4.2. Use Hölder's inequality to prove Minkowski's integral inequality.

4.3. Verify that kernels $\{K_\delta\}$ in the proof of Theorem 4.5 satisfies (1), (2), (3) stated at the beginning of this section.

4.4. Suppose $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, then $f * g \in C^0(\mathbb{R}^n)$.

4.5. Show that $f(x) = e^{-\frac{1}{x^2} - x^2}\chi_{(0,\infty)}$ belongs to $\mathcal{S}(\mathbb{R})$, and $g(x) = f(x-a)f(b-x) \in C_0^\infty(\mathbb{R})$, where $a < b$ are fixed.

4.6. Given bounded open sets $G_1 \subset G_2$ such that $\overline{G}_1 \subset G_2$. Construct a function $f \in C_0^\infty$ such that $f = 1$ on G_1 and $f = 0$ on G_2^c.

5. Linear Transformations

This section is brief introduction to the concepts of bounded linear operators and dual spaces. Those who unacquainted with undefined concepts here are referred to any standard textbook on linear algebra. Those who discontented with brevity of discussions herein are referred to any standard textbook on functional analysis. For concepts intimately related to our discussions for L^p spaces, we prove them here.

In this section we consider only normed spaces, even though many concepts can be extended to more general topological vector spaces. For convenience, we shall use the same notation $\|\cdot\|$ for norms of various spaces, and we shall simply write X for $(X, \|\cdot\|)$, for instance. It is often evident which norm we are referring to. When it is necessary to avoid confusion, we denote its norm by $\|\cdot\|_X$.

Definition 5.1. Given two normed spaces X and Y, a linear transformation (operator) $T : X \to Y$ is said to be *bounded* if there exists some $M > 0$ such that

$$\|Tx\| \le M\|x\| \quad \text{for any } x \in X.$$

Denote the space of bounded linear operators from X to Y by $B(X, Y)$.

Theorem 5.1. *A linear operator $T : X \to Y$ is bounded if and only if it is continuous.*

Proof. Clearly any bounded linear operator is Lipschitz continuous. If T is continuous, then $T^{-1}(B_1(0)) \supset B_\delta(0)$ for some $\delta > 0$. Then, whenever $\|x\| \le 1$, we have

$$\|Tx\| = \frac{2}{\delta}\left\|T\left(\frac{\delta}{2}x\right)\right\| \le \frac{2}{\delta}.$$

For general x, we have $\|Tx\| = \|T\big(\frac{x}{\|x\|}\big)\|\,\|x\| \le \frac{2}{\delta}\|x\|$. Thus T is bounded. □

It is a simple exercise to show that

$$\sup_{\|x\|=1}\|Tx\| = \sup_{x \ne 0}\frac{\|Tx\|}{\|x\|} = \sup_{\|x\|\le 1}\|Tx\| = \inf\{M : \|Tx\| \le M\|x\| \ \forall x \in X\}.$$

Their common value is denoted by $\|T\|$. This notation is justified in the following theorem.

Theorem 5.2. *$B(X, Y)$ with the function $T \mapsto \|T\|$ is a normed space. If Y is a Banach space, then so is $B(X, Y)$.*

PROOF. We only verify the triangle inequality for $\|\cdot\|$, for the other two axioms of norm are obvious here. Given S, $T \in B(X,Y)$, and $x \in V$.

$$\|(S+T)x\| = \|Sx + Tx\| \leq \|Sx\| + \|Tx\| \leq \|S\|\,\|x\| + \|T\|\,\|x\|.$$

This implies the triangle inequality $\|S+T\| \leq \|S\| + \|T\|$.

Given a Cauchy sequence $\{T_n\}$ in $B(X,Y)$. For any $x \in V$, the sequence $\{T_n x\}$ is a Cauchy sequence in Y, and so it converges. Let $T : X \to Y$ be defined by $Tx = \lim_{n\to\infty} T_n x$. Clearly T is linear. Given $\varepsilon > 0$, choose N such that $\|T_n - T_m\| < \varepsilon$ for all $n, m \geq N$. Fix $n \geq N$ and $x \in X$,

$$\|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| \leq \varepsilon \|x\|.$$

Therefore,

$$\|Tx\| \leq \|Tx - T_N x\| + \|T_N x\| \leq (\varepsilon + \|T_N\|)\|x\|.$$

This shows that T is bounded. Furthermore,

$$\|T_n - T\| = \sup_{\|x\|=1} \|T_n x - Tx\| \leq \varepsilon \quad \text{for any } n \geq N.$$

This implies that $\|T_n - T\| \to 0$ as $n \to \infty$, since $\varepsilon > 0$ is arbitrary. □

A simple fact worth mentioning is: any composition of bounded linear operators is a bounded linear operator. Indeed, if $S \in B(X,Y)$, $T \in B(Y,Z)$, then $TS \in B(X,Z)$ and $\|TS\| \leq \|T\|\,\|S\|$ since

$$\|TSx\| \leq \|T\|\,\|Sx\| \leq \|T\|\,\|S\|\,\|x\| \quad \text{for any } x \in X.$$

EXAMPLE 5.1. Given $g \in L^1(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Define the convolution operator $G : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ by $G(f) = f * g$. Lemma 4.1 tells us that G is a bounded linear operator and $\|G\| \leq \|g\|_1$. More generally, if $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $g \in L^q(\mathbb{R}^n)$, then Young's convolution inequality (exercise 4.1) tells us that $G : L^p(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ is a bounded linear operator and $\|G\| \leq \|g\|_q$.

DEFINITION 5.2. A *linear functional* f on a normed space X over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) is a linear transformation from X to $\mathbb{F}$. The space $B(X, \mathbb{F})$ of bounded linear functionals on X is called the *dual space* of X. It is usually denoted by X^*.

COROLLARY 5.3. *The dual space X^* of any normed space X over $\mathbb{R}$ or $\mathbb{C}$ is a Banach space.*

DEFINITION 5.3. Given $T \in B(X,Y)$. The *adjoint* of T, denoted by T^*, is a linear operator from Y^* to X^* defined by

$$(T^* y^*)(x) = y^*(Tx),$$

where $y^* \in Y^*$, $x \in X$. In other words, $T^* y^* = y \circ T$.

It is straightforward to verify that T^* is linear. $T^* y^*$ is indeed a bounded linear functional on X since it is simply composition of y^* and T. Moreover, $T^* \in B(Y^*, X^*)$ since

$$\|T^* y^*\| = \|y^* \circ T\| = \sup_{\|x\|=1} \|y^*(Tx)\| \leq \sup_{\|x\|=1} \|y^*\|\,\|Tx\| \leq \|T\|\|y^*\| \quad \text{for any } y^* \in Y^*.$$

A frequently used notation for $x^*(x)$ is $\langle x^*, x\rangle$, where $x^* \in X^*$ and $x \in X$.

REMARK 5.1. If $S \in B(X,Y)$, $T \in B(Y,Z)$, then $(TS)^* = S^* T^*$. This follows trivially from the definition of adjoint:

$$(TS)^* z^* = z^* \circ (TS) = (z^* \circ T) \circ S = S^*(T^* z^*) = (S^* T^*) z^* \quad \text{for any } z^* \in Z^*.$$

Definition 5.4. Two normed spaces X, Y are said to be *isomorphic* if there exists a bijective $T \in B(X,Y)$ with inverse $T^{-1} \in B(Y,X)$. Such an operator T is called an *isomorphism.* We say X, Y are *isometrically isomorphic* if there exists an isomorphism $T : X \to Y$ which is also an isometry.

Two normed spaces are considered equivalent if they are isometrically isomorphic. This clearly defines an equivalence relation on the class of Banach spaces.

Now we state some facts without proof. They can be served as incentives to consider isometric isomorphisms and separable Banach spaces.

Example 5.2. Given any normed space X, there exists a canonical isometric isomorphism τ from X to a subspace of X^{**}. It is defined by $\tau(x)(x^*) = x^*(x)$. The mapping τ is often called the *canonical embedding* from X to X^{**}. For convenience, the term $\tau(x)$ is often written x, and thus we regard elements in X as elements in X^{**}.

Definition 5.5. We say a normed space X is *reflexive* if the canonical embedding $\tau : X \to X^{**}$ is an isomorphism.

Note that, by Corollary 5.3, any reflexive normed space is a Banach space.

Example 5.3. Every separable Banach is isometrically isomorphic to a closed subspace of $C^0[0,1]$. This is known as the *Banach-Mazur theorem.* Note that the space $\mathbb{Q}[t]$ of polynomials with rational coefficients is a countable dense subset of $C^0[0,1]$ (by Weierstrass approximation theorem, or Stone-Weierstrass theorem). The Banach-Mazur theorem tells us that $C^0[0,1]$ is the "largest" separable Banach space.

Exercises.

5.1. Verify identities above Theorem 5.2.

5.2. Given $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $K \in L^q(\mathbb{R}^n \times \mathbb{R}^n)$. Consider the integral operator T defined by

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad f \in L^p(\mathbb{R}^n).$$

(a) Given $f \in L^p(\mathbb{R}^n)$. Show that for almost every x, the function $K(x,y)f(y)$ is integrable with respect to y.
(b) Show that T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, and $\|T\| \le \|K\|_q$.

5.3. Assume the following axiom[1]:

For any linear subspace X_0 of X, $\ell \in X_0^*$, there exists $\tilde{\ell} \in X^*$ such that $\tilde{\ell}\,|_{X_0} = \ell$ and $\|\tilde{\ell}\| = \|\ell\|$.

Now prove the following statements under this assumption.

(a) For any $x \in X$, $\|x\| = \sup\{|x^*(x)| : x^* \in X^*,\ \|x^*\| \le 1\}$.
(*Hint*: Consider a function which sends αx to $\alpha\|x\|$, where α is a scalar.)
(b) For any $T \in B(X,Y)$, $\|T\| = \sup\{|y^*(Tx)| : y^* \in Y^*,\ \|y^*\| \le 1,\ x \in X,\ \|x\| \le 1\}$.
(c) $\|T\| = \|T^*\|$ for any $T \in B(X,Y)$.
(d) The canonical embedding $\tau : X \to X^{**}$ in Example 5.2 is an isometric isomorphism from X to $\tau(X)$.

[1]It is a special case of a more general statement known as the *Hahn-Banach theorem*, which is a consequence of the axiom of choice.

6. Dual Space of L^p

In this section we characterize the dual space of $L^p(I)$, where I is an interval. A more general theorem will be proved after general measures and integrals were introduced.

The key ingredients of the proof are the following two lemmas. The first one says that, roughly speaking, any bounded linear functional on $L^p(I)$ induces a canonical indefinite integral. Generalization of this lemma to higher dimensional spaces requires further knowledge about indefinite integrals.

LEMMA 6.1. *Let I be a bounded interval, $\bar{I} = [a, b]$, $1 \le p < \infty$. For any $G \in L^p(I)^*$, there exists $g \in L^1(I)$ such that $G(\chi_A) = \int_A g$ for any measurable set $A \subset I$.*

PROOF. Without loss of generality, assume $I = [a, b]$. Let $\mathcal{B}$ be the σ-algebra of measurable sets in I. First note that $G(\chi_A) = G(\chi_B)$ if $A\Delta B$ has measure zero. Let $\phi(s) = G(\chi_{[a,s]})$, $s \in I$. Then for any subinterval $[s, t] \subset I$,

$$\phi(t) - \phi(s) = G(\chi_{[a,t]}) - G(\chi_{[a,s]}) = G(\chi_{[a,t]} - \chi_{[a,s]}) = G(\chi_{[s,t]}).$$

Given nonoverlapping subintervals $\{[a_i, b_i] : i = 1, \cdots, n\}$ of I with total length δ.

$$\begin{aligned}
\sum_{i=1}^n |\phi(b_i) - \phi(a_i)| &= \sum_{i=1}^n (\phi(b_i) - \phi(a_i))\,\mathrm{sgn}\,(\phi(b_i) - \phi(a_i)) \\
&= \sum_{i=1}^n G(\chi_{[a_i,b_i]})\,\mathrm{sgn}\,(\phi(b_i) - \phi(a_i)) \\
&= G\left(\sum_{i=1}^n \mathrm{sgn}\,(\phi(b_i) - \phi(a_i))\,\chi_{[a_i,b_i]}\right) \\
&\le \|G\| \left\|\sum_{i=1}^n \mathrm{sgn}\,(\phi(b_i) - \phi(a_i))\,\chi_{[a_i,b_i]}\right\|_p \le \|G\|\delta^{1/p}.
\end{aligned}$$

This implies that $\phi \in AC[a, b]$. By the fundamental theorem of calculus, $g = \phi' \in L^1(I)$ and

$$G(\chi_{[s,t]}) = \phi(t) - \phi(s) = \int_s^t g(u)\,du \quad \text{for any subinterval } [s, t] \subset I.$$

It follows that $G(\chi_A) = \int_A g$ for any $A \in \mathcal{B}$ (exercise 6.1). $\square$

LEMMA 6.2. *Let $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $E \subset \mathbb{R}$ be measurable. Suppose $g \in L^1(E)$ and for some $M > 0$,*

$$\left|\int_E \varphi g\right| \le M\|\varphi\|_p$$

for any simple function φ. Then $g \in L^q(E)$ and $\|g\|_q \le M$.

PROOF. Assume $1 < p < \infty$. Let ψ_n be a sequence of nonnegative simple functions such that $\psi_n \nearrow |g|^q$. Let $\varphi_n = \psi_n^{\frac{1}{p}}\,\mathrm{sgn}(g)$. Then φ_n is a simple function and

$$\varphi_n g = \psi_n^{\frac{1}{p}}|g| \ge |\psi_n|^{\frac{1}{p}+\frac{1}{q}} = |\psi_n|.$$

Then,

$$\int_E \psi_n \le \int_E \varphi_n g \le M\|\varphi_n\|_p = M\left(\int_E \psi_n\right)^{\frac{1}{p}},$$

implying that $\int_E \psi_n \le M^q$. By the monotone convergence theorem, $\int_E |g|^q \le M^q$. The case $p = 1$ is left to the reader (exercise 6.2). $\square$

THEOREM 6.3. (RIESZ REPRESENTATION THEOREM FOR $L^p(I)$, $I \subset \mathbb{R}$)
Suppose $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $I \subset \mathbb{R}$ *is an interval. For any* $G \in L^p(I)^*$, *there exists unique* $g \in L^q(I)$ *such that*

$$G(f) = \int_E fg \quad \textit{for any } f \in L^p(I).$$

Moreover, $\|G\| = \|g\|_q$. *The map* $G \mapsto g$ *is an isometric isomorphism from* $L^p(I)^*$ *to* $L^q(I)$.

PROOF. Denote the σ-algebra of measurable sets in I by $\mathcal{B}$. Consider the case $\bar{I} = [a, b]$. By Lemma 6.1, there exists $g \in L^1(I)$ such that $G(\chi_A) = \int_A g$ for any $A \in \mathcal{B}$.

Given any simple function $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$, where $\{E_i\} \subset \mathcal{B}$ are disjoint.

$$G(\varphi) = \sum_{i=1}^n a_i G(\chi_{E_i}) = \sum_{i=1}^n a_i \int_{E_i} g = \sum_{i=1}^n a_i \int_E \chi_{E_i} g = \int_I \varphi g.$$

Therefore,

$$\left|\int_I \varphi g\right| = |G(\varphi)| \le \|G\|\,\|\varphi\|_p.$$

By Lemma 6.2, $g \in L^q(I)$ and $\|g\|_q \le \|G\|$. Consider the linear functional $\tilde{G}$ on $L^p(I)$ defined by $\tilde{G}(f) = \int_I fg$. It is clearly continuous, by Hölder's inequality. Since simple functions are dense in $L^p(I)$, we see that $G = \tilde{G}$. Hölder's inequality also tells us that $\|G\| \le \|g\|_q$, and so $\|g\|_q = \|G\|$. The function $g \in L^q(I)$ is unique, for if $G(f) = \int_E f\tilde{g}$ for all $f \in L^p(I)$, we would have

$$\int_I f(g - \tilde{g}) = 0 \quad \text{for all } f \in L^p(I).$$

Then $g - \tilde{g}$ gives the zero functional on $L^p(I)$, so that $\|g - \tilde{g}\|_q = 0$. That is, $g = \tilde{g}$ in $L^q(I)$.

Now we consider unbounded interval I. Let $\{I_n\}$ be an increasing sequence of bounded intervals such that $\bigcup_{n=1}^\infty I_n = I$. Then for any n, there exists unique $g_n \in L^q(I)$ such that $g_n = 0$ on I_n^c and

$$G(f) = \int_I f g_n \quad \text{for any } f \in L^p(I) \text{ with } f = 0 \text{ on } I_n^c.$$

Moreover, $\|g_n\|_q \le \|G\|$. By uniqueness, $g_{n+1} = g_n$ a.e. on I_n for each n. We may assume without loss of generality that $g_{n+1} = g_n$ on I_n. Then the function

$$g(x) := g_n(x), \quad x \in I_n$$

is well-defined on I. Furthermore, $|g_n(x)| \nearrow |g(x)|$ as $n \to \infty$. By the monotone convergence theorem,

$$\int_I |g|^q = \lim_{n\to\infty} \int_I |g_n|^q \le \|G\|^q.$$

In particular, $g \in L^q(I)$, $\|g\|_q \leq \|G\|$. For any $f \in L^p(I)$, let $f_n = f\,\chi_{I_n}$. Then $|f_n| \leq |f|$, $f_n \to f$ pointwise on I as $n \to \infty$. It follows that $\|g_n - g\|_q \to 0$, $\|f_n - f\|_p \to 0$ (see exercise 3.6), and hence

$$\begin{aligned} \|f_n g_n - fg\|_1 &\leq \|(f_n - f)g_n\|_1 + \|f(g_n - g)\|_1 \\ &\leq \|f_n - f\|_p \, \|g_n\|_q + \|f\|_p \, \|g_n - g\|_q \ \to \ 0 \quad \text{as } n \to \infty. \end{aligned}$$

Then, by continuity of G on $L^p(I)$,

$$\int_I fg = \lim_{n\to\infty} \int_I f_n g_n = \lim_{n\to\infty} G(f_n) = G(f).$$

Moreover, $\|G\| \leq \|g\|_q$, and so $\|g|_q = \|G\|$. Uniqueness of such g follows as in the case of bounded I. This completes the proof. $\square$

Corollary 6.4. *If $1 < p < \infty$, then $L^p(I)$ is reflexive.*

This is an immediate consequence of the Riesz representation theorem for $L^p(I)$.

Exercises.

6.1. Let $I \subset \mathbb{R}$ be a bounded interval. Given $1 \leq p < \infty$, $G \in L^p(I)^*$, $g \in L^1(I)$. Suppose $G(\chi_J) = \int_J g$ on any subinterval $J \subset I$. Show that $G(\chi_A) = \int_A g$ for any measurable set $A \subset I$.

6.2. Prove Lemma 6.2 for the case $p = 1$.

6.3. Let $I \subset \mathbb{R}$ be an interval. Suppose $g \in L^\infty(I)$. Show that for any $\varepsilon > 0$ there exists $f \in L^1(I)$, $\|f\|_1 \neq 0$, such that

$$\int_I fg \geq \|f\|_1(\|g\|_\infty - \varepsilon).$$

6.4. Let $I = [0, 1]$. Consider the linear functional G on $C(I)$ defined by $G(f) = f(1)$. Use this linear functional and the assumption in Exercise 5.3 to show that $L^1(I)$ is not isometrically isomorphic to $L^\infty(I)^*$.

6.5. Determine a representation for $(\ell^p)^*$, $1 \leq p < \infty$.