CHAPTER 2

Series Expansion for Gravitational Potential

2.1. Legendre polynomials and Newtonian potential

Developing the Newtonian potential due to a point mass as a suitable convergent series is useful in evaluating potential due to a continuous body, and in approximating solutions of (1.1.1). The purpose of this chapter is to introduce Adrien-Marie Legendre's method of power series expansion in his 1782 work on the potential due to spheroids. It is a powerful tool to compute the potential due to more general symmetric celestial bodies.

Consider point $\mathbf{P}(r, \phi, \theta)$ in $\mathbb{R}^3 \setminus \{0\}$ in spherical coordinates, where $r \in (0, \infty)$ is the radial distance, $\phi \in [0, 2\pi)$ is the azimuthal angle, and $\theta \in [0, \pi]$ is the polar angle. They relate rectangular coordinates (x, y, z) by

$$(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta).$$

Let $\mathbf{P'}(r', \phi', \theta')$ be another point in $\mathbb{R}^3 \setminus \{0\}$. Viewing $\mathbf{P}$ and $\mathbf{P'}$ as nonzero vectors, there is well-defined angle $\psi \in [0, \pi]$ between them. Clearly

$$\begin{aligned} |\mathbf{P} - \mathbf{P'}|^2 &= r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')) \\ &= r^2 + r'^2 - 2rr'\cos\psi. \end{aligned} \tag{2.1.1}$$

Therefore, the potential at $\mathbf{P}$ due to a unit mass at $\mathbf{P'}$ is

$$\begin{aligned} \frac{1}{|\mathbf{P} - \mathbf{P'}|} &= \left(r^2 + r'^2 - 2rr'\cos\psi\right)^{-\frac{1}{2}} \\ &= \frac{1}{r}\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\psi\right)^{-\frac{1}{2}} \\ &= \frac{1}{r'}\left(1 + \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\psi\right)^{-\frac{1}{2}}. \end{aligned} \tag{2.1.2}$$

By expanding this potential as series in either r'/r or r/r', the following sequence of polynomials naturally arise in coefficient terms.

Definition 2.1.1. *Legendre polynomials* $\{P_n\}_{n=0}^{\infty}$ are defined by

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n, \ \ n \in \mathbb{N}. \tag{2.1.3}$$

In particular,

$$P_1(x) = x, \ P_2(x) = \frac{1}{2}(3x^2 - 1), \ P_3(x) = \frac{1}{2}(5x^3 - 3x), \ \cdots.$$

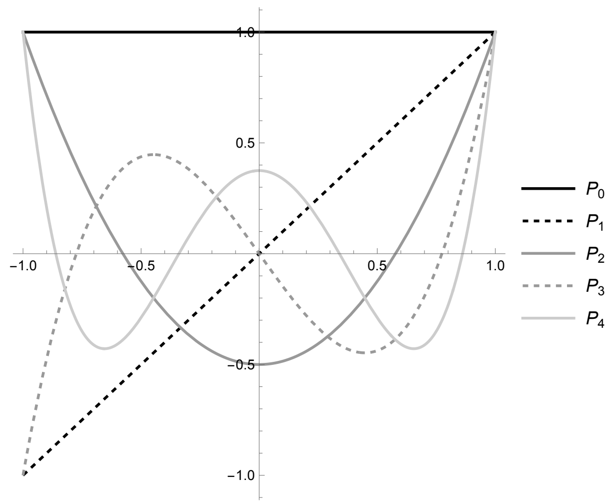

FIGURE 1. Legendre polynomials

There are several alternative definitions for Legendre polynomials, and the above definition is also known as *Rodrigue's formula*. Here are some nice features of Legendre polynomials:

THEOREM 2.1.1. *Let* $\{P_n\}_{n=0}^{\infty}$ *be Legendre polynomials. Then*

(a) $P_n(x)$ *is a polynomial of degree* n. *It is even if* n *is even, odd if* n *is odd. Its leading coefficient is* $\frac{(2n)!}{2^n(n!)^2}$.

(b) $\{P_n\}_{n=0}^{\infty}$ *is an orthogonal basis of* $L^2[-1,1]$ *and*

$$\|P_n\|_{L^2} = \sqrt{\frac{2}{2n+1}}.$$

(c) $\{P_n\}_{n=0}^{\infty}$ *are solutions of Legendre's differential equation:*

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) \ = \ 0.$$

(d) $|P_n(x)| < 1$ *whenever* $|x| < 1$ *and* $P_n(1) = 1$ *for any* $n \in \mathbb{N}$.

Proof for Theorem 2.1.1 is left as exercise. This theorem suggest two alternative definitions for Legendre polynomials, one is through Legendre's differential equations, the other is through the Gram-Schmidt process. The following neat application of Theorem 2.1.1 provides a recursive formulation for Legendre polynomials, which can be served as yet another alternative definition for Legendre polynomials.

THEOREM 2.1.2. (BONNET'S RECURSION FORMULA)
Let $\{P_n\}_{n=0}^{\infty}$ *be Legendre polynomials. Then*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \textit{for any } n \in \mathbb{N}.$$

Proof. For any integer $n \geq 0$ the family $\{P_0, P_1, \cdots, P_{n+1}\}$ is a basis for the space of polynomials of degree less than or equal to $n+1$. Therefore

$$xP_n(x) \;=\; \sum_{k=0}^{n+1} \alpha_k P_k(x) \quad \text{for some } \alpha_0, \alpha_1, \cdots, \alpha_{n+1} \in \mathbb{R}. \tag{2.1.4}$$

By comparing leading coefficients in (2.1.4), we find

$$\frac{(2n)!}{2^n (n!)^2} \;=\; \frac{(2n+2)!}{2^{n+1}((n+1)!)^2}\alpha_{n+1} \;\;\Rightarrow\;\; \alpha_{n+1} \;=\; \frac{n+1}{2n+1}.$$

Since Legendre polynomials are mutually orthogonal in $L^2[-1,1]$, for $0 \leq j \leq n+1$ we have

$$\int_{-1}^{1} xP_n(x)P_j(x)dx \;=\; \sum_{k=0}^{n+1} \alpha_k \int_{-1}^{1} P_k(x)P_j(x)dx \;=\; \alpha_j \left(\frac{2}{2j+1}\right).$$

In particular,

$$\int_{-1}^{1} xP_n(x)P_{n+1}(x)dx \;=\; \frac{2(n+1)}{(2n+1)(2n+3)}.$$

Note that coefficients $\{\alpha_k\}$ depend on the preselected n but this last identity is valid for any $n \geq 0$. Considering $n \geq 1$, by setting $j = n-1$ we obtain

$$\alpha_{n-1}\left(\frac{2}{2n-1}\right) \;=\; \int_{-1}^{1} xP_n(x)P_{n-1}(x)dx \;=\; \frac{2n}{(2n-1)(2n+1)} \;\;\Rightarrow\;\; \alpha_{n-1} = \frac{n}{2n+1}.$$

Now it is sufficient to show that all α_j but α_{n+1}, α_{n-1} are zero. By setting $j = n$ and observing that $xP_n(x)^2$ is an odd function, we find $\alpha_n = 0$. If $n \geq 2$ and $j < n-1$, then $xP_j(x)$ has degree less than n, so in $L^2[-1,1]$ it is orthogonal to $P_n(x)$, and thus $\alpha_j = 0$. This finishes the proof. $\square$

We are now ready to show main theorem of this section which reveals the connection of Legendre polynomials with series expansions of (2.1.2).

Theorem 2.1.3. (Generating Function for Legendre Polynomials) *The function $(1-2st+t^2)^{-1/2}$ is the generating function for Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ in the sense that*

$$\frac{1}{\sqrt{1-2st+t^2}} \;=\; \sum_{k=0}^{\infty} P_k(s)\, t^k \quad \textit{whenever} \quad |s| \leq 1,\; 0 \leq t < 1. \tag{2.1.5}$$

Proof. The case $t = 0$ is obvious, so we consider $t \in (0,1)$. Let $F(s,t)$ be the lefthand side of (2.1.5). From the binomial series expansion of $(1-x)^{-1/2}$ about $x = 0$ we find

$$F(s,t) \;=\; \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2}(2st-t^2)^k, \quad \text{provided } |2st-t^2| < 1.$$

The series converges absolutely whenever $|2st - t^2| < 1$, and uniformly on its compact subsets in the st-plane. As a power series in t the coefficient of t^k is a polynomial in s, denote it by $g_k(s)$. The first two coefficients can be easily determined: $g_0(s) = 1$, $g_1(s) = s$. Observe that

$$(1-2st+t^2)\frac{\partial F(s,t)}{\partial t} + (t-s)F(s,t) \;=\; 0.$$

Therefore

$$(1-2st+t^2)\sum_{k=1}^{\infty} k g_k(s)t^{k-1} + (t-s)\sum_{k=0}^{\infty} g_k(s)t^k \ = \ 0.$$

As a power series in t, the coefficient for t^n with $n \in \mathbb{N}$ is

$$(n+1)g_{n+1}(s) - (2n+1)sg_n(s) + ng_{n-1}(s).$$

According to Bonnet's recursion formula, the polynomial g_n is precisely P_n.

We have proved that (2.1.5) holds provided $|2st - t^2| < 1$. This implies that, given $|s| \le 1$, (2.1.5) holds for $0 < t < \sqrt{2}-1$. Moreover, $F(s,t)$ is analytic in $t \in (0,1)$ since

$$1-2st+t^2 \ \ge \ (1-t)^2 \ > \ 0 \quad \text{for } 0<t<1.$$

By Theorem 2.1.1 (d), the series on the righthand side of (2.1.5) is analytic in $t \in (0,1)$ as well. These two analytic functions on $t \in (0,1)$ coincide on the open interval $(0, \sqrt{2}-1)$, they must coincide on the whole interval $(0,1)$. This completes the proof. □

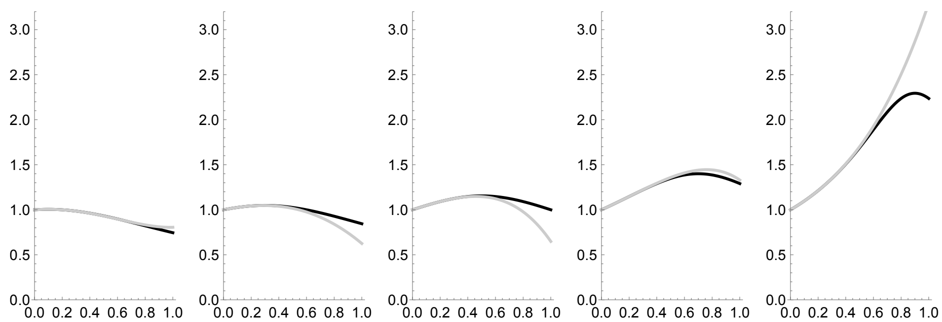

FIGURE 2. The generating function (black) as function of $t \in [0,1)$ and the fourth order approximation (gray) by Legendre's theorem. From left to right, $s = 0.1,\ 0.3,\ 0.5,\ 0.7,\ ;0.9$.

By Theorem 2.1.3 and (2.1.2) we find

$$\frac{1}{|\mathbf{P}-\mathbf{P}'|} \ = \ \begin{cases} \dfrac{1}{r}\displaystyle\sum_{k=0}^{\infty}\left(\frac{r'}{r}\right)^k P_k(\cos\psi) & \text{if } \ 0 \le r' < r, \\ \dfrac{1}{r'}\displaystyle\sum_{k=0}^{\infty}\left(\frac{r}{r'}\right)^k P_k(\cos\psi) & \text{if } \ 0 \le r < r'. \end{cases} \tag{2.1.6}$$

The formula (2.1.2) includes the case $r = r'$ but (2.1.6) does not.

2.2. Integral formula for Legendre polynomials

Domain of Legendre polynomials can be extended to the whole complex plane. As complex functions, they have several succinct integral representations.

THEOREM 2.2.1. *Given any positively oriented simple closed curve γ enclosing $z \in \mathbb{C}$. For any $z \in \mathbb{C}$, Legendre polynomials $\{P_n\}$ as complex analytic functions satisfy*

$$\begin{aligned} P_n(z) &= \frac{1}{2^{n+1}\pi i} \oint_\gamma \frac{(w^2-1)^n}{(w-z)^{n+1}} dw \qquad \text{(Schläfli integral)} \\ &= \frac{1}{\pi} \int_0^\pi \left(z + (z^2-1)^{\frac{1}{2}} \cos\theta \right)^n d\theta. \qquad \text{(Laplace integral)} \end{aligned}$$

PROOF. The first identity, the *Schläfli integral*, follows immediately from the Rodrigues' formula (2.1.3) and the Cauchy integral formula.

By Theorem 2.1.1, we have $P_n(-1) = (-1)^n$, so the second identity is obvious when $z = \pm 1$. Consider $z \neq \pm 1$, then

$$\left\{ w \in \mathbb{C} : \ |w - z| = |z^2 - 1|^{\frac{1}{2}} \right\}$$

is a circle around z. Take any analytic branch of $(z^2-1)^{\frac{1}{2}}$, then parametrize the circle by

$$\left\{ w = z + (z^2-1)^{\frac{1}{2}} e^{i\theta} : \ \theta \in [-\pi, \pi] \right\}.$$

The second identity follows by substituting the variable w by θ in the Schläfli integral:

$$\begin{aligned} P_n(z) &= \frac{1}{2^{n+1}\pi} \int_{-\pi}^{\pi} \left(\frac{(z - 1 + (z^2-1)^{\frac{1}{2}} e^{i\theta})(z + 1 + (z^2-1)^{\frac{1}{2}} e^{i\theta})}{(z^2-1)^{\frac{1}{2}} e^{i\theta}} \right)^n d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(z + (z^2+1)^{\frac{1}{2}} \cos\theta \right)^n d\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \left(z + (z^2+1)^{\frac{1}{2}} \cos\theta \right)^n d\theta. \end{aligned}$$

□

(... to be continued)

2.3. Legendre functions and spherical harmonics

Legendre polynomials can be generalized as follows. For any nonnegative integers n and any integer m with $|m| \leq n$, define the *associated Legendre functions* $\{P_n^m\}$ by

$$P_n^m(x) = \frac{(-1)^m (1-x^2)^{\frac{|m|}{2}}}{2^n n!} \frac{d^{n+|m|}}{dx^{n+|m|}} (x^2-1)^n, \quad n \in \mathbb{N}. \tag{2.3.1}$$

They arise naturally in solving the Laplace equation in spherical coordinates.

THEOREM 2.3.1. *Associated Legendre functions $\{P_n^m\}$ as complex analytic functions satisfy*

$$P_n^m(z) = \frac{(n+m)!}{n!\,\pi} \int_0^\pi \left(z + (z^2-1)^{\frac{1}{2}} \cos\theta \right)^n \cos m\theta \, d\theta.$$

THEOREM 2.3.2. (SPHERICAL HARMONIC ADDITION THEOREM)
Let $\{P_n\}$ be Legendre polynomials and $\{P_n^m\}$ be associated Legendre functions. Suppose $\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. *Then*

$$P_n(\cos\psi) = P_n(\cos\theta) P_n(\cos\theta') + 2 \sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(\cos\theta) P_n^k(\cos\theta') \cos(k(\phi - \phi')).$$

(proof based on integral formula, to be added.)

COROLLARY 2.3.3. *Under the assumption of Theorem 2.3.2, the average of $P_n(\cos\psi)$ over ϕ' is*

$$\frac{1}{2\pi}\int_0^{2\pi} P_n(\cos\psi)d\phi' \quad = \quad P_n(\cos\theta)\, P_n(\cos\theta').$$