CHAPTER 3

Kepler's Laws of Planetary Motions

3.1. Kepler problem and its integrals of motion

Here the *Kepler problem* is referred to the Newtonian two-body problem with the mass center fixed at the origin. It is named after Johannes Kepler (1571–1630) for his discovery of three celebrated laws of planetary motions based on astronomical observations of the Danish astronomer Tycho Brahe (1546–1601). The first and second laws were published in 1609, the last one was published in 1619. Keper's three laws of planetary motions are

First Law: *The orbit of every planet is an ellipse with the sun at one of two foci.*

Second Law: *A line joining a planet and Sun sweeps out equal areas during equal intervals of time.*

Third Law: *The ratio of the square of a planet's orbital period with the cube of the semi-major axis of its orbit is the same as that of other planets.*

We will begin with integrals of motion for the problem, discuss important features of conics in the next section, and then reformulate and prove these planetary laws.

The equation of motion for the Kepler problem is given by

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U, \quad k = 1, 2,$$

where $x_k \in \mathbb{R}^d$ is the position of the mass point m_k, and

$$U(x_1, x_2) = \frac{\alpha}{|x_1 - x_2|}\,, \ \alpha = m_1 m_2,$$

is the potential energy. The mass center is assumed to be fixed at the origin, so the position and velocity of one mass point determine the position and velocity of the other. By setting $x = x_2 - x_1$, $\mu = m_1 m_2/(m_1 + m_2)$, called the *reduced mass*, the new coordinate x is related to x_1 and x_2 by

$$\mu x \ = \ -m_1 x_1 \ = \ m_2 x_2.$$

Since the potential U depends only on x, the kinetic energy K depends only on $\dot{x}$, we slightly abuse notations and denote $U(x_1, x_2)$ by $U(x)$, $K(\dot{x}_1, \dot{x}_2)$ by $K(\dot{x})$, then the Lagrangian and Hamiltonian are

$$L(x, \dot{x}) = K(\dot{x}) + U(x) = \frac{\mu}{2}|\dot{x}|^2 + \frac{\alpha}{|x|},$$
$$H(x, y) = K\big(\frac{y}{\mu}\big) - U(x) = \frac{1}{2\mu}|y|^2 - \frac{\alpha}{|x|},$$

where $y = m\dot{x}$. The equation of motion becomes:

$$\boxed{\mu\ddot{x} \;=\; -\frac{\alpha x}{|x|^3}\,.} \tag{3.1.1}$$

Solutions of (3.1.1) are called *Keplerian orbits*.

Kinetic energies K_1 and K_2 of bodies m_1 and m_2 are respectively

$$K_1 \;=\; \frac{m_1 m_2^2}{2(m_1+m_2)}|\dot{x}|^2,\;\; K_2 \;=\; \frac{m_1^2 m_2}{2(m_1+m_2)}|\dot{x}|^2 \quad\Longrightarrow\quad K_1 : K_2 \;=\; m_2 : m_1.$$

In particular, when one mass is much smaller than the other, almost all kinetic energy is due to the light body. For example, while considering a satellite orbiting around Earth, in practice the kinetic energy is always referring to that of the satellite.

Let us begin with a naïve guess. Due to rotational invariance we try simple circular solutions of the form $x(t) = \mathfrak{a}e^{i\omega t}$, where $\mathfrak{a}$ and ω are nonzero constants. Then we immediately find:

$$x(t) = \mathfrak{a}e^{i\omega t} \text{ is a solution for (3.1.1)} \quad\Longleftrightarrow\quad \mathfrak{a}^3\omega^2 \;=\; \frac{\alpha}{\mu}. \tag{3.1.2}$$

In this case the kinetic energy is clearly half of the potential energy. Another simple solution is

$$x(t) \;=\; \left(\frac{9\alpha}{2\mu}\right)^{\frac{1}{3}} t^{\frac{2}{3}}\mathbf{u}, \quad \text{where } \mathbf{u} \text{ is a unit vector.} \tag{3.1.3}$$

It is easy to check that this is the only solution for (3.1.1) of the form $c\,t^{\lambda}\mathbf{u}$. It is not obvious how general solutions are like, so we seek for reduction of the system.

Before proceeding further let us be more precise about what we mean by "solution". A *classical solution* for (3.1.1) should be at least twice differentiable. As the righthand side of (3.1.1) becomes singular at $x = 0$, which corresponds to collision of m_1 and m_2, we demand a classical solution to stay inside of $\mathbb{R}^d \setminus \{0\}$. At any point x_0 in $\mathbb{R}^d \setminus \{0\}$ the righthand side of (3.1.1) is locally Lipschitz continuous, so that classical solutions for (3.1.1) are locally unique. A simple bootstrap argument shows classical solutions are of class C^∞. We may also accept solutions in weaker sense: a *weak solution* for (3.1.1) is only required to be continuous throughout its time domain and be twice differentiable almost everywhere. Without specified otherwise, "almost everywhere", abbreviated as "a.e.", is always in the sense of Lebesgue measure. With the same bootstrap argument we see that weak solutions are of class C^∞ a.e. Weak solutions may, of course, experience collisions.

Physicists often formulate the system (3.1.1) in $\mathbb{R}^2$ or $\mathbb{C}$ because of the conservation of angular momentum. Let us not to limit the space dimension d at this moment. From mathematical point of view the initial value problem

$$\mu\ddot{x} \;=\; -\frac{\alpha x}{|x|^3}, \quad x(0) \;=\; x_0\,(\neq 0), \quad \dot{x}(0) \;=\; v_0$$

has a solution in the linear span of $\{x_0,\, v_0\}$, which is either one- or two-dimensional. By uniqueness theorem of ordinary differential equations we know that with higher d we still have the same solution. This explains why classical solutions of (3.1.1) are always planar, regardless of what $d \geq 2$ is, and are rectilinear if v_0 is zero or parallel to x_0. Weak solutions can be concatenations of rectilinear Keplerian orbits, not necessarily coplanar.

Now, consider $d = 2$ and write $\mathbb{R}^2$ as $\mathbb{C}$ whenever convenient. Equation (3.1.1) gives rise to a 4-dimensional dynamical system. Integrals of motion discussed earlier tell us that there are six integrals of motion which reduces (1.1.1) with $N = 2$, $d = 2$ into a 2-dimensional dynamical system. What we have not yet reduct are angular and energy integrals.

The angular momentum is

$$\mathcal{C} \;=\; m_1 x_1 \wedge \dot{x}_1 + m_2 x_2 \wedge \dot{x}_2 \;=\; \mu x \wedge \dot{x}.$$

In terms of rectangular coordinates $x = \xi + i\eta$ or polar coordinates $x = re^{i\theta}$, the scalar angular momentum is

$$C \;=\; \mu(\xi\dot{\eta} - \eta\dot{\xi}) \;=\; \mu r^2 \dot{\theta}. \tag{3.1.4}$$

In terms of polar coordinates and the angular momentum integral, the energy integral becomes

$$H \;=\; K(r, \dot{r}, \dot{\theta}) - U(r) \;=\; \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\alpha}{r} \;=\; \frac{\mu}{2}\left(\dot{r}^2 + \frac{C^2}{\mu^2 r^2}\right) - \frac{\alpha}{r}.$$

These two integrals of motions show that both $\dot{r}$ and $\dot{\theta}$ can be expressed in terms of r, as soon as C and H are determined. They define two 3-manifolds in the $(r, \theta, \dot{r}, \dot{\theta})$ space.

With respect to variables $(r, \theta, \dot{r}, \dot{\theta})$, gradients of angular momentum and energy are respectively

$$(2\mu r\dot{\theta}, 0, 0, \mu r^2), \quad \left(\mu r\dot{\theta}^2 + \frac{\alpha}{r^2}, 0, \mu\dot{r}, \mu r^2\dot{\theta}\right).$$

They are linearly independent except when $\dot{r} = 0$ and $\mu r\dot{\theta}^2 = \frac{\alpha}{r^2}$. This happens exactly when r, $\dot{\theta}$ are constants and $r^3\dot{\theta}^2 = \alpha/\mu$, which is precisely at the circular solution we just obtained. Therefore, by the implicit function theorem, C and H define a 2-dimensional smooth manifold except at those circular solutions.

Another integral of motion, called *eccentricity vector*, was discovered by Jakob Hermann (1710), generalized to modern form by Johann Bernoulli (1710), then rediscovered by Lagrange (1781), Laplace (1799), Hamilton (1847), and Gibbs (1901). It is more commonly, and improperly, known as the *Laplace-Runge-Lenz vector* among physicists. This integral of motion was not widely known among physicists in early days, resulted in misnaming and repeated rediscoveries. The vector is defined by

$$\mathcal{E} \;=\; \frac{1}{\alpha}\dot{x} \wedge \mathcal{C} - \frac{x}{|x|}. \tag{3.1.5}$$

The verification for conservation of the eccentricity vector is a routine exercise left to the reader. The name "eccentricity vector" will be justified after we derive Kepler's laws.

3.2. Some preliminaries on conics

Here we quickly review definition and some properties of conics. We shall put some emphasis on relations between true and eccentric anomalies, which are very useful in the deduction of Kepler's laws and the Kepler equation, and in our later discussions for the Keplerian action functional.

Given a line

$$L = \{(x, y) \in \mathbb{R}^2 : \ \alpha x + \beta y + \gamma = 0\}, \quad (\alpha, \beta) \neq (0, 0),$$

and a point $F(x_0, y_0) \notin L$ on the xy-plane. A *conic* is the locus of points $P(x, y)$ such that the distance between P and L is a positive constant multiple of the distance between P and F; i.e.

$$\operatorname{dist}(P, F) \;=\; \mathfrak{e}\operatorname{dist}(P, L) \quad \text{for some } \mathfrak{e} > 0.$$

We call F a *focus*, L a *directrix*, and $\mathfrak{e}$ the *eccentricity.* The equation for the conic is

$$\sqrt{(x-x_0)^2+(y-y_0)^2} \;=\; \frac{\mathfrak{e}\,|\alpha x+\beta y+\gamma|}{\sqrt{\alpha^2+\beta^2}}.$$

By suitable translation and scaling, we only need to consider the case

$$(x_0,y_0)=(0,0),\;\; \alpha^2+\beta^2=1, \;\text{and}\;\; \gamma>0.$$

It is convenient to set $(\alpha,\beta)=-(\cos\theta_0,\sin\theta_0)$. By doing so the equation of the conic becomes

$$\sqrt{x^2+y^2} \;=\; \mathfrak{e}\,|-(\cos\theta_0,\sin\theta_0)\cdot(x,y)+\gamma|.$$

A branch of the conic is

$$\sqrt{x^2+y^2} \;=\; -\mathfrak{e}(\cos\theta_0,\sin\theta_0)\cdot(x,y)+\gamma.$$

In terms of polar coordinates $(x,y)=(r\cos\theta, r\sin\theta)$, we find

$$r=-r\mathfrak{e}\cos(\theta-\theta_0)+\gamma.$$

Then we obtain the polar equation for a branch of the conic:

$$r=\frac{\gamma}{1+\mathfrak{e}\cos(\theta-\theta_0)}.$$

We may now include circles as conics with $\mathfrak{e}=0$. A conic is an ellipse if and only if $\mathfrak{e}\in[0,1)$, a parabola if and only if $\mathfrak{e}=1$, and a branch of hyperbola if and only if $\mathfrak{e}>1$.

The other branch of the conic is

$$\sqrt{x^2+y^2} \;=\; \mathfrak{e}(\cos\theta_0,\sin\theta_0)\cdot(x,y)-\gamma,$$

with which we obtain the polar equation for the other branch of the conic:

$$r=\frac{-\gamma}{1-\mathfrak{e}\cos(\theta-\theta_0)}.$$

This is valid only when $\mathfrak{e}>1$ and $\cos(\theta-\theta_0)>1/\mathfrak{e}$, which corresponds the other branch of the hyperbola. The angle $\theta-\theta_0$ is called the *true anomaly.*

In above the positive constant γ is the *semi-latus rectum.* A more conventional notation for the semi-latus rectum is $\mathfrak{p}$, which we shall adopt from this point on. When $\mathfrak{e}\neq 1$, the *semi-major axis* $\mathfrak{a}$ and *semi-minor axis* $\mathfrak{b}$ are given by

$$\mathfrak{a}=\frac{\mathfrak{p}}{1-\mathfrak{e}^2},\quad \mathfrak{b}=\frac{\mathfrak{p}}{\sqrt{1-\mathfrak{e}^2}}. \tag{3.2.1}$$

With this definition, we have $\mathfrak{a}<0$ and $\mathfrak{b}$ is purely imaginary ($\mathfrak{b}=\frac{-\mathfrak{p}i}{\sqrt{\mathfrak{e}^2-1}}$) when $1<\mathfrak{e}$.[1]

Let $\theta_0=0$ for convenience. Then θ is the true anomaly and the equation of the conic in rectangular coordinates is

$$\sqrt{x^2+y^2}=-\mathfrak{e}x+\mathfrak{p}.$$

[1]It should be alerted that many people define the semi-major axis as $|\mathfrak{a}|$ and semi-minor axis as $|\mathfrak{b}|$. With this definition, parametrizations for hyperbolic orbits in our subsequently discussions need some adjustments.

One convenient way of writing the conic equation is

$$\mathfrak{e} = 1 \quad \Rightarrow \quad y^2 + 2\mathfrak{p}x = \mathfrak{p}^2, \tag{3.2.2}$$

$$\mathfrak{e} \neq 1 \quad \Rightarrow \quad \frac{(x+\mathfrak{a}\mathfrak{e})^2}{\mathfrak{a}^2} + \frac{y^2}{\mathfrak{b}^2} = 1. \tag{3.2.3}$$

The center of the conic with $\mathfrak{e} \neq 1$ is $(-\mathfrak{a}\mathfrak{e}, 0)$. For $\mathfrak{e} \neq 1$, another convenient way of writing is

$$\sqrt{(x+2\mathfrak{a}\mathfrak{e})^2 + y^2} \pm \sqrt{x^2+y^2} \;=\; \pm 2\mathfrak{a}.$$

One can easily check that the point $\tilde{F} + (-2\mathfrak{a}\mathfrak{e}, 0)$ is also a focus of the conic. This conic equation leads to the familiar characterization of ellipse (resp. hyperbola) as the locus of points where the sum (resp. difference) of distances to the two foci is constant.

When $0 \le \mathfrak{e} < 1$, a natural parametrization for the conic equation (3.2.3) is

$$(x, y) = (-\mathfrak{a}\mathfrak{e} + \mathfrak{a}\cos\psi, \mathfrak{b}\sin\psi), \quad \psi \in [0, 2\pi). \tag{3.2.4}$$

The $\psi \in [0, 2\pi)$ is called the *eccentric anomaly*. Observe that $r(1 + \mathfrak{e}\cos\theta) = \mathfrak{p} = \mathfrak{a}(1 - \mathfrak{e}^2)$, moving $r\mathfrak{e}\cos\theta = \mathfrak{e}x = \mathfrak{e}(-\mathfrak{a}\mathfrak{e} + \mathfrak{a}\cos\psi)$ to the right side yields

$$r \;=\; \frac{\mathfrak{a}(1-\mathfrak{e}^2)}{1+\mathfrak{e}\cos\theta} \;=\; \mathfrak{a}(1 - \mathfrak{e}\cos\psi). \qquad (0 \le \mathfrak{e} < 1) \tag{3.2.5}$$

Here are some simple and useful relations between the true anomaly θ and eccentric anomaly ψ. Their derivations need nothing but simple trigonometric identities. Details are left as exercises.

$$\boxed{\begin{aligned} \cos\psi &= \frac{\cos\theta + \mathfrak{e}}{1+\mathfrak{e}\cos\theta} \\ \sin\psi &= \frac{\sqrt{1-\mathfrak{e}^2}\sin\theta}{1+\mathfrak{e}\cos\theta} \end{aligned}} \iff \boxed{\begin{aligned} \cos\theta &= \frac{\cos\psi - \mathfrak{e}}{1-\mathfrak{e}\cos\psi} \\ \sin\theta &= \frac{\sqrt{1-\mathfrak{e}^2}\sin\psi}{1-\mathfrak{e}\cos\psi} \end{aligned}} \tag{3.2.6}$$

It follows that

$$\begin{aligned} \tan\frac{\theta}{2} &= \sqrt{\frac{1+\mathfrak{e}}{1-\mathfrak{e}}}\tan\frac{\psi}{2}, \\ \frac{d\psi}{d\theta} &= \frac{1-\mathfrak{e}\cos\psi}{\sqrt{1-\mathfrak{e}^2}} = \frac{r}{\mathfrak{p}/\sqrt{1-\mathfrak{e}^2}}. \end{aligned} \tag{3.2.7}$$

When $1 < \mathfrak{e}$, the branch of the hyperbola closer to the focus F can be also parametrized by (3.2.4), but now with an imaginary ψ. Let $\psi_{\mathrm{H}} = -i\psi$, called the *hyperbolic eccentric anomaly*, or *hyperbolic anomaly* for simplicity. Then $\cos\psi = \cosh(\psi_{\mathrm{H}})$, $\sin\psi = i\sinh(\psi_{\mathrm{H}})$, and (3.2.4), (3.2.5) become

$$(x, y) \;=\; (-\mathfrak{a}\mathfrak{e} + \mathfrak{a}\cosh\psi_{\mathrm{H}}, -i\mathfrak{b}\sinh\psi_{\mathrm{H}}), \quad \psi_{\mathrm{H}} \in \mathbb{R}. \tag{3.2.8}$$

$$r \;=\; \frac{-\mathfrak{a}(\mathfrak{e}^2-1)}{1+\mathfrak{e}\cos\theta} \;=\; -\mathfrak{a}(\mathfrak{e}\cosh\psi_{\mathrm{H}} - 1). \qquad (1 < \mathfrak{e}) \tag{3.2.9}$$

Relations similar to (3.2.6) and (3.2.7) hold:

$$\boxed{\begin{aligned}\cosh\psi_{\mathrm{H}} &= \frac{\mathfrak{e}+\cos\theta}{1+\mathfrak{e}\cos\theta}\\ \sinh\psi_{\mathrm{H}} &= \frac{\sqrt{\mathfrak{e}^2-1}\,\sin\theta}{1+\mathfrak{e}\cos\theta}\end{aligned}} \iff \boxed{\begin{aligned}\cos\theta &= \frac{\mathfrak{e}-\cosh\psi_{\mathrm{H}}}{\mathfrak{e}\cosh\psi_{\mathrm{H}}-1}\\ \sin\theta &= \frac{\sqrt{\mathfrak{e}^2-1}\,\sinh\psi_{\mathrm{H}}}{\mathfrak{e}\cosh\psi_{\mathrm{H}}-1}\end{aligned}} \tag{3.2.10}$$

$$\begin{aligned}\tan\frac{\theta}{2} &= \sqrt{\frac{\mathfrak{e}+1}{\mathfrak{e}-1}}\tanh\frac{\psi_{\mathrm{H}}}{2},\\ \frac{d\psi_{\mathrm{H}}}{d\theta} &= \frac{\mathfrak{e}\cosh\psi_{\mathrm{H}}-1}{\sqrt{\mathfrak{e}^2-1}} = \frac{r}{\mathfrak{p}/\sqrt{\mathfrak{e}^2-1}}.\end{aligned} \tag{3.2.11}$$

3.3. Derivations of Kepler's laws of planetary motion

In terms of polar coordinates $x(t)=r(t)e^{i\theta(t)}$ we decompose the velocity and acceleration vectors into radial and tangential parts:

$$\begin{aligned}\dot{x} &= \dot{r}e^{i\theta}+r\dot{\theta}\,ie^{i\theta},\\ \ddot{x} &= (\ddot{r}-r\dot{\theta}^2)e^{i\theta}+(2\dot{r}\dot{\theta}+r\ddot{\theta})\,ie^{i\theta}.\end{aligned}$$

The Lagrangian and (3.1.1) can thus be written

$$L(r,\theta,\dot{r},\dot{\theta}) = K(r,\dot{r},\dot{\theta})+U(r) = \frac{\mu}{2}(\dot{r}^2+r^2\dot{\theta}^2)+\frac{\alpha}{r}\,, \tag{3.3.1}$$

$$\begin{cases}\mu(\ddot{r}-r\dot{\theta}^2) = -\dfrac{\alpha}{r^2}\\ \mu r^2\dot{\theta} = C.\end{cases} \tag{3.3.2}$$

The constant C is the scalar angular momentum (3.1.4). The term $\frac{1}{2}r^2\dot{\theta}=\frac{C}{2\mu}$ is precisely the *areal velocity* (see figure 1); i.e. the changing rate of the area swept out by the position vector x. This proves *Kepler's second law* which asserts that areal velocity is a constant.

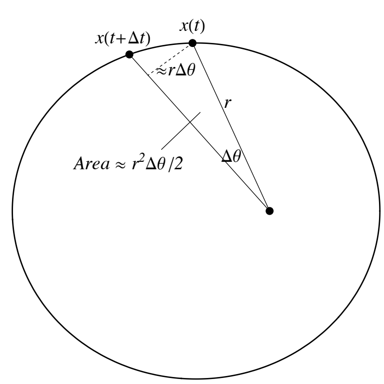

FIGURE 1. Geometric description of areal velocity.

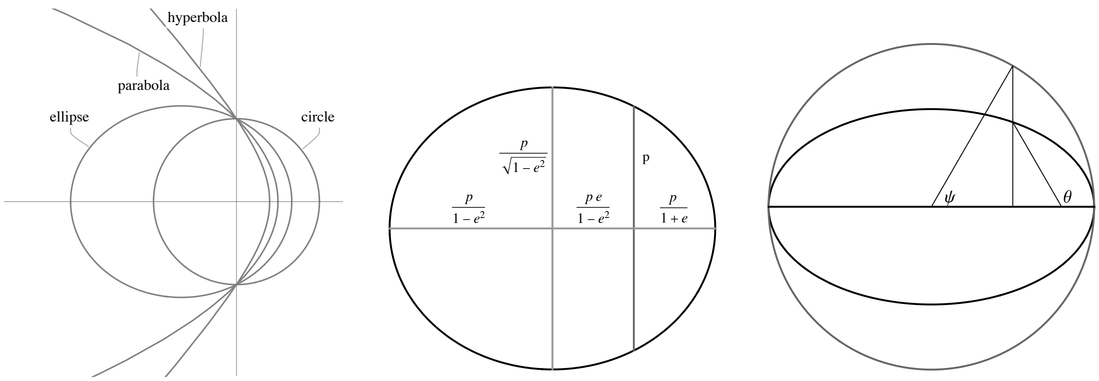

FIGURE 2. Left: Keplerian orbits with attractive focus at the origin. Center: Geometric relations between major axis, minor axis, and latus rectum. Right: Geometric relations between true and eccentric anomalies.

By substituting the second equation in (3.3.2) into the first, the problem is reduced to solving

$$\mu\ddot{r} = -\frac{\alpha}{r^2} + \frac{C^2}{\mu r^3} = -U'_{\text{eff}}(r),\ U_{\text{eff}}(r) = -U(r) + \frac{C^2}{2\mu r^2}.$$

U_{eff} is called the *effective* (or *amended*) *potential*.

Observe that $C = 0$ if and only if the motion is rectilinear. In the case $C \neq 0$, r and $\dot{\theta}$ are nonzero and, by setting $u = \frac{1}{r}$, one deduce easily from (3.3.2) that

$$\frac{du}{d\theta} = -\frac{\mu}{C}\dot{r}, \tag{3.3.3}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu\alpha}{C^2}. \tag{3.3.4}$$

The second equation (3.3.4) is known as the *Clairaut equation*. Solutions of the Clairaut equation are of the form

$$u(\theta) = \frac{\mu\alpha}{C^2} + B\cos(\theta - \theta_0),\ B \geq 0,\ \theta_0 \in [0, 2\pi).$$

Therefore r and θ are related by

$$r(\theta) = \frac{\mathfrak{p}}{1 + \mathfrak{e}\cos(\theta - \theta_0)}\ ,\quad \text{where}\ \ \mathfrak{e} = \frac{BC^2}{\mu\alpha},\ \mathfrak{p} = \frac{C^2}{\mu\alpha}. \tag{3.3.5}$$

Equation (3.3.5) implies that all Keplerian orbits are conics (see figure 2) with one focus at the mass center of the system. This is *Kepler's first law*.

It would be convenient to express the areal velocity in terms of $\mathfrak{p}$ and masses:

$$\frac{1}{2}r^2\dot{\theta} \ = \ \frac{1}{2}\sqrt{\frac{\mathfrak{p}\alpha}{\mu}}. \tag{3.3.6}$$

Constants $\mathfrak{p}$, $\mathfrak{e}$ are exactly the semi-latus rectum and eccentricity, respectively. When there are two foci (i.e. $\mathfrak{e} \neq 1$), we call the mass center the *attractive focus*, and the other focus the *non-attractive* or *unoccupied focus*.

It follows easily from (3.3.5) that u solves the differential equation

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2u}{\mathfrak{p}} + \frac{\mathfrak{e}^2 - 1}{\mathfrak{p}}.$$

Using this, together with (3.3.1), (3.3.2), (3.3.3), the energy integral H can be written:

$$H = \frac{\mu}{2}\left(\dot{r}^2 + \frac{C^2}{\mu^2 r^2}\right) - \frac{\alpha}{r} = \frac{\alpha\mathfrak{p}}{2}\left(\left(\frac{d}{d\theta}\frac{1}{r}\right)^2 + \frac{1}{r^2}\right) - \frac{\alpha}{r} = \frac{\alpha(\mathfrak{e}^2 - 1)}{2\mathfrak{p}} = -\frac{\alpha}{2\mathfrak{a}}. \tag{3.3.7}$$

Here $\mathfrak{a}$ is the semi-major axis. Therefore, Keplerian orbits are periodic if and only if the energy is negative.

The shape and size of the conic are determined by geometric quantities $\mathfrak{e}$ and $\mathfrak{p}$, or by physical quantities H and C. Relations between $(\mathfrak{e}, \mathfrak{p})$, and (H, C) will be frequently recalled, so we summarize them here:

$$\boxed{\begin{aligned} \mathfrak{e} &= \sqrt{\frac{2HC^2}{\mu\alpha^2} + 1} \\ \mathfrak{p} &= \frac{C^2}{\mu\alpha} \end{aligned}} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} H &= \frac{\alpha(\mathfrak{e}^2 - 1)}{2\mathfrak{p}} \\ C &= \sqrt{\mu\alpha\mathfrak{p}} \end{aligned}} \tag{3.3.8}$$

Motivated from observations above, we say a Keplerian orbit is *elliptic* (or *elliptical*) if its energy is negative; *parabolic* if its energy is zero; *hyperbolic* if its energy is positive. This definition matches the identity of H in (3.3.8) for planar orbits, but the definition is more general since rectilinear orbits are included as well.

Suppose x is a Keplerian orbit with minimum period T. Let t_0 be a moment at which $\theta(t_0) = \theta_0$. For convenience we usually set $t_0 = \theta_0 = 0$. By (3.3.2), (3.2.5), (3.2.7) we find

$$T = 2\int_0^{\frac{T}{2}} dt = 2\int_0^{\pi} \frac{dt}{d\theta}\frac{d\theta}{d\psi} d\psi = \frac{2\mu\mathfrak{a}^2\sqrt{1 - \mathfrak{e}^2}}{C}\int_0^{\pi} 1 - \mathfrak{e}\cos\psi \, d\psi = 2\pi\mathfrak{a}^{\frac{3}{2}}\sqrt{\frac{\mu}{\alpha}}.$$

This identity implies *Kepler's third law* which states that T^2 is proportional to $\mathfrak{a}^3$:

$$T^2 = 4\pi^2\left(\frac{\mu}{\alpha}\right)\mathfrak{a}^3, \quad \text{or} \quad \mathfrak{a}^3\omega^2 = \frac{\alpha}{\mu} \text{ (the total mass).} \tag{3.3.9}$$

This include (3.1.2) as a special case.

Another simple way of deriving Kepler's first law, due to Hermann and Lagrange, is via the conservation of the eccentricity vector $\mathcal{E}$ (3.1.5). Observe that

$$(\dot{x} \wedge \mathcal{C}) \cdot x \;=\; (x \wedge \dot{x}) \cdot \mathcal{C} \;=\; \frac{C^2}{\mu}.$$

By writing $x = re^{i\theta}$, where θ is the angle between $\mathcal{E}$ and x, then

$$r|\mathcal{E}|\cos\theta \;=\; |\mathcal{E}||x|\cos\theta \;=\; \mathcal{E}\cdot x \;=\; \frac{C^2}{\mu\alpha} - |x| \;=\; \frac{C^2}{\mu\alpha} - r.$$

Therefore

$$r \;=\; \frac{C^2/\mu\alpha}{1 + |\mathcal{E}|\cos\theta}$$

This implies Kepler's first law, and $|\mathcal{E}|$ is the eccentricity of the orbit. This justifies the naming of the eccentricity vector.