Introduction to Celestial Mechanics
Homework 1, due 09/03/20

1. Draw the phase diagram for the equation of an ideal pendulum

$$\ddot{x} + \sin x = 0.$$

2. Let $S(E)$ be the area enclosed by the closed phase curve corresponding to the energy level E. Show that the period T of motion along this curve is equal to

$$T = \frac{dS}{dE}.$$

3. Show that the phase curves of

$$\begin{aligned} \ddot{x}_1 + x_1 &= 0, \\ \ddot{x}_2 + x_2 &= 0 \end{aligned}$$

are great circles of three-dimensional spheres which are level sets of the total energy.

4. Let π_{E_0} be a level set given in the previous problem. Show that the set of phase curves on this surface forms a two-dimensional sphere. The Hopf map

$$w = \frac{x_1 + iy_1}{x_2 + iy_2}$$

maps π_{E_0} to this two-dimensional sphere.

5. Consider the Lissajous curves given by

$$\begin{aligned} \ddot{x}_1 + x_1 &= 0, \\ \ddot{x}_2 + \omega^2 x_2 &= 0. \end{aligned}$$

Show that the curves are closed if ω is rational; and they dense in a rectangle if ω is irrational.

Introduction to Celestial Mechanics
Homework 2, due 09/04/17

1. Suppose that the potential energy of a central field is a homogeneous function of degree ν; i.e. $U(\alpha x) = \alpha^\nu U(x)$ for any $\alpha > 0$. Show that if a curve γ is the orbit of a motion, then the homothetic curve $\alpha\gamma$ is also an orbit. Determine the ratio of the circulation times along these orbits. Apply this result to the Kepler problem ($\nu = -1$) and harmonic oscillators ($\nu = 2$).

2. Read A.Albouy's lecture note on the two-body problem:
http://www.math.nthu.edu.tw/~kchen/teaching/NewtonBernoulli.pdf
Use the *eccentric anomaly* defined in (16) to give a short proof for Kepler's third law of planetary motion.

3. Consider the circular restricted three-body problem with primary masses $m_1 = 1-\mu$, $m_2 = \mu$. Show that collinear libration points L_1, L_2, L_3 are linearly unstable.

4. Given n positive masses $m_1, \cdots, m_n$. Let M be the total mass. Suppose that the mass center of a path $x = (x_1, \cdots, x_n)$ in $(\mathbb{R}^d)^n$ is fixed at the origin. Show that the moment of inertia $\mathcal{I}(x) = \sum_{k=1}^n m_k |x_k|^2$ and the kinetic energy $K(\dot{x})$ can be written

$$\mathcal{I}(x) = \frac{1}{M}\sum_{i<j} m_i m_j |x_i - x_j|^2, \qquad K(\dot{x}) = \frac{1}{2M}\sum_{i<j} m_i m_j |\dot{x}_i - \dot{x}_j|^2 .$$

5. Let $q_k \in \mathbb{R}^2 \cong \mathbb{C}$ be the position of mass m_k, $k = 1, \ldots, n$. Introduce the rotating coordinate system $(x_k, y_k) = z_k = e^{-i\omega t} q_k$. Show that the equations of motion in terms of z_k for the n-body problem are

$$\begin{aligned} m_k \ddot{x}_k &= 2\omega m_k \dot{y}_k + \frac{\partial}{\partial x_k} V, \\ m_k \ddot{y}_k &= -2\omega m_k \dot{x}_k + \frac{\partial}{\partial y_k} V, \end{aligned}$$

where

$$V(x,y) = V(z) = \sum_{i<j} \frac{m_i m_j}{|z_i - z_j|} + \frac{\omega^2}{2}\sum_{k=1}^{N} m_k |z_k|^2 .$$