3. The L^p Space

In this section we consider a space $L^p(E)$ which resembles ℓ^p on many aspects. After general concepts of measure and integral were introduced, we will see that these two spaces can be viewed as special cases of a more general L^p space.

DEFINITION 3.1. Given a measurable set $E \subset \mathbb{R}^n$. For $0 < p < \infty$, define the space $L^p(E)$ and the real-valued function $\|\cdot\|_p$ on $L^p(E)$ by

$$L^p(E) = \{f : f \text{ is measurable on } E \text{ and } \int_E |f|^p < \infty\}, \qquad \|f\|_p = \left(\int_E |f|^p\right)^{\frac{1}{p}}.$$

The *essential supremum* of a measurable function f on E is defined by

$$\operatorname*{ess\,sup}_E f = \inf\{\alpha \in (-\infty, \infty] : m(\{f > \alpha\}) = 0\}.$$

The space $L^\infty(E)$ and the real-valued function $\|\cdot\|_\infty$ on $L^\infty(E)$ are given by

$$L^\infty(E) = \{f : f \text{ is measurable on } E \text{ and } \operatorname*{ess\,sup}_E |f| < \infty\}, \qquad \|f\|_\infty = \operatorname{ess\,sup} |f|.$$

Functions in $L^\infty(E)$ are said to be *essentially bounded*.

The measurable function f in the definition of $L^p(E)$ for $0 < p < \infty$ can be complex-valued, but functions in $L^\infty(E)$ are assumed to be real-valued. We leave it to the readers to check that $m(f > \operatorname{ess\,sup}_E f) = 0$ for any $f \in L^\infty(E)$ (Exercise 3.1). In other words, $f \leq \operatorname{ess\,sup}_E f$ and $|f| \leq \|f\|_\infty$ almost everywhere.

For any $0 < p \leq \infty$, two functions f_1, f_2 in $L^p(E)$ are considered equivalent if $f_1 = f_2$ almost everywhere on E. The space of equivalence classes, still denoted by $L^p(E)$, are called *$L^p(E)$ classes* or *$L^p(E)$ spaces*.

Similar to ℓ^p, the space $L^p(E)$ is a vector space for any $0 < p \leq \infty$. Indeed, $\|\alpha f\|_p = |\alpha|\|f\|_p$ for any scalar α, $\|\alpha f\|_p = |\alpha|\|f\|_p$ and

$$\begin{aligned} f, g \in L^p(E) \quad &\Rightarrow \quad |f+g|^p \leq (2\max\{|f|, |g|\})^p \leq 2^p(|f|^p + |g|^p), \\ f, g \in L^\infty(E) \quad &\Rightarrow \quad \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty. \end{aligned}$$

The second line follows by observing that

$$\left.\begin{array}{l} |f| \leq \|f\|_\infty \text{ almost everywhere} \\ |g| \leq \|g\|_\infty \text{ almost everywhere} \end{array}\right\} \Rightarrow |f+g| \leq \|f\|_\infty + \|g\|_\infty \text{ almost everywhere.}$$

When $1 \leq p \leq \infty$, the function $\|\cdot\|_p$ is a norm on $L^p(E)$. This follows from the theorem below, the proof for which is similar to that of ℓ^p.

THEOREM 3.1. *Given $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be measurable functions on $E \subset \mathbb{R}^n$.*

(a) (HÖLDER'S INEQUALITY FOR L^p) *If $f \in L^p(E)$, $g \in L^q(E)$, then $fg \in L^1(E)$ and*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

(b) (MINKOWSKI'S INEQUALITY FOR L^p)

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

PROOF. (a) The cases $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$ are obvious. Consider $1 < p, q < \infty$. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $fg = 0$ almost everywhere on E, and the asserted inequality is obvious. We may now assume $0 < \|f\|_p, \|g\|_q < \infty$.

Let $F = \frac{f}{\|f\|_p}$, $G = \frac{g}{\|g\|_q}$. By Young's inequality,

$$\begin{aligned} \int_E |FG| &\le \int_E \frac{|F|^p}{p} + \frac{|G|^q}{q} = \frac{\|F\|_p^p}{p} + \frac{\|G\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1, \\ \|fg\|_1 &= \int_E |fg| = \|f\|_p \|g\|_q \int_E |FG| \le \|f\|_p \|g\|_q. \end{aligned}$$

(b) The case $p = 1$ is obvious, and the case $p = \infty$ has been proved. Now we consider $1 < p < \infty$. Note that $q = \frac{p}{p-1}$. Minkowski's inequality follows easily from (a):

$$\begin{aligned} \|f+g\|_p^p &= \int_E |f+g|^p \\ &\le \int_E |f+g|^{p-1}|f| + \int_E |f+g|^{p-1}|g| \\ &= \left(\int_E |f+g|^p\right)^{\frac{p-1}{p}} \left(\int_E |f|^p\right)^{\frac{1}{p}} + \left(\int_E |f+g|^p\right)^{\frac{p-1}{p}} \left(\int_E |g|^p\right)^{\frac{1}{p}} \\ &= \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p). \end{aligned}$$

□

The special case $p = q = 2$ of the Hölder inequality is also known as the *Cauchy-Schwarz inequality*. The assumption $1 \le p \le \infty$ is necessary. For example, let $E = [0, 1]$, $f = \chi_{[0,\frac{1}{2}]}$, $g = \chi_{[\frac{1}{2},1]}$. Then

$$\|f\|_p + \|g\|_p = \left(\frac{1}{2}\right)^{\frac{1}{p}} + \left(\frac{1}{2}\right)^{\frac{1}{p}} = 2^{1-\frac{1}{p}} < 1 = \|f+g\|_p.$$

COROLLARY 3.2. *Suppose* $0 < p < q < \infty$, $m(E) < \infty$. *Then*

$$\left(\frac{1}{m(E)} \int_E |f|^p\right)^{\frac{1}{p}} \le \left(\frac{1}{m(E)} \int_E |f|^q\right)^{\frac{1}{q}}.$$

In particular, $L^q(E) \subset L^p(E)$.

PROOF. Let $r = \frac{q}{q-p}$, then $\frac{1}{q/p} + \frac{1}{r} = 1$. Therefore,

$$\int_E |f|^p \le \left(\int_E (|f|^p)^{\frac{q}{p}}\right)^{\frac{p}{q}} \left(\int_E 1^r\right)^{\frac{1}{r}} = \left(\int_E |f|^q\right)^{\frac{p}{q}} m(E)^{\frac{q-p}{q}}.$$

Then the corollary follows from

$$\|f\|_p = \left(\int_E |f|^p\right)^{\frac{1}{p}} \le \|f\|_q \, m(E)^{\frac{q-p}{qp}} = \|f\|_q \, m(E)^{\frac{1}{p}-\frac{1}{q}}.$$

□

EXAMPLE 3.1. Consider $f(x) = x^r$, $r \neq 0$, defined on $[0, \infty)$.

When $r < 0$, $f \in L^p[1, \infty)$ if and only if $p > -\dfrac{1}{r}$, $f \in L^p[0, 1)$ if and only if $0 < p < -\dfrac{1}{r}$.

When $r > 0$, $f \notin L^p[1, \infty)$ for any $p > 0$, $f \in L^p[0, 1)$ for any $p > 0$.

This example shows that the assumption $m(E) < \infty$ is necessary in the above corollary, and $L^q(E) \subsetneq L^p(E)$ if $0 < p < q < \infty$ and $E = [1, \infty)$.

EXAMPLE 3.2. The function $\log x$ belongs to $L^p(0, 1)$ for any $0 < p < \infty$ but it is not in $L^\infty(0, 1)$.

THEOREM 3.3. (RIESZ-FISHER) *For any* $1 \leq p \leq \infty$, *the space* $(L^p(E), \|\cdot\|_p)$ *is a Banach space.*

PROOF. Consider $p = \infty$ first. Note that convergence in $L^\infty(E)$ means uniform convergence outside a set of measure zero.

Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(E)$. For each $n, m \in \mathbb{N}$, $|f_n - f_m| \leq \|f_n - f_m\|_\infty$ except on a set $Z_{n,m}$ of measure zero. Let $Z = \bigcup_{n,m \in \mathbb{N}} Z_{n,m}$, then Z has measure zero and

$$|f_n - f_m| \leq \|f_n - f_m\|_\infty \quad \text{on } E \setminus Z$$

In particular, for any $x \in E \setminus Z$, $\{f_n(x)\}$ converges. Let $f(x) = \lim_{n \to \infty} f_n(x)$ for $x \in E \setminus Z$ and set $f(x) = 0$ on Z. Then

$$f_n \to f \text{ uniformly on } E \setminus Z.$$

This implies that f_n converges to f in $L^\infty(E)$, and so $L^\infty(E)$ is complete.

Now we consider $1 \leq p < \infty$. By Theorem 1.3, we only have to show that every absolutely convergent series converges to some element in $L^p(E)$.

Let $\sum_{k=1}^\infty f_k$ be an absolutely convergent series. Then $\sum_{k=1}^\infty \|f_k\|_p = M$ is finite. Let

$$g_n = \sum_{k=1}^n |f_k|, \quad s_n = \sum_{k=1}^n f_k.$$

By Minkowski's inequality, $\|g_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq M$. Thus $\int_E g_n^p \leq M^p$ for any n. For any $x \in E$, the function $g_n(x)$ is increasing in n, and so g_n converges pointwise to some function $g : E \to [0, \infty]$. The function g is measurable and, by Fatou's lemma,

$$\int_E g^p \leq \liminf_{n \to \infty} \int_E g_n^p \leq M^p.$$

Therefore g is finite almost everywhere and $g \in L^p(E)$. When $g(x)$ is finite, $\sum_{k=1}^\infty f_k(x)$ is absolutely convergent. Let $s(x)$ be its value, and set $s(x) = 0$ elsewhere. Then the function s is defined everywhere, measurable on E, and

$$\sum_{k=1}^n f_k = s_n \to s \quad \text{almost everywhere on } E.$$

Since $|s_n(x)| \leq g(x)$ for all n, we have $|s(x)| \leq g(x)$, where hence $s \in L^p(E)$ and $|s_n(x) - s(x)| \leq 2g(x) \in L^p(E)$. By the Lebesgue dominated convergence theorem,

$$\int_E |s_n - s|^p \to 0 \text{ as } n \to \infty.$$

This proves that $\sum_{k=1}^\infty f_k$ converges to $s \in L^p(E)$, and thus proves completeness of $L^p(E)$. ☐

THEOREM 3.4. *If $1 \le p < \infty$, then $L^p(E)$ is separable.*

PROOF. Consider $E = \mathbb{R}^n$. Consider the collection of cubes of the form $[k_1, k_1+1] \times \ldots \times [k_n, k_n+1]$, $k_1 \ldots, k_n \in \mathbb{Z}$. Bisect each of these cubes into 2^n congrument subcubes, and repeat this process. The collection of all these cubes are called dyadic cubes. Let $\mathcal{D}$ be the set of finite linear combinations of characteristic functions on these dyadic cubes with rational coefficients. Clearly $\mathcal{D}$ is countable. All we need to prove is that $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n)$. That is, given $f \in L^p(\mathbb{R}^n)$, there exists a sequence $f_k \in \mathcal{D}$ such that $\|f_k - f\|_p \to 0$ as $k \to \infty$.

It suffices to consider the case $f \ge 0$ since

$$f = f^+ - f^-, \quad \|f_k - f\|_p \le \|f_k^+ - f^+\|_p + \|f_k^- - f^-\|_p \quad \text{(by Minkowski's inequality)}.$$

In fact, it suffices to consider the case $f \ge 0$ with compact support since

$$\int_{\mathbb{R}^n} |f_k - f|^p = \lim_{m\to\infty} \int_{[-m,m]^n} |f_k - f|^p$$

Let $\{g_k\}$ be an increasing sequence of nonnegative simple functions such that $g_k \nearrow f$, $f \ge 0$ has compact support. Then, by the monotone convergence theorem,

$$g_k \in L^p(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} |g_k - f|^p \to 0 \quad \text{as } k \to \infty.$$

Therefore, it suffices to consider the case $f \ge 0$, f is a simple function with compact support.

For a simple function $f = \sum_{k=1}^{N} a_k \chi_{E_k}$,

$$\int_{\mathbb{R}^n} |f - g|^p = \sum_{k=1}^{N} \int_{E_k} |a_k - g|^p \quad \text{for any } g \in L^p(\mathbb{R}^n).$$

From this observation, it suffices to consider the case when f is the characteristic function of some bounded measurable set E. There exists a G_δ set G containing E with $m(G \setminus E) = 0$, so that we may consider only the case E being a G_δ set.

Let $E = \bigcap_{k=1}^{\infty} \mathcal{O}_k$, $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots$ is a nested sequence of bounded open sets. Then, by the monotone convergence theorem,

$$\int_{\mathbb{R}^n} |\chi_{\mathcal{O}_k} - \chi_E|^p \to 0 \quad \text{as } k \to \infty.$$

Therefore, it suffices to consider $f = \chi_{\mathcal{O}}$, where $\mathcal{O}$ is a bounded open set. But in this case, $f = \sum_{k=1}^{\infty} \chi_{c_k}$ for some dyadic cubes c_k. This proves $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n)$.

Now consider arbitrary measurable set $E \subset \mathbb{R}^n$. Let $\mathcal{D}' = \{g \cdot \chi_E : g \in \mathcal{D}\}$. Then $\mathcal{D}'$ is a countable set consisting of finite linear combinations of characteristic functions on dyadic cubes which intersect with E and with rational coefficients.

Given $f \in L^p(E)$. Let $\tilde{f} = f$ on E, $\tilde{f} = 0$ on $\mathbb{R}^n \backslash E$. Choose $\{f_k\} \subset \mathcal{D}$ such that $\int_{\mathbb{R}^n} |f_k - \tilde{f}|^p \to 0$ as $k \to \infty$. Then

$$\int_E |f_k \cdot \chi_E - \tilde{f}|^p = \int_{\mathbb{R}^n} |f_k \cdot \chi_E - \tilde{f}|^p \to 0 \text{ as } k \to \infty.$$

This proves that $\mathcal{D}'$ is dense in $L^p(E)$. $\square$

Given $h \in \mathbb{R}^n$. Let $\tau_h f(x) = f(x+h)$ be the translation operator. Similar to the case $L^1(\mathbb{R}^n)$, we have continuity of variable translations with respect to $\|\cdot\|_p$:

THEOREM 3.5. *If $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then*

$$\lim_{h\to\infty} \|\tau_h f - f\|_p = 0.$$

PROOF. Let C_p be the collection of $L^p(\mathbb{R}^n)$ functions satisfying this property. It follows easily from the Minkowski inequality that it is a subspace of $L^p(\mathbb{R}^n)$.

Given $E \subset \mathbb{R}^n$ with $m(E) < \infty$. By the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)|^p dx = m(E \setminus E_h) + m(E_h \setminus E) \to 0 \text{ as } h \to 0,$$

where $E_h = E - h = \{e - h : e \in E\}$. This shows that $\chi_E \in C_p$, and as well as simple functions in $L^p(\mathbb{R}^n)$. Suppose $f \in L^p(\mathbb{R}^n)$ is nonnegative. Choose simple functions $f_k \ge 0$, $f_k \nearrow f$. Then $f_k \in L^p(\mathbb{R}^n)$ and, by the monotone convergence theorem, $\|f_k \to f\|_p \to 0$ as $k \to \infty$. Therefore,

$$\begin{aligned} \|\tau_h f - f\|_p &\le \|\tau_h f - \tau_h f_k\|_p + \|\tau_h f_k - f_k\|_p + \|f_k - f\|_p \\ &\le \|\tau_h f_k - f_k\|_p + 2\|f - f_k\|_p \end{aligned}$$

Let $h \to 0$, then let $k \to \infty$, we find that $\limsup_{h\to\infty} \|\tau_h f - f\|_p = 0$. This proves that $f \in C_p$. This actually implies $C_p = L^p(\mathbb{R}^n)$ since any $f \in L^p(\mathbb{R}^n)$ is the difference of two nonnegative measurable functions in $L^p(\mathbb{R}^n)$. □

Exercises.

3.1. Given any $f \in L^\infty(E)$. Show that $m(\{f > \operatorname{ess\,sup}_E f\}) = 0$.

3.2. Use the generalized Young's inequality in Exercise 2.2 to formulate a generalization of Hölder's inequality for $L^p(E)$.

3.3. Suppose $m(E) < \infty$. Show that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. How about if $m(E) = \infty$?

3.4. Let f be a real-valued measurable function on E. Define the *essential infimum* on E by

$$\operatorname*{ess\,inf}_E f = \sup\{\alpha \in (-\infty, \infty] : m(\{f < \alpha\}) = 0\}.$$

Show that, if $f \ge 0$, then $\operatorname{ess\,inf}_E f = 1/\operatorname{ess\,sup}_E(1/f)$.

3.5. Consider $L^p(E)$ with $0 < p < 1$. Verify that $\rho_p(f,g) = \int_E |f-g|^p$ is a metric on $L^p(E)$. Prove that $(L^p(E), \rho_p)$ is a complete separable metric space.

3.6. Given $0 < p < \infty$, $f_n \in L^p(E)$. Suppose f_n converges to f almost everywhere. Prove that each of the following conditions implies $\|f_n - f\|_p \to 0$ as $n \to \infty$.

(a) There exists some $g \in L^p(E)$ such that $|f_n| \le g$ for any n.
(b) $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

3.7. For what kind of $f \in L^p(E)$ and $g \in L^q(E)$, $\frac{1}{p} + \frac{1}{q} = 1$, do we have equality for the Hölder inequality? For what kind of $f, g \in L^p(E)$ do we have equality for the Minkowski inequality?

3.8. Consider $1 < p < \infty$. Give a proof for the Minkowski inequality using convexity of x^p, and without using Hölder's inequality.

3.9. Show that $L^\infty(E)$ is not separable whenever $m(E) > 0$.