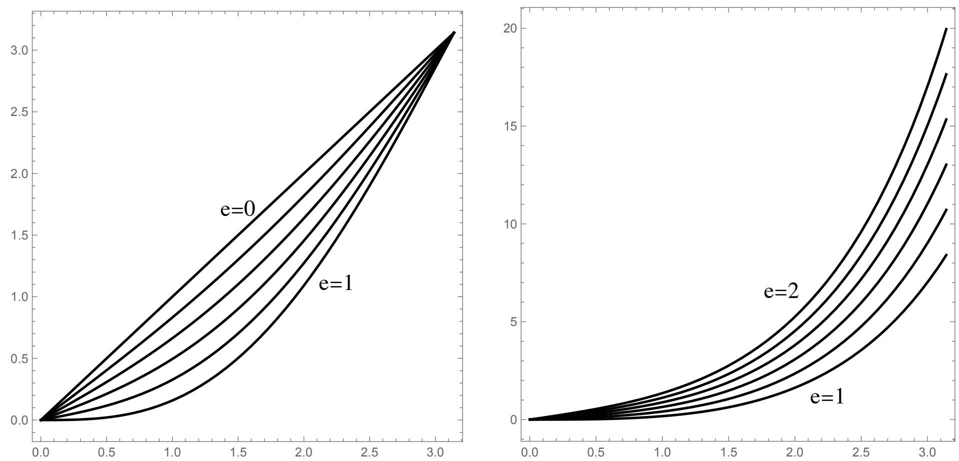

FIGURE 1. Kepler's equation. Left: Graph of M as function of ψ, eccentricity from 0 to 1 with increment 0.2. Right: Graph of M_{H} as function of ψ_{H}, eccentricity from 1 to 2 with increment 0.2.

the calculation for the area of sector swept by the position vector can be expressed in terms of $\mathfrak{p}$:

$$\frac{1}{2}\int_0^{\theta(\tau)} r^2\,d\theta = \frac{\mathfrak{p}^2}{8}\int_0^{\theta(\tau)}\left(1+\tan^2\frac{\theta}{2}\right)^2 d\theta = \frac{\mathfrak{p}^2}{4}\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right).$$

Substitutes for eccentric and mean anomalies are

$$\psi_{\mathrm{P}} \;=\; \tan\frac{\theta}{2}, \quad M_{\mathrm{P}} \;=\; 2\sqrt{\frac{\alpha}{\mu\mathfrak{p}^3}}\,t. \tag{4.1.3}$$

Again, here t is the time since the last periapsis passage. We call ψ_{P} the *parabolic eccentric anomaly* or simply the *parabolic anomaly*, and call M_{P} the *parabolic mean anomaly.*

THEOREM 4.1.3. (THE PARABOLIC KEPLER EQUATION)
For parabolic Keplerian orbits, the parabolic mean anomaly M_P and the parabolic anomaly ψ_P are related by

$$M_P \;=\; \psi_P + \frac{1}{3}\psi_P^3.$$

A natural question arises: How do we express eccentric (or parabolic, hyperbolic) anomaly Ψ in terms of the mean anomaly M? Resolving this problem allows an explicit description of position vector as function of time. This is the subject of the next two sections.

4.2. Lagrange's method of power series

A direct and simple way of solving Kepler's equations is by iteration, due to Lagrange in 1771. Assume $0 \le \mathfrak{e} \ll 1$. Consider the Taylor series of ψ in powers of $\mathfrak{e}$.

$$\begin{aligned} \psi &= M + \mathfrak{e}\sin\psi \\ &= M + O(\mathfrak{e}) \quad \text{as} \quad \mathfrak{e} \to 0. \\ \Rightarrow \psi &= M + \mathfrak{e}\sin(M + O(\mathfrak{e})) \\ &= M + \mathfrak{e}\sin M + O(\mathfrak{e}^2). \\ \Rightarrow \psi &= M + \mathfrak{e}\sin(M + \mathfrak{e}\sin M + O(\mathfrak{e}^2)) \\ &= M + \mathfrak{e}\sin M + \frac{\mathfrak{e}^2}{2}\sin 2M + O(\mathfrak{e}^3). \end{aligned}$$

The last line uses $\sin(M + \mathfrak{e}\sin M + O(\mathfrak{e}^2)) = \sin M + \cos M \sin M \mathfrak{e} + O(\mathfrak{e}^2)$, obtained by expanding the lefthand side as series of $\mathfrak{e}$.

Continue this process,

$$\begin{aligned} \psi &= M + \mathfrak{e}\sin(M + \mathfrak{e}\sin M + \frac{\mathfrak{e}^2}{2}\sin 2M + O(\mathfrak{e}^2)) \\ &= M + \mathfrak{e}\sin M + \frac{\mathfrak{e}^2}{2}\sin 2M + \frac{\mathfrak{e}^3}{2}\cos M \sin 2M + O(\mathfrak{e}^4) \\ &= M + \mathfrak{e}\sin M + \frac{\mathfrak{e}^2}{2}\sin 2M + \mathfrak{e}^3\left(\frac{3}{8}\sin(3M) \; - \frac{1}{8}\sin M\right) + O(\mathfrak{e}^4) \\ &= \cdots \\ &= M + \sum_{k=1}^{\infty} a_k(M)\mathfrak{e}^k. \end{aligned}$$

The coefficient a_k is a linear combination of $\sin(jM)$ with $0 \le j \le k$. Radlus of convergence $\mathfrak{e}^*$ is approximately 0.662743, called the *Laplace limit.*

4.3. Bessel's method of Fourier series

The *Bessel functions* J_n *of the first type* are given by

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta.$$

They solve the *Bessel differential equation*:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

and have the following simple and fast converging series expansion:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n}k!(n+k)!}. \tag{4.3.1}$$

Bessel functions of the first type arise naturally in solving the Laplace equation using separation of variables. Their origin is Bessel's approach for the Kepler equation using Fourier series.

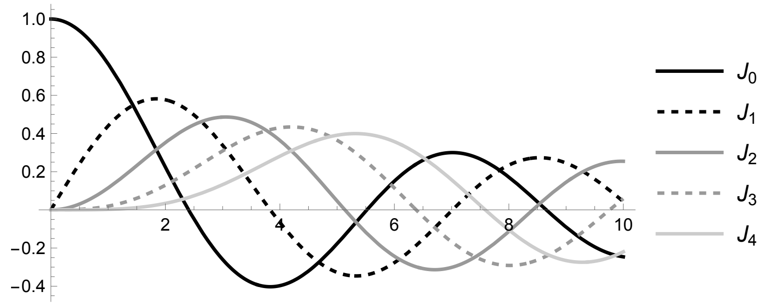

FIGURE 2. Bessel functions of the first kind

Observing the solution of the Bessel differential equation in integral form is nontrivial but the verification is simple:

$$\begin{aligned}
J_n'(x) &= \frac{1}{\pi}\int_0^\pi \sin(n\theta - x\sin\theta)\sin\theta d\theta \\
&= \frac{1}{\pi}\left[-\sin(n\theta - x\sin\theta)\cos\theta\Big|_{\theta=0}^{\theta=\pi} + \int_0^\pi \cos(n\theta - x\sin\theta)(n - x\cos\theta)\cos\theta d\theta\right] \\
&= \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)(n - x\cos\theta)\cos\theta d\theta \\
J_n''(x) &= \frac{-1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\sin^2\theta d\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
& x^2 J_n''(x) + xJ_n'(x) + (x^2+n^2)J_n(x) \\
=& \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\big(-x^2\sin^2\theta + x(n - x\cos\theta)\cos\theta + x^2 - n^2\big)d\theta \\
=& \frac{-n}{\pi}\int_0^\infty \cos(n\theta - x\sin\theta)(nx\cos\theta - n^2)d\theta \\
=& \frac{n}{\pi}\sin(n\theta - x\sin\theta)\Big|_{0=0}^{\pi=\theta} = 0.
\end{aligned}$$

From Bessel's differential equation one can use the method of power series to verigy (4.3.1).

Let $\psi = g(M)$. Then $g(0) = 0$, $g(\pi) = \pi$, and g is an odd function. Therefore $g(M) - M$ is an odd function which equals zeros at 0 and π. This allows us to uniformly approximate it by Fourier sine series. Set

$$g(M) - M = \sum_{n=1}^{\infty} b_n(\mathfrak{e})\sin(nM), \quad 0 \le M \le \pi.$$

Fourier coefficients $b_n(\mathfrak{e})$ are determined by

$$\begin{aligned} b_\pi(\mathfrak{e}) &= \frac{2}{\pi}\int_0^\pi (g(M)-M)\sin(nM)dM \\ &= \frac{2}{n\pi}\int_0^\pi (g'(M)-1)\cos(nM)dM \quad \text{(use integration by parts)} \\ &= \frac{2}{n\pi}\int_0^\pi \cos nM dg(M) \end{aligned}$$

By Kepler's equation,

$$M = \psi - \mathfrak{e}\sin\psi = g(M) - \mathfrak{e}\sin g(M),$$

we conclude

$$b_n(\mathfrak{e}) = \frac{2}{n\pi}\int_0^\pi \cos(n\psi - n\mathfrak{e}\sin\psi)d\psi = \frac{2}{n}J_n(n\mathfrak{e}).$$

Therefore

$$\psi = M + \sum_{n=1}^{\infty}\left(\frac{2}{n}J_n(n\mathfrak{e})\right)\sin nM. \tag{4.3.2}$$

This is Bessel's solution for the Kepler equation.