4. Dense Subspaces of L^p

In the proof of Theorem 3.4 we constructed a countable collection of step functions which is dense in $L^p(E)$. These step functions are linear combinations of characteristic functions on some dyadic cubes. This implies that the space of simple functions is also dense in $L^p(\mathbb{R}^n)$. In this section we prove that the space of smooth functions with compact supports, and the space of functions with rapidly decreasing derivatives are also dense in $L^{(}(\mathbb{R}^n)$.

Let us begin by recalling that $(L^1(\mathbb{R}^n), *)$ is a commutative algebra without identity, but there are "good kernels" $\{K_\delta\}$ which approximate the identity in the sense that, for any $f \in L^1(\mathbb{R}^n)$,

$$f * K_\delta \to f \ \text{ a.e. and } \ \|f * K_\delta - f\|_1 \to 0 \quad \text{as } \delta \searrow 0.$$

For example, kernels satisfying the following conditions are approximations to the identity:

(1) $\int K_\delta = 1$ for any $\delta > 0$.
(2) There exist some $C > 0$ such that $|K_\delta| \le \frac{C}{\delta^n}$ for any $\delta > 0$.
(3) There exist some $C' > 0$ such that $|K_\delta(x)| \le \frac{C'\delta}{|x|^{n+1}}$ for any $\delta > 0$, $x \in \mathbb{R}^n \setminus \{0\}$.

Conditions (2) and (3) imply

(4) There exist some $C > 0$ such that $\int K_\delta = 1$ and $\int |K_\delta| < C$ for any $\delta > 0$.
(5) For any $\eta > 0$, $\int_{|x|\ge\eta} |K_\delta(x)|dx \to 0$ as $\delta \searrow 0$.

Recall also that, for any $f, g \in L^1(\mathbb{R}^n)$, $f * g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1.$$

We may extend these results to $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, based on two auxiliary inequalities:

Lemma 4.1. *Given* $1 \le p \le \infty$.

(a) *If* $f \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, *then* $f * g \in L^p(\mathbb{R}^n)$ *and*

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

(b) (Minkowski's integral inequality)
If $1 < p < \infty$, $f \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$, *then*

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x,y)dx \right|^p dy\right]^{\frac{1}{p}} \le \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x,y)|^p dy\right]^{\frac{1}{p}} dx.$$

Both inequalities are simple applications of Hölder's inequality. The first one is a special case of *Young's convolution inequality* (see exercise 4.1). We leave details and proofs as exercises.

Lemma 4.2. *Given* $1 \le p \le \infty$. *If* $f \in L^p(\mathbb{R}^n)$, $g \in C_0^m(\mathbb{R}^n)$, $0 \le m \le \infty$, *then* $f * g \in C_0^m(\mathbb{R}^n)$ *and* $D^\alpha(f * g) = f * D^\alpha g$ *for any multi-index* α *with* $|\alpha| \le m$.

PROOF. Let $q = \frac{p}{p-1}$ be the conjugate exponent of p. Consider $m = 0$. Given $h \in \mathbb{R}^n$,

$$\begin{aligned}|(f * g)(x+h) - (f * g)(x)| &= \left|\int_{\mathbb{R}^n} f(x+h-y)g(y)dy - \int_{\mathbb{R}^n} f(x-y)g(y)dy\right| \\ &= \left|\int_{\mathbb{R}^n} f(x-y)g(y+h)dy - \int_{\mathbb{R}^n} f(x-y)g(y)dy\right| \\ &= \left|\int_{\mathbb{R}^n} f(x-y)[g(y+h) - g(y)]dy\right| \\ &\leq \|f\|_p \|\tau_h g - g\|_q \quad \text{(by Hölder's inequality).}\end{aligned}$$

By Theorem 3.5, the last term converges to zero as $h \to 0$. Note that when $p = 1$, the term $\|\tau_h g - g\|_q$ converges to zero as $h \to 0$ since g is uniformly continuous. This proves $f * g \in C^0(\mathbb{R}^n)$.

Consider $m = 1$. Given $t > 0$. By the mean-value theorem, there exist $s \in [0, t]$ such that

$$\begin{aligned}(f * g)(x + te_i) - (f * g)(x) &= \int_{\mathbb{R}^n} f(y)\left[g(x + te_i - y) - g(x - y)\right] dy \\ &= \int_{\mathbb{R}^n} f(y)\frac{\partial}{\partial x_i} g(x + se_i - y)tdy \\ &= f * \frac{\partial}{\partial x_i} g(x + se_i)t.\end{aligned}$$

Since $\frac{\partial}{\partial x_i} g \in C^0_0(\mathbb{R}^n)$, $f * \frac{\partial}{\partial x_i} g \in C^0(\mathbb{R}^n)$, we see that $\frac{\partial}{\partial x_i}(f * g)$ exists and equals $f * \frac{\partial}{\partial x_i} g$. This implies $f * g \in C^1(\mathbb{R}^n)$ since i is arbitrary. The proof for general m follows by induction. □

THEOREM 4.3. *Given $1 \leq p < \infty$. Let $\{K_\delta\}$ be kernels satisfying (1),(2),(3). Then for any $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$,*

$$\|f * K_\delta - f\|_p \to 0 \quad \text{as} \quad \delta \searrow 0.$$

PROOF. Observe that

$$|f * K_\delta(x) - f(x)| \leq \int_{\mathbb{R}^n} |f(x-y) - f(x)||K_\delta(y)|dy.$$

By Minkowski's integral inequality,

$$\begin{aligned}\|f * K_\delta - f\|_p &\leq \left[\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(x-y) - f(x)||K_\delta(y)|dy\right]^p dx\right]^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p |K_\delta(y)|^p dx\right]^{\frac{1}{p}} dy \\ &= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p dx\right]^{\frac{1}{p}} |K_\delta(y)|dy \\ &= \int_{\mathbb{R}^n} \|\tau_{-y} f - f\|_p |K_\delta(y)|dy.\end{aligned}$$

Given $\varepsilon > 0$, choose $\eta > 0$ such that $\|\tau_{-y} f - f\| < \varepsilon$ whenever $|y| < \eta$. Then

$$\|f * K_\delta - f\|_p \leq \varepsilon \int_{|y|<\eta} |K_\delta(y)|dy + \int_{|y|\geq\eta} 2\|f\|_p |K_\delta(y)|dy.$$

The theorem follows by letting $\delta \to 0$ and then $\varepsilon \to 0$. □

The *support* $\mathrm{supp}(f)$ of a measurable function f is defined as the intersection of set in

$$\mathcal{K}_f = \{K \subset \mathbb{R}^n : K \text{ is closed and } f = 0 \text{ a.e. on } K^c\}.$$

LEMMA 4.4. *Given measurable functions f and g. We have $f * g = 0$ on $(\mathrm{supp}(f) + \mathrm{supp}(g))^c$. In particular, $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$.*

PROOF. Observe that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathrm{supp}(g)} f(x-y)g(y)dy = 0$$

if $x - y \notin \mathrm{supp}(f)$ for every $y \in \mathrm{supp}(g)$. That is, $f * g = 0$ on $(\mathrm{supp}(f) + \mathrm{supp}(g))^c$. □

THEOREM 4.5. *$C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.*

PROOF. Given $f \in L^p(\mathbb{R}^n)$. For any $N \in \mathbb{N}$, let

$$f_N = \begin{cases} f(x) & \text{if } |f(x)|, \|x\| < N \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f_N converges almost everywhere to f as $N \to \infty$, and $|f(x) - f_N(x)|^p \le 2^p |f(x)|^p$. By Lebesgue dominated convergence theorem, $\|f - f_N\|_p \to 0$ as $N \to \infty$. From this observation, it suffices to consider the case of bounded f with compact support.

Choose a nonnegative function $K \in C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} K = 1$. Let

$$K_\delta(x) = \frac{1}{\delta^n} K\left(\frac{x_1}{\delta}, \cdots, \frac{x_n}{\delta}\right) \quad \text{for } \delta > 0.$$

Then the family $\{K_\delta\}$ satisfies conditions (1), (2), (3) stated at the beginning of this section (check it!). By Lemma 4.2 and Lemma 4.4, $f * K_\delta \in C_0^\infty(\mathbb{R}^n)$. By Theorem 4.3, $\|f * K_\delta - f\|_p \to 0$ as $\delta \searrow 0$. This completes the proof. □

DEFINITION 4.1. The *Schwartz class* $\mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \sup_{x \in \mathbf{R}^n} |x^\alpha D^\beta f(x)| < \infty \quad \text{for any multi-indices } \alpha, \beta\}.$$

Roughly speaking, the Schwartz class consists of smooth functions whose derivatives decrease to zero faster than the inverse of any polynomial. This function space is of special importance in Fourier analysis and distribution theory.

COROLLARY 4.6. *For any $1 \le p < \infty$, the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$.*

PROOF. It follows easily from Theorem 4.5, $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, and the observation that $\mathcal{S}(\mathbb{R}^n)$ is a subspace of $L^p(\mathbb{R}^n)$. □

Exercises.

4.1. Prove the Lemma 4.1(a). More generally, given $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and given $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, prove the following *Young's convolution inequality*:

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Hint: Find suitable p', q' such that $\frac{1}{p'} + \frac{1}{q} + \frac{1}{r} = 1$, then apply generalized Hölder's inequality.

4.2. Use Hölder's inequality to prove Minkowski's integral inequality.

4.3. Verify that kernels $\{K_\delta\}$ in the proof of Theorem 4.5 satisfies (1), (2), (3) stated at the beginning of this section.

4.4. Suppose $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, then $f * g \in C^0(\mathbb{R}^n)$.

4.5. Show that $f(x) = e^{-\frac{1}{x^2}}\big(1 - e^{-\frac{1}{x^2}}\big)\chi_{(0,\infty)}$ belongs to $\mathcal{S}(\mathbb{R})$, and $g(x) = f(x-a)f(b-x) \in C_0^\infty(\mathbb{R})$, where $a < b$ are fixed.

4.6. Given bounded open sets $G_1 \subset G_2$ such that $\overline{G}_1 \subset G_2$. Construct a function $f \in C_0^\infty$ such that $f = 1$ on G_1 and $f = 0$ on G_2^c.

5. Linear Transformations

This section is brief introduction to the concepts of bounded linear operators and dual spaces. Those who unacquainted with undefined concepts here are referred to any standard textbook on linear algebra. Those who discontented with brevity of discussions herein are referred to any standard textbook on functional analysis. For concepts intimately related to our discussions for L^p spaces, we prove them here.

In this section we consider only normed spaces, even though many concepts can be extended to more general topological vector spaces. For convenience, we shall use the same notation $\|\cdot\|$ for norms of various spaces, and we shall simply write X for $(X, \|\cdot\|)$, for instance. It is often evident which norm we are referring to. When it is necessary to avoid confusion, we denote its norm by $\|\cdot\|_X$.

Definition 5.1. Given two normed spaces X and Y, a linear transformation (operator) $T : X \to Y$ is said to be *bounded* if there exists some $M > 0$ such that

$$\|Tx\| \;\le\; M\|x\| \quad \text{for any } x \in X.$$

Denote the space of bounded linear operators from X to Y by $B(X, Y)$.

Theorem 5.1. *A linear operator $T : X \to Y$ is bounded if and only if it is continuous.*

Proof. Clearly any bounded linear operator is Lipschitz continuous. If T is continuous, then $T^{-1}(B_1(0)) \supset B_\delta(0)$ for some $\delta > 0$. Then, whenever $\|x\| \le 1$, we have

$$\|Tx\| = \frac{2}{\delta}\left\|T\left(\frac{\delta}{2}x\right)\right\| \le \frac{2}{\delta}.$$

For general x, we have $\|Tx\| = \|T\big(\frac{x}{\|x\|}\big)\|\,\|x\| \le \frac{2}{\delta}\|x\|$. Thus T is bounded. □

It is a simple exercise to show that

$$\sup_{\|x\|=1}\|Tx\| \;=\; \sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} \;=\; \sup_{\|x\|\le 1}\|Tx\| \;=\; \inf\{M : \|Tx\| \le M\|x\| \;\forall x \in X\}.$$

Their common value is denoted by $\|T\|$. This notation is justified in the following theorem.

Theorem 5.2. *$B(X, Y)$ with the function $T \mapsto \|T\|$ is a normed space. If Y is a Banach space, then so is $B(X, Y)$.*