2022/5/26

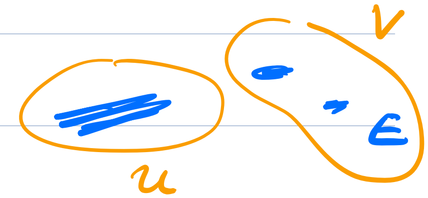
Def. We say $E \subset \mathbb{R}^n$ is <u>disconnected</u> if

$\exists$ nonempty open sets U, V s.t.

(i) $E \subset U \cup V$.

(ii) $U \cap V = \emptyset$.

(iii) $E \cap U \neq \emptyset$. $E \cap V \neq \emptyset$.



In this case, we say $\{U, V\}$ is a <u>separation</u> of E, or $\{U, V\}$ <u>separate</u> E.

If E is not disconnected (i.e. $\nexists$ separation), then we say E is <u>connected</u>.

<u>Example</u>. $\emptyset$ is connected.

Singletons are connected. (singleton - set w/ only one element)

$[0,1] \setminus \{\frac{1}{2}\}$ is disconnected since $(-\infty, \frac{1}{2}), (\frac{1}{2}, \infty)$ is a separation.

$\mathbb{Q}$ is disconnected since $(-\infty, \pi), (\pi, \infty)$ is separation.

<u>Prop 1</u>. Suppose $\{U, V\}$ is a separation for E and $F \subset E$ is a connected subset of E. Then $F \subset U$ or $F \subset V$.

pf. Suppose o.w. $F \not\subset U$. and $F \not\subset V$.

$$\begin{cases} F \subset E \subset U \cup V. & U \cap V = \emptyset. \\ F \cap U \neq \emptyset. & F \cap V \neq \emptyset \end{cases}$$

$\therefore$ $\{U, V\}$ is a separation of F.

i.e. F is disconnected. $\times$

<u>Prop 2</u>. Suppose E is connected, $E \subset A \subset \bar{E}$.
Then A is connected. (In particular, $\bar{E}$ is connected)

pf. Suppose o.w. then $\exists$ separation $\{U, V\}$ for A.

<u>Claim</u>: $E \cap U \neq \phi$.

If not, $E \cap U = \phi$, then $\exists\, x \in A \cap U$ s.t.

$x \in A \setminus E \subset \bar{E} \setminus E \subset \bar{E} \setminus E^{\circ} = \partial E$

$\because$ U is open

$\therefore$ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U$.

$\Rightarrow B_\varepsilon(x) \cap E = \phi$ $\times$.

Similarly, $E \cap V \neq \phi$.

$E \subset U \cup V$ since $E \subset A$.

$\therefore$ $\{U, V\}$ is separation for E. $\times$.

$\therefore$ $\nexists$ separation for A.

i.e. A is connected.

Def. An interval in $\mathbb{R}$ is a set of the form
(a,b), $(a,b]$, $[a,b)$, $[a,b]$.

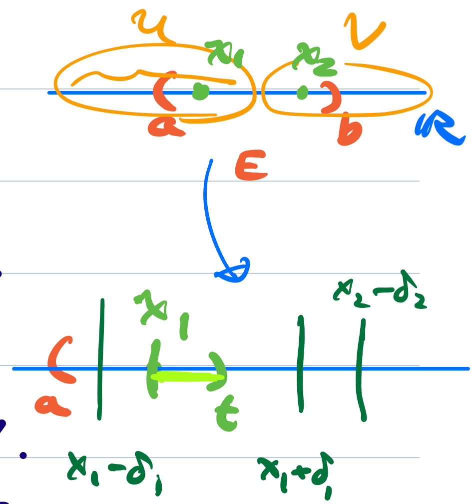
Prop 3. A subset E of $\mathbb{R}$ is connected if and only if it is an interval.

pf. "$\Rightarrow$" Assume E is connected.
Suppose E is not an interval.
$\Rightarrow \exists\ x_1 < x < x_2$, $x_1, x_2 \in E$, $x \notin E$.
$\Rightarrow \{(-\infty, x), (x, \infty)\}$ is a separation for E. $\divideontimes$.
"$\Leftarrow$". Assume E is an interval.
Consider $E = (a,b)$.
By Prop 2, if we can show E is connected, then so are $(a,b]$, $[a,b)$, $[a,b]$.

Suppose $E=(a,b)$ has a separation $\{U,V\}$.

Let $x_1 \in E\cap U$. $x_2 \in E \cap V$.

W.l.o.g., may assume $x_1 < x_2$.



$\because$ $E\cap U$ is open

$\therefore$ $\exists\, \delta_1 > 0$ s.t. $(x_1-\delta_1, x_1+\delta_1) \subset E\cap U$.

$\because$ $E\cap V$ is open

$\therefore$ $\exists\, \delta_2 > 0$ s.t. $(x_2-\delta_2, x_2+\delta_2) \subset E\cap V$.

Let $y = \sup\{t\in\mathbb{R} : [x_1, t) \subset E\cap U\}$

$\Rightarrow$ $x_1 < x_1+\delta_1 \le y \le x_2-\delta_2 < x_2$.

$\because$ E is an interval.

$\therefore$ $y \in E \Rightarrow y \in E\cap U$ or $E\cap V$.

If $y\in E\cap U$, then $\exists\,\delta>0$ s.t. $(y-\delta, y+\delta)\subset E\cap U$.

$\Rightarrow$ $[x_1, y+\delta) \subset E\cap U$, contradicts the def. of y.

If $y \in E\cap V$, then $\exists\,\delta>0$ s.t. $(y-\delta, y+\delta) \subset E\cap V$.

$\Rightarrow$ $[x_1, y) \not\subset E\cap U$ $\divideontimes$.

$\therefore E$ has no separation. i.e. E is connected.

QED.

Theorem 6. Continuous functions map connected sets to connected sets.

pf. Suppose $A \subset \mathbb{R}^n$ is connected. $f: A \to \mathbb{R}^m$ is continuous.

Suppose $f(A)$ is disconnected; i.e. $\exists$ separation U, V for $f(A)$. Then $U, V \neq \phi$.

(i) $f(A) \subset U \cup V$.

(ii) $U \cap V = \phi$.

(iii) $f(A) \cap U \neq \phi$. $f(A) \cap V \neq \phi$.

Then $A \subset f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. (by (i)).

$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \phi$ (by (ii))

$f^{-1}(U) \cap A = f^{-1}(U \cap f(A)) \neq \phi$. $f^{-1}(V) \cap A \neq \phi$ (by (iii)).

$f^{-1}(U), f^{-1}(V)$ are open since f is continuous.

$\therefore \{f^{-1}(U), f^{-1}(V)\}$ is a separation for A. $\times$.

$\therefore f(A)$ is connected. **QED**.

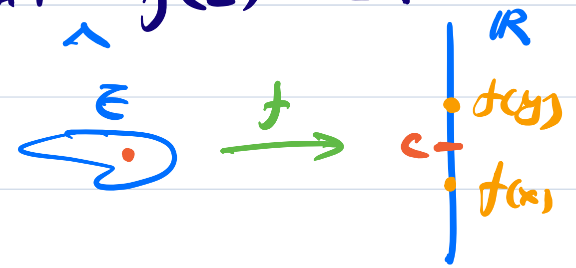
Corollary 1. If $f: E \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and E is connected, then $f(E)$ is an interval.

(by Theorem 6 and Prop. 3).

Corollary 2. (Intermediate Value Theorem, IVT)

If $f: E \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and E is connected, and if $f(x) < f(y)$ for some $x, y \in E$, then $\forall c \in (f(x), f(y))$, $\exists z \in E$ s.t. $f(z) = c$.

(by Corollary 1)



Chap 10. Metric Spaces.

§1. Introduction.

Euclidean space $\mathbb{R}^n$

- Linear structure (algebraic structure)
- Topological structure. (use "norm" to define open sets)
- Order structure

(total order in $\mathbb{R}$. partial order in $\mathbb{R}^n$).

Def. A metric space is a set X w/ a function $\rho : X \times X \to \mathbb{R}$ s.t.

(i) $\rho(x,y) \geq 0$ and $= 0$ if and only if $x = y$.

(ii) $\rho(x,y) = \rho(y,x)$

(iii) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ (Triangle inequality).

This ρ is called the metric of the metric space.

Examples. (1) $(\mathbb{R}, |\cdot|)$ is a metric space, where the metric is defined by $\rho(x,y) = |x-y|$.

(2) $(\mathbb{R}^n, \|\cdot\|)$ also defines a metric space by
$$\rho(x,y) = \|x-y\|.$$
$\|\cdot\|$ can be replaced by $\|\cdot\|_1$, $\|\cdot\|_\infty$, etc.
$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|.$$
(i) $\|x-y\|_1 \geq 0$ & $= 0 \Leftrightarrow x = y$.

(ii) $\|x-y\|_1 = \|y-x\|_1$

(iii) $\|x-y\|_1 \leq \|x-z\|_1 + \|z-y\|_1$ (triangle ineq. for norm)

(3) More generally, for any normed vector space $(X, \|\cdot\|)$, $\rho(x,y) = \|x-y\|$ defines a metric on X.

$\therefore$ Any normed vector space is a metric space.

(4) $(C[a,b], \|\cdot\|_\infty)$ w/

$$\|f\|_\infty \text{ (or } \|f\|_{C^0}) = \sup_{x\in[a,b]} |f(x)|.$$

$C[a,b] = \{ f : [a,b] \to \mathbb{R},\ f \text{ is continuous}\}.$

$\rho(f,g) = \|f-g\|_\infty$, defines a metric on $C[a,b]$.

(5) X : nonempty set.

$$\rho(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y. \end{cases}$$

(i) $\rho(x,y) \geq 0$ & $=0 \iff x=y$.

(ii) $\rho(x,y) = \rho(y,x)$

(iii) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$

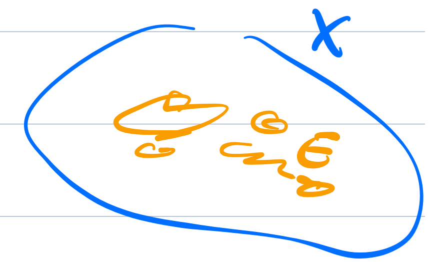
l.h.s. $=1$. means $x\neq y$. $\Rightarrow$ $z\neq x$ or $z\neq y$. $\forall z$

$\Rightarrow$ r.h.s. $=1$ or 2.

This ρ is called the discrete metric.

(6) (X, ρ) — metric space. $E \subset X$ is nonempty.

$\Rightarrow$ (E, ρ) is also a metric space, call it a subspace of the metric space (X, ρ).



Def. Given a metric space (X, ρ)

A set of the form

$$B_r(x) = \{ y \in X : \rho(x, y) < r \}$$

is called an open ball. We call x the center of $B_r(x)$, r the radius of $B_r(x)$.

A set of the form

$$\overline{B_r(x)} = \{ y \in X : \rho(x, y) \leq r \}$$

is called a closed ball, for which x is the center and r is the radius.

$B_r(x)$ is also called the r-neighborhood of x.

We say $E \subset X$ is <u>open</u> if

$\forall x \in E.\ \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset E$.

We say E is <u>closed</u> if E^c is open.

<u>Example.</u> (1) X. ϕ are both open and closed.

(2) Open balls are open.

Closed balls are closed.

(3) Singletons are closed.

<u>Theorem.</u> Arbitrary union and finite intersection of open sets are open.

Finite union and arbitrary intersection of closed sets are closed.

(pf. — same as $\mathbb{R}^n$).