CHAPTER 1

Newton's Equations and Symmetry

1.1. Newton's equations and equivalent formulations

The *Law of Universal Gravitation* states, in Isaac Newton's own words (*Principia*, 1687), that

> The gravitational attraction force between two point masses is directly proportional to the product of their masses and inversely proportional to the square of their separation distance. The force is always attractive and acts along the line joining them.

The *inverse square law* in this proposition was first proposed by French astronomer Ismaël Bullialdus (*Astronomia philolaica*, 1645), adopted by English scientist Robert Hooke (*Micrographia*, 1666) in his study of relationships between the height of the atmosphere and the barometric pressure. [1] The law is now most commonly attributed to Newton for recognizing it as the key to celestial mechanics.

Suppose we are given two mass points $m_1,\ m_2$ and let $x_1,\ x_2 \in \mathbb{R}^d$ be their positions. The space dimension d is assumed to be within $\{1, 2, 3\}$ for physical reasons, but there are occasions where allowing bodies to move in a higher dimensional space would lead to better understandings for motions in lower dimensional spaces. We will meet such occasions as we encounter central configurations. The law of universal gravitation can be expressed as

$$F_{12} \ = \ \frac{Gm_1m_2(x_1 - x_2)}{|x_1 - x_2|^3},$$

where F_{12} is the force exerted on m_2 due to m_1, and

$$G \ \approx \ 6.67430 \times 10^{-11}\ m^3/(kg \cdot sec^2)$$

is a positive constant known as the *universal gravitational constant.* The constant G varies as one scales mass, length, or time unit. Throughout this book we set $G = 1$ for convenience.

Given $N(\geq 2)$ mass points $m_1, \cdots, m_N$ in space $\mathbb{R}^d$ with positions $x_1, \cdots, x_N$. The *Newtonian N-body problem*, abbreviated as the *N-body problem* herein, concerns the motion of these mass points in accordance with the law of universal gravitation. By Newton's second law and superposing forces

[1] The law appeared in variant forms in Hooke's letters to Newton in 1679-1680. Soon after the publication of *Philosophiæ Naturalis Principia Mathematica*, often referred to as simply *Principia*, Hooke accused Newton of plagiarizing the notion of inverse square law. Newton acknowledged in a scholium of *Principia* its connection with Christopher Wren, Robert Hooke, and Edmond Halley, but did not acknowledge that Hooke told him anything new or original. Indeed, the law has appeared in preliminary forms in Newton's manuscripts in the 1660s, showing explicit influence by Descartes' work in 1644 on linear momentum, perhaps not owing to communications with Hooke. See Whiteside, D. T., "Before the Principia: The Maturing of Newton's Thoughts on Dynamical Astronomy, 1664–1684". *Journal for the History of Astronomy* (1970).

due to every mass point, equations of motions for this system are

$$m_k \ddot{x}_k \;=\; \sum_{i\neq k} \frac{m_i m_k (x_i - x_k)}{|x_i - x_k|^3}, \quad k = 1, \ldots, N. \tag{1.1.1}$$

Equations of motions (1.1.1) for the N-body problem are called *Newton's equations*. By cancelling out m_k from both sides in the identity, the lefthand side of (1.1.1) becomes the instantaneous acceleration of mass point m_k, so the system makes sense if we allow zero masses. When some but not all masses are zero, the N-body problem is also called the *restricted N-body problem*. It provides an ideal model for motions of astroids, comets, natural and artificial satellites. Unless specified otherwise, the term *N-body problem* is always referring to systems with positive masses.

Newton's equations (1.1.1) is an approximation for real motions of celestial bodies from several perspectives. First it treats celestial bodies as point masses, and that is justified by *Newton's shell theorem* which states that the gravitational field outside a spherical shell with uniform density is the same as if the entire mass is concentrated at its center. Secondly, closely connected with the previous one, tidal forces and energy dissipations have been neglected. Additionally, it has not consider relativistic effects. These are oftentimes considered small perturbations of (1.1.1). Indeed, been tested by numerous experiments and practical missions, (1.1.1) together with some of its perturbed versions provide satisfactory models for the evolution of astrophysical systems without supermassive body. Studying (1.1.1) has been the central part of celestial mechanics since the publication of *Principia*, stimulated developments of several subfields of mathematics, and became an independent mathematical discipline.

Let $x = (x_1, \cdots, x_N) \in \mathbb{R}^{dN}$ and

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|}$$

be the *potential* (also called the *self-potential*). The *Newtonian potential energy* is $-U$. It can be easily checked that the righthand side of (1.1.1) is precisely $\frac{\partial U}{\partial x_k}$, the gradient of U with respect to position x_k. A succinct expression for (1.1.1) is

$$\boxed{\mathbf{M}\ddot{x} \;=\; \nabla U(x),} \quad \text{or} \quad \left\{ \begin{array}{rcl} \dot{x} & = & \mathbf{M}^{-1} y \\ \dot{y} & = & \nabla U(x)\,, \end{array} \right. \tag{1.1.2}$$

where $\mathbf{M}$ is the $dN \times dN$ diagonal matrix

$$\mathbf{M} = diag(\underbrace{m_1, \cdots, m_1}_{d \text{ copies}}, \underbrace{m_2, \cdots, m_2}_{d \text{ copies}}, \cdots, \underbrace{m_N, \cdots, m_N}_{d \text{ copies}}).$$

The expression (1.1.2) assumes all masses are positive.

Alternatively, the system (1.1.1) can be written

$$\boxed{\frac{d}{dt}\frac{\partial}{\partial \dot{x}} L(x, \dot{x}) \;=\; \frac{\partial}{\partial x} L(x, \dot{x}),} \tag{1.1.3}$$

where $L(x, \dot{x})$ is the *Lagrangian* given by

$$K(\dot{x}) = \frac{1}{2}\sum_{k=1}^{N} m_k |\dot{x}_k|^2\,, \quad L(x, \dot{x}) = K(\dot{x}) + U(x)\,.$$

Here $K(\dot{x})$ is called the *kinetic energy.* This is known as the *Lagrangian formulation* for the N-body problem.

Another formulation, known as the *Hamiltonian formulation* for the N-body problem, is

$$\begin{cases} \dot{x} &= \dfrac{\partial}{\partial y} H(x,y) \\ \dot{y} &= -\dfrac{\partial}{\partial x} H(x,y)\,, \end{cases} \tag{1.1.4}$$

where $H(x,y)$ is the *Hamiltonian* (or total energy) given by

$$H(x,y) \;=\; K(\mathbf{M}^{-1}y) - U(x).$$

In more general context, mechanical systems of the form (1.1.2), (1.1.3), (1.1.4) are respectively called *Newtonian*, *Lagrangian*, and *Hamiltonian systems.* It would be helpful to have the following simple examples at disposal.

EXAMPLE 1.1.1. Consider two mass points m, M moving toward each other along a straight line. Suppose they are separated by an initial distance $R \gg 0$, then local in time the equation for their separation r can be approximated by

$$\ddot{r} \;=\; -\frac{GM}{R^2}. \tag{1.1.5}$$

For example, on the surface of the earth, modeled by spherically symmetric ball, with estimated total mass $5.9722 \times 10^{24}\,kg$ and average radius $6.371 \times 10^6\,m$, the absolute value of the righthand side of the above equation is approximately $9.8203\,m/sec^2$. This is the *gravitational acceleration*, also called the *free fall acceleration*, on the surface of the earth. The conventional standard value is defined as $g = 9.80665\,m/sec^2$.

The equation (1.1.5) can be expressed as the Newtonian system

$$\ddot{r} \;=\; -\frac{\partial V}{\partial r}, \quad V(r) = gr.$$

It can be also written as a Lagrangian system with Lagrangian $L(r,\dot{r}) = \dot{r}^2/2 - V(r)$, or a Hamiltonian system with Hamiltonian $H(x,y) = y^2/2 + V(x)$.

EXAMPLE 1.1.2. . The equation of motion for a one-dimensional harmonic oscillator is

$$m\ddot{x} \;=\; -kx, \quad m,\, k > 0 \text{ are constants.}$$

It can be expressed as the Newtonian system

$$m\ddot{x} \;=\; -\frac{\partial V}{\partial x}, \quad V(x) = kx^2/2.$$

It can be also written as a Lagrangian system with Lagrangian $L(x,\dot{x}) = m\dot{x}^2/2 - V(x)$, and a Hamiltonian system with Hamiltonian $H(x,y) = y^2/(2m) + V(x)$.

Formulating a mechanical system as either a Newtonian system, Lagrangian system, or Hamiltonian system has their individual advantages. As a Newtonian system the underlying space is an Euclidean space on which the system is invariant under Galilean transformations (see section 1.2). As a Lagrangian system the underlying space, called the *configuration space*, is a tangent bundle on which symmetries are characterized by one-parameter group of diffeomorphisms (see section 1.3), and the induced flow has a variational structure based on which a contemporary existence theory were developed. As a Hamiltonian system the underlying space, called the *phase space*, is a cotangent bundle which has a canonical symplectic structure based on which several stability theories were established, and symmetries within are characterized by symplecmorphisms. In this book we put special emphasis on the existence theory based on Lagrangian formulation.

1.2. Integrals of motion and symmetries

The dynamical system (1.1.4) is $2dN$-dimensional, which has considerably high degree of freedom even when there are only three bodies in the plane. To understand the system (1.1.1), we need to characterize its invariant set, and to this end we begin with finding lower dimensional spaces (manifolds) on which the flow resides. This amounts to finding integrals of motion.

Given a system of ordinary differential equations, a function on the underlying space (configuration space for a Lagrangian system, phase space for a Hamiltonian system) and time is an *integral of motion* (also called *constant of motion*, *first integral*) if solution curves stay on constant levels of that function.[2] Ideally if we are able to find sufficiently many integrals of motion, then we can describe all solution curves without explicitly solving the system of differential equations.

For Hamiltonian systems it is rather obvious that the Hamiltonian H is an integral of motion: if (x, y) solves (1.1.4), then

$$\frac{d}{dt}H(x,y) = \frac{\partial}{\partial x}H(x,y)\cdot\dot{x} + \frac{\partial}{\partial y}H(x,y)\cdot\dot{y} = -\dot{y}\cdot\dot{x} + \dot{x}\cdot\dot{y} = 0.$$

For Lagrangian systems like (1.1.3) we have something similar (omit $(x, \dot{x})$ for simplicity):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}L\cdot\dot{x} - L\right) = \left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L\right)\cdot\dot{x} + \frac{\partial}{\partial \dot{x}}L\cdot\ddot{x} - \frac{\partial}{\partial x}L\cdot\dot{x} - \frac{\partial}{\partial \dot{x}}L\cdot\ddot{x} = 0.$$

One simple example is the one-dimensional harmonic oscillator in Example 1.1.2. It is a Lagrangian system and Hamiltonian system with

$$L(x,\dot{x}) = \frac{1}{2}\left(m\dot{x}^2 - kx^2\right), \quad H(x,y) = \frac{1}{2}\left(\frac{y^2}{m} + kx^2\right).$$

Level sets of H are ellipses, and upon choosing coordinates

$$(x,y) = \left(\frac{1}{\sqrt{k}}\, r\cos\theta, \sqrt{m}\; r\sin\theta\right)$$

the Hamiltonian becomes $H(r,\theta) = r^2/2$.

The righthand side of (1.1.1) is the total force exerted to mass m_k. Denote it by $\mathbf{F}_k$, then

$$\mathbf{F}_k \;=\; \sum_{i\neq k} F_{ik}, \quad F_{ik} \;=\; \frac{m_i m_k (x_i - x_k)}{|x_i - x_k|^3}.$$

Following from the anti-symmetries $F_{ik} = -F_{ki}$ and $x_i \wedge x_k = -x_k \wedge x_i$,

$$\sum_{k=1}^{N}\mathbf{F}_k \;=\; 0, \quad \sum_{k=1}^{N}\mathbf{F}_k \wedge x_k \;=\; 0.$$

[2]Sometimes the term "integral of motion" is reserved for time-dependent constants of motions. Given the fact that new time variable can be introduced and meanwhile the original time variable can be treated as part of the space variables, here we consider them synonyms.

That is,

$$\frac{d^2}{dt^2}\sum_{k=1}^{N} m_k x_k \;=\; 0, \quad \frac{d}{dt}\sum_{k=1}^{N} m_k x_k \wedge \dot{x}_k \;=\; 0.$$

from which we conclude that

$$\sum_{k=1}^{N} m_k x_k \;=\; At + B, \quad \sum_{k=1}^{N} m_k x_k \wedge \dot{x}_k \;=\; C.$$

for some A, B, $C \in \mathbb{R}^d$.

For the spatial problem (i.e. $d = 3$), we have a total of 10 integrals of motion:

$$\begin{cases} A \;=\; \sum_{k=1}^{N} y_k & \text{(3 linear momentum integrals)} \\ B \;=\; \sum_{k=1}^{N} (m_k x_k - y_k t) & \text{(3 center of mass integrals)} \\ C \;=\; \sum_{k=1}^{N} x_k \times y_k & \text{(3 angular momentum integrals)} \\ H \;=\; K(\mathbf{M}^{-1} y) - U(x) & \text{(1 energy integral)} \end{cases} \tag{1.2.1}$$

For planar problems there is 1 angular momentum integral, resulted in a total of 6 integrals of motion.

Knowing that there are (1.1.1) has 10 integrals of motion for the spatial case, it is intriguing to see that the system has 10 symmetries. This is not coincidence, as to be seen in the next section. To be more accurate, (1.1.1) is invariant under the action of a 10-dimensional group of affine transformations, called *Galilean transformations*, on $\mathbb{R}^3 \times \mathbb{R}$ (space-time). They are generated by the followings. To avoid confusion with previously adopted symbols, for these generating Galilean transformations we use bold face for positions and velocity vectors in $\mathbb{R}^3$.

$$\begin{cases} (\mathbf{x}, t) \mapsto (\mathbf{x} + \mathbf{c}, t), \;\; \mathbf{c} \in \mathbb{R}^3 & \text{(translation in space)} \\ (\mathbf{x}, t) \mapsto (\mathbf{x}, t + s), \;\; s \in \mathbb{R} & \text{(translation in time)} \\ (\mathbf{x}, t) \mapsto (A\mathbf{x}, t), \;\; A \in SO(3) & \text{(rotation in space)} \\ (\mathbf{x}, t) \mapsto (\mathbf{x} + \mathbf{v}t, t), \;\; \mathbf{v} \in \mathbb{R}^3 & \text{(uniform motion in space)} \end{cases} \tag{1.2.2}$$

Those actions on $\mathbb{R}^3$ act on $(\mathbb{R}^3)^N$ component-wise; that is, for $\mathbf{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^3)^N$, $\mathbf{c} \in \mathbb{R}^3$, $A \in SO(3)$, by slightly abusing notations we set

$$\begin{aligned} \mathbf{x} + \mathbf{c} \;&=\; (x_1 + \mathbf{c}, \cdots, x_N + \mathbf{c}), \\ A\mathbf{x} \;&=\; (Ax_1, \cdots, Ax_N). \end{aligned}$$

It is a simple exercise to show that Galilean transformations map solution curves of (1.1.1) to solution curves. Two solutions are considered identical if one can be transform to the other via a Galilean transformation.

Not all of Galilean symmetries in (1.1.1) were carried over to (1.1.3) and (1.1.4). The Lagrangian and Hamiltonian do not involve $\ddot{x}$ while Newton's equations (1.1.1) do not involve $\dot{x}$. This makes the uniform motion of space an invariance of (1.1.1) but not for (1.1.3) and (1.1.4). This invariance means space coordinates can be changed to another one which is moving at a constant velocity relative to the original one, by doing so the kinetic energy and hence the Lagrangian and Hamiltonian would be different from the original ones. This invariance corresponds to the integral of motion B in (1.2.1), as can be seen by differentiating it with respect to time. Other than this symmetry, the first three types of Galilean transformations in (1.2.2) are also invariance of the Lagrangian and Hamiltonian. In the next section we shall see how these symmetries correspond other integrals of motion in (1.2.1).

1.3. Noether's theorem

Consider a Lagrangian $L(x,\dot{x})$ defined on the tangent bundle TM of some smooth n-manifold M, and the corresponding Lagrangian system is

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L(x,\dot{x}) \;=\; \frac{\partial}{\partial x}L(x,\dot{x}). \tag{1.3.1}$$

Let h be a diffeomorphism on M. Denote the linear map $Dh(x) : T_xM \to T_{hx}M$ by h_* (the *pushforward* of h). We say L *admits* the diffeomorphism h if $L(x,\dot{x}) = L(hx, h_*\dot{x})$. It is easy to see that if x is a solution for (1.3.1), then so is hx if L admits h.

THEOREM 1.3.1. (NOETHER'S THEOREM, AUTONOMOUS VERSION)
Suppose the Lagrangian $L(x,\dot{x})$ admits a one-parameter group $\{h^s\}$ of diffeomorphisms, and h^s is of class C^1 in s. Then

$$\frac{\partial}{\partial \dot{x}}L(x,\dot{x})\cdot\frac{\partial}{\partial s}h^s(x)\big|_{s=0}$$

is an integral of motion for the Lagrangian system (1.3.1).

PROOF. Let ϕ be a solution to (1.3.1) and regard $L\left(h^s\phi, h^s_*\dot{\phi}\right)$ as a function of s and t. Then

$$0 = \frac{\partial}{\partial s}L(h^s\phi, h^s_*\dot{\phi}) = \frac{\partial}{\partial x}L(h^s\phi, h^s_*\dot{\phi})\cdot\frac{\partial}{\partial s}h^s\phi + \frac{\partial}{\partial \dot{x}}L(h^s\phi, h^s_*\dot{\phi})\cdot\frac{\partial}{\partial s}h^s_*\dot{\phi}.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \dot{x}}L(\phi,\dot{\phi})\cdot\frac{\partial}{\partial s}h^s\phi\right) &= \left(\frac{\partial}{\partial t}\frac{\partial}{\partial \dot{x}}L(\phi,\dot{\phi})\right)\cdot\frac{\partial}{\partial s}h^s\phi + \frac{\partial}{\partial \dot{x}}L(\phi,\dot{\phi})\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial s}h^s\phi \\ &= \frac{\partial}{\partial x}L(\phi,\dot{\phi})\cdot\frac{\partial}{\partial s}h^s\phi + \frac{\partial}{\partial \dot{x}}L(\phi,\dot{\phi})\cdot\frac{\partial}{\partial s}h^s_*\dot{\phi}\end{aligned}$$

Observe that h^0 is the identity map on M, so that h^0_* is the identity map on each fiber of TM. The theorem follows by setting $s = 0$ in identities above. □

Noether's theorem can be applied to non-autonomous Lagrangian systems as well. Here is how it works. Suppose we are given the following Lagrangian system:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L(x,\dot{x},t) \;=\; \frac{\partial}{\partial x}L(x,\dot{x},t). \tag{1.3.2}$$

If x is a solution, then we have

$$\frac{\partial}{\partial t}L(x,\dot{x},t) = -\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}L(x,\dot{x},t)\cdot\dot{x} - L(x,\dot{x},t)\right). \tag{1.3.3}$$

We have seen the righthand side of (1.3.3) at the beginning of the previous section.

Let $\tilde{L}(x,t,x',t')$ be the following Lagrangian with new time variable τ:

$$\tilde{L}(x,t,x',t') = L(x,\dot{x},t)\,t'. \tag{1.3.4}$$

The prime sign $'$ represents the derivative with respect to τ. This definition of $\tilde{L}$ makes sense because $\dot{x} = x'/t'$. The new Lagrangian $\tilde{L}$ is autonomous by viewing (x,t) as new space variable. Its partial derivatives with respect to x' and t' are familiar:

$$\begin{aligned} \frac{\partial}{\partial x'}\tilde{L}(x,t,x',t') &= \frac{\partial}{\partial x'}L\left(x,\frac{x'}{t'},t\right)t' = \frac{\partial}{\partial \dot{x}}L\left(x,\frac{x'}{t'},t\right) = \frac{\partial}{\partial \dot{x}}L\left(x,\dot{x},t\right) \\ \frac{\partial}{\partial t'}\tilde{L}(x,t,x',t') &= \frac{\partial}{\partial t'}\left[L\left(x,\frac{x'}{t'},t\right)t'\right] = -\frac{\partial}{\partial \dot{x}}L\left(x,\frac{x'}{t'},t\right)\cdot\frac{x'}{t'} + L\left(x,\frac{x'}{t'},t\right) \\ &= -\frac{\partial}{\partial \dot{x}}L\left(x,\dot{x},t\right)\cdot\dot{x} + L\left(x,\dot{x},t\right). \end{aligned} \tag{1.3.5}$$

By (1.3.3) and (1.3.5), we see that (x,t) solves the autonomous Lagrangian system for $\tilde{L}(x,t,x',t')$ if x solves the non-autonomous Lagrangian system (1.3.2):

$$\begin{aligned} \frac{d}{d\tau}\frac{\partial}{\partial x'}\tilde{L}(x,t,x',t') &= \left(t'\frac{d}{dt}\right)\frac{\partial}{\partial \dot{x}}L\left(x,\dot{x},t\right) = \frac{\partial}{\partial x}\tilde{L}(x,t,x',t'), \\ \frac{d}{d\tau}\frac{\partial}{\partial t'}\tilde{L}(x,t,x',t') &= t'\frac{d}{dt}\left[-\frac{\partial}{\partial \dot{x}}L\left(x,\dot{x},t\right)\cdot\dot{x} + L\left(x,\dot{x},t\right)\right] = \frac{\partial}{\partial t}\tilde{L}(x,t,x',t'). \end{aligned} \tag{1.3.6}$$

This observation tells us that *we can always transform a non-autonomous Lagrangian system to an autonomous one.*

For a non-autonomous Lagrangian $L(x,\dot{x},t)$, we say L *admits* a diffeomorphism h on $M\times\mathbb{R}$ if its associated autonomous Lagrangian $\tilde{L}(x,t,x',t')$ in (1.3.4) admits h. To be more precise, denote the space component of h by h_1, time variable by h_2, then L admits h if

$$L\left(x,\frac{x'}{t'},t\right)t' = L\left(h_1(x,t),\frac{h_{1*}(x',t')}{h_{2*}(x',t')},h_2(x,t)\right)h_{2*}(x',t'). \tag{1.3.7}$$

This includes autonomous case as a special case, where $L(x,\dot{x})$ admits diffeomorphism h on $M\times\mathbb{R}$ if

$$L\left(x,\frac{x'}{t'}\right)t' = L\left(h_1(x,t),\frac{h_{1*}(x',t')}{h_{2*}(x',t')}\right)h_{2*}(x',t'). \tag{1.3.8}$$

We are now ready to state the non-autonomous version of Noether's theorem:

THEOREM 1.3.2. (NOETHER'S THEOREM, NON-AUTONOMOUS VERSION) *Suppose the Lagrangian $L(x,\dot{x},t)$ admits a one-parameter group $\{h^s\}$ of diffeomorphisms on $M\times\mathbb{R}$, and h^s is of class C^1 in s. Then*

$$\left(\frac{\partial}{\partial \dot{x}}L(x,\dot{x},t), -\frac{\partial}{\partial \dot{x}}L(x,\dot{x},t)\cdot\dot{x} + L(x,\dot{x},t)\right)\cdot\frac{\partial}{\partial s}h^s(x,t)\big|_{s=0}$$

is an integral of motion for the Lagrangian system (1.3.2).

PROOF. Let $\tilde{L}(x,t,x',t')$ be as in (1.3.4). By the autonomous version of Noether's theorem, the following is an integral of motion for $\tilde{L}$:

$$\left(\frac{\partial}{\partial x'}\tilde{L}(x,t,x',t'), \frac{\partial}{\partial t'}\tilde{L}(x,t,x',t')\right)\cdot\frac{\partial}{\partial s}h^s(x,t)\big|_{s=0}.$$

This is, by (1.3.5), precisely the function asserted in the theorem. □

Examples below demonstrate the usage of Noether's theorem for the N-body problem (1.1.1).

EXAMPLE 1.3.1. Fix $\mathbf{c} \in \mathbb{R}^3$. Consider diffeomorphisms h^s on $\mathbb{R}^{3N}$ defined by

$$h^s(x) = x + s\mathbf{c}.$$

Its derivative h^s_* is the identity, making it obvious that L admits $\{h^s\}$. By Noether's theorem,

$$(m_1\dot{x}_1, \cdots, m_N\dot{x}_N) \cdot (\mathbf{c}, \cdots, \mathbf{c}) = \sum_{k=1}^{N} m_k\dot{x}_k \cdot \mathbf{c}$$

is an integral of motion. Since $\mathbf{c}$ is arbitrary, we conclude the conservation of linear momentum.

EXAMPLE 1.3.2. Consider diffeomorphisms h^s on $\mathbb{R}^{3N}$ defined by

$$h^s(x) = A_s x, \quad A_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{pmatrix}.$$

Then L admits $\{h^s\}$. By Noether's theorem, one can easily deduce that

$$\sum_{k=1}^{N} m_k(x_k \times \dot{x}_k) \cdot e_1$$

is an integral of motion. Here $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Replacing A_s by rotations about x_2- and x_3-axis, then we may replace e_1 above by e_2 and e_3. Thus we conclude the angular momentum

$$\sum_{k=1}^{N} m_k(x_k \times \dot{x}_k)$$

is an integral of motion.

EXAMPLE 1.3.3. Consider diffeomorphisms h^s on $\mathbb{R}^{3N} \times \mathbb{R}$ defined by

$$h^s(x,t) = (x, t+s).$$

Then h^s_* is the identity map. With h in (1.3.8) replaced by h^s, the L admits $\{h^s\}$ since the righthand side of (1.3.8) is precisely

$$L\left(x, \frac{x'}{t'}\right) t'$$

By the non-autonomous Noether theorem,

$$(m_1\dot{x}_1, \cdots, m_N\dot{x}_N, -K(\dot{x}) + U(x)) \cdot (\underbrace{0, \cdots, 0}_{\in\mathbb{R}^{3N}}, 1) = U(x) - K(\dot{x})$$

is an integral of motion, and that is precisely the energy integral.