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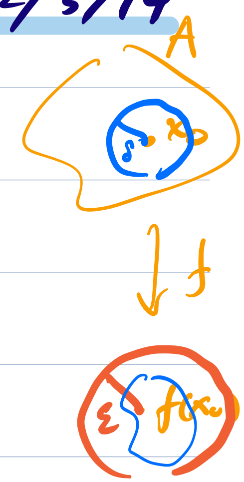
§ 4. Continuous functions.



Def. Consider $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$. $x_0 \in A$.

We say f is continuous at x_0 if

$\forall \varepsilon > 0$. $\exists \delta > 0$ s.t.

$\|x - x_0\| < \delta$. $x \in A$. imply $\|f(x) - f(x_0)\| < \varepsilon$.

i.e. $\lim\limits_{x \to x_0} f(x)$ exists and equals $f(x_0)$.

If f is cont. at every $x_0 \in A$, then we say f is continuous (on A).

Remark. (Sequential characterization of Continuity)

f cont. at $x_0 \Leftrightarrow \forall$ seq. $\{x_k\} \subset A$ s.t. $\lim\limits_{k \to \infty} x_k = x_0$

we have $\lim\limits_{k \to \infty} f(x_k) = f(x_0)$.

Theorem! (Equivalent Descriptions for Continuity)

Consider $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. The followings are equivalent.

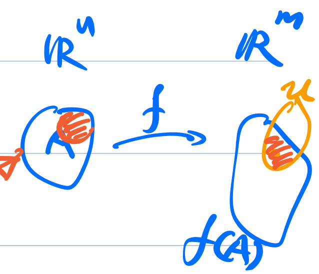
(a) f is continuous (on A)

(b) $f^{-1}(\mathcal{U})$ is open (in A) for any open set $\mathcal{U}$.

(c) $f^{-1}(V)$ is closed (in A) for any closed set V.

(d) $f(\overline{W}) \subset \overline{f(W)}$ for any $W \subset A$.

(e) $\forall x_0 \in A$. $\forall$ seq. $\{x_k\}$ in A converging to x_0, we have $\lim_{k\to\infty} f(x_k) = f(x_0)$.



pf. (a) $\Rightarrow$ (b). Given open set $\mathcal{U} \subset \mathbb{R}^m$.

The case $f^{-1}(\mathcal{U}) = \phi$ is obvious.

Consider $f^{-1}(\mathcal{U}) \neq \phi$. Given $x \in f^{-1}(\mathcal{U})$ i.e. $f(x) \in \mathcal{U}$

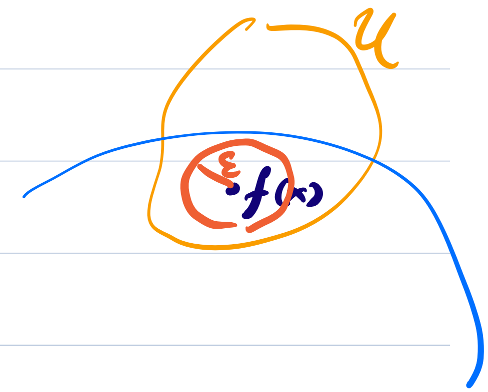
$\because$ $\mathcal{U}$ is open.

$\therefore$ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subset \mathcal{U}$.

Choose $\delta > 0$ s.t.

$$f(B_\delta(x)) \subset B_\varepsilon(f(x)) \subset \mathcal{U}$$

$$\Rightarrow \quad B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(\mathcal{U}).$$



$\therefore$ $f^{-1}(\mathcal{U})$ is open.

(b) $\Rightarrow$ (c). Given closed set $V \subset \mathbb{R}^m$.

$$f^{-1}(V) = f^{-1}(\mathbb{R}^m \setminus V^c) = f^{-1}(\mathbb{R}^m) \setminus f^{-1}(V^c)$$

$$= A \setminus \underbrace{f^{-1}(\underbrace{V^c}_{\text{open}})}_{\text{open}} \text{ is closed (in } A).$$

(c) $\Rightarrow$ (d). Given $W \subset A$.

$\overline{f(W)}$ is closed $\Rightarrow$ $f^{-1}(\overline{f(W)})$ is closed

$$W \subset f^{-1}(f(W)) \subset f^{-1}(\overline{f(W)})$$

$\Rightarrow$ $\overline{W} \subset f^{-1}(\overline{f(W)})$ since $f^{-1}(\overline{f(W)})$ is closed.

$\Rightarrow$ $f(\overline{W}) \subset \overline{f(W)}$.

(d) $\Rightarrow$ (e). Suppose not.

i.e. $\exists\, x_0 \in A$. $\exists$ seq. $\{x_k\}$ conv. to x_0 but $f(x_k)$ does not conv. to $f(x_0)$.

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$. $f(x_n) \notin B_\varepsilon(f(x_0))$ for some $n \geq N$

$\Rightarrow \exists$ subseq. $\{f(x_{n_k})\}_{k=1}^{\infty}$ s.t. $f(x_{n_k}) \notin B_\varepsilon(f(x_0))$ $\forall k$.

Let $W = \{x_{n_k}\}_{k=1}^{\infty}$. $\Rightarrow x_{n_k} \to x_0$ as $k \to \infty$ & $x_0 \in \overline{W}$.

$\Rightarrow f(x_0) \in f(\overline{W}) \subset \overline{f(W)} = \overline{\{f(x_{n_k})\}_{k=1}^{\infty}}$ $\times$.

(e) $\Rightarrow$ (a), by Remark on sequential characterization of continuity.

<u>QED</u>.

Example. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x \log y}{\sqrt{x^2+(y-1)^2}} & (x,y) \neq (0,1) \quad (0,0) \\ 0 & \text{o.w.} \end{cases}$$

Is f continuous at $(0,1)$?

Let $(x,y) = (r\cos\theta, 1 + r\sin\theta)$.

$\Rightarrow (x,y) \to (0,1) \Leftrightarrow r \to 0$.

$$f(r\cos\theta, 1+r\sin\theta) = \frac{\cancel{r}\cos\theta \log(1+r\sin\theta)}{\cancel{r}}$$

$$= \cos\theta \log(1+r\sin\theta) \longrightarrow 0 \text{ as } r \to 0$$

$$0 = f(0,1)$$

$\therefore f$ is continuous at $(0,1)$.

Theorem 2. Continuous functions map compact sets to compact sets.

pf. Given cont. function $f: K \subset \mathbb{R}^n \to \mathbb{R}^m$, K is compact.

Given any open cover $\{U_\alpha\}_{\alpha \in I}$ for $f(K)$.

$\because$ f is continuous.

$\therefore$ $f^{-1}(U_\alpha)$ is open $\forall \alpha \in I$.

$\Rightarrow$ $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for K.

$\because$ K is compact.

$\therefore$ $\exists$ finite subcover $\{f^{-1}(U_1), \dots, f^{-1}(U_N)\}$.

i.e. $K \subset \bigcup_{i=1}^{N} f^{-1}(U_i)$.

$\Rightarrow$ $f(K) \subset \bigcup_{i=1}^{N} U_i$, $\{U_1, \dots, U_N\}$ is a finite subcover for $f(K)$. $\therefore$ $f(K)$ is compact.

QED.

Def. We say $f : A \to f(A)$ is a homeomorphism if it is cont. 1-1, onto, and $f^{-1} : f(A) \to A$ is also continuous.

"1-1 & onto" $\equiv$ "bijection".

"1-1" $\equiv$ "injection".

"onto" $\equiv$ "surjection".

bijection = injection + surjection.

Theorem 3. Continuous bijection on a compact set is a homeomorphism.

pf. Suppose $f : K \to f(K)$ is continuous, K is compact.

We only need to show that f^{-1} is continuous.

Given closed set $V \subset K \Rightarrow V$ is compact

$\Rightarrow f(V)$ is compact. $\Rightarrow f(V)$ is closed.

$\Rightarrow (f^{-1})^{-1} = f$ maps closed set V to closed set. $\therefore f^{-1}$ is cont.

QED.

Theorem 4. (Extreme Value Theorem)

Suppose $K \subset \mathbb{R}^n$ is compact. $f : K \to \mathbb{R}$ is continuous. Then $\inf_K f$, $\sup_K f$ are finite and $\exists\ \underline{x}, \bar{x} \in K$ s.t.

$$f(\underline{x}) = \inf_K f, \quad f(\bar{x}) = \sup_K f$$

pf. $\because$ K: compact. f is cont.

$\therefore$ $f(K)$ is a compact.

$\therefore$ $f(K)$ is closed and bounded, by Heine-Borel theorem.

$\Rightarrow$ $\inf_K f$, $\sup_K f$ are finite

Let $\sup_K f = M$. $\Rightarrow \exists$ seq. $\{x_k\}$ in K s.t.

$$M - \frac{1}{k} < f(x_k) \leq M.$$

$\Rightarrow$ $f(x_k)$ conv. to M as $k \to \infty$

$\because$ K is seq. compact. (note : compact $\equiv$ sequentially compact)

$\therefore$ $\{x_k\}$ has conv. subseq. $\{x_{k_j}\}_{j=1}^{\infty}$ w/ limit $\bar{x} \in K$.

and $f(x_{k_j}) \to M$ as $j \to \infty$.

$\because$ f is continuous,

$\therefore$ $f(x_{k_j}) \to f(\bar{x})$ as $j \to \infty$.

$\Rightarrow$ $f(\bar{x}) = M = \sup_K f$, by uniqueness of limit.

The proof for existence of $\underline{x} \in K$ s.t. $f(\underline{x}) = \inf_K f$ is similar.

QED.

Def. We say $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x, y \in A$, $\|x-y\| < \delta$ imply $\|f(x) - f(y)\| < \varepsilon$.

Theorem 5. Continuous functions on compact sets are uniformly continuous.

pf. Let $f: K \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and K is compact.

Given $\varepsilon > 0$, $x \in K$, $\exists \delta(x) > 0$ s.t.

$\|x-y\| < \delta(x)$, $y \in K$ imply $\|f(x) - f(y)\| < \frac{\varepsilon}{2}$.

$\{ B_{\frac{\delta(x)}{2}}(x) : x \in K \}$ is an open cover for K.

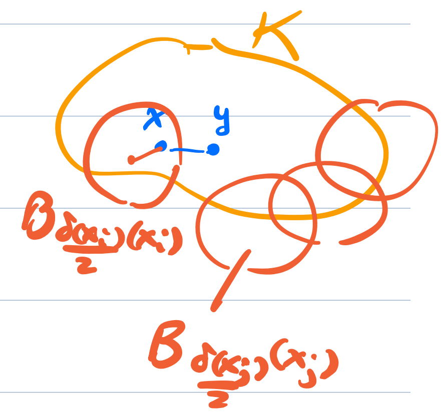
$\Rightarrow \exists$ finite subcover $\{ B_{\frac{\delta(x_1)}{2}}(x_1), \cdots, B_{\frac{\delta(x_N)}{2}}(x_N) \}$ for K.

Let $\delta = \min\left\{ \frac{\delta(x_1)}{2}, \dots, \frac{\delta(x_N)}{2} \right\}$.

If $\|x-y\| < \delta$, $x, y \in K$, then

$\exists\, x_i$ s.t. $\|x - x_i\| < \frac{\delta(x_i)}{2} < \delta(x_i)$

and
$$\|y - x_i\| \le \|y - x\| + \|x - x_i\| < \delta + \frac{\delta(x_i)}{2} \le \delta(x_i)$$



$$\therefore \|f(x) - f(y)\| \le \|f(x) - f(x_i)\| + \|f(x_i) - f(y)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ f is uniformly continuous on K.

QED.