CHAPTER 8

Introduction to Banach Spaces

1. Uniform and Absolute Convergence

As a preparation we begin by reviewing some familiar properties of Cauchy sequences and uniform limits in the setting of metric spaces.

DEFINITION 1.1. A *metric space* is a pair (X, ρ), where X is a set and ρ is a real-valued function on $X \times X$ which satisfies that, for any x, y, $z \in X$,

(a) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$,
(b) $\rho(x, y) = \rho(y, x)$,
(c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. (Triangle inequality)

The function ρ is called the *metric* on X.

Any metric space has a natural topology induced from its metric. A subset U of X is said to be *open* if for any $x \in U$ there exists some $r > 0$ such that $B_r(x) \subset U$. Here $B_r(x) = \{y \in X : \rho(x, y) < r\}$ is the open ball of radius r centered at x. It is an easy exercise to show that open balls are indeed open and the collection of open sets is indeed a topology, called the *metric topology.*

On the contrary, there are topological spaces whose topology can be defined by some metric. In this case we say the topology is *metrizable.*

DEFINITION 1.2. A sequence $\{x_n\}$ in a metric space (X, ρ) is said to be a *Cauchy Sequence* if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \rho(x_n, x_m) < \varepsilon \text{ whenever } n, m \geq N.$$

The metric space (X, ρ) is said to be *complete* if every Cauchy sequence is convergent.

DEFINITION 1.3. Let (X, ρ) be a metric space. For any nonempty set $A \subset X$, the *diameter* of the set A is defined by

$$\operatorname{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}.$$

The set A is said to be *bounded* if its diameter is finite. Otherwise, we say it is *unbounded.*

Let S be a nonempty set. We say a function $f : S \to X$ is *bounded* if its image $f(S)$ is a bounded set. Equivalently, it is bounded if for any $x \in X$, there exists $M > 0$ such that $\rho(f(s), x) \leq M$ for any $s \in S$. We say f is *unbounded* if it is not bounded.

DEFINITION 1.4. Given a sequence $\{f_n\}$ of functions from S to X. We say $\{f_n\}$ *converges pointwise* to the function $f : S \to X$ if

$$\forall\, s \in S, \forall\, \varepsilon > 0, \exists\, N_s \in \mathbb{N} \quad \text{such that} \quad \rho(f_n(s), f(s)) < \varepsilon,\ \forall\, n \geq N.$$

In this case, the function f is called the *pointwise limit.*

We say $\{f_n\}$ *converges uniformly* to a function $f : S \to X$ if the above N_s is independent of s; that is,

$$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N} \quad \text{such that} \quad \rho(f_n(s), f(s)) < \varepsilon,\ \forall\, n \geq N,\ \forall\, s \in S.$$

The function f is called the *uniform limit* of $\{f_n\}$.

When S is countably infinite, the function f above is a sequence in X, and $\{f_n\}$ is a sequence of sequences in X; or in other words, $\{f_n(m)\}$ is a double sequence in X.

The next two theorems highlight some important features of Cauchy sequences and uniform convergence.

THEOREM 1.1. (Cauchy Sequences) *Consider sequences in a metric space* (X, ρ).

(a) *Any convergent sequence is a Cauchy sequence.*
(b) *Any Cauchy sequence is bounded.*
(c) *If a subsequence of a Cauchy sequence converges, then the Cauchy sequence converges to the same limit.*

PROOF. (a) Suppose $\{x_n\}$ converges to x. Given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\rho(x_n, x) < \varepsilon/2$ for any $n \geq N$. The sequence $\{x_n\}$ is Cauchy because

$$\rho(x_n, x_m) < \rho(x_n, x) + \rho(x, x_m) < \varepsilon \quad \text{for any } n, m \geq N.$$

(b) Let $\{x_n\}$ be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < 1$ for all $n, m \geq N$. Then for any $x \in X$,

$$\begin{aligned} \rho(x_n, x) &\leq \rho(x_n, x_N) + \rho(x_N, x) \\ &< \max\{\rho(x_1, x_N), \rho(x_2, x_N), \cdots, \rho(x_{N-1}, x_N), 1\} + \rho(x_N, x), \end{aligned}$$

where the last equation is a finite bound independent of n.

(c) Let $\{x_n\}$ be a Cauchy sequence with a subsequence $\{x_{n_k}\}$ converging to x. Given $\varepsilon > 0$, choose K, $N \in \mathbb{N}$ such that

$$\rho(x_{n_k}, x) < \frac{\varepsilon}{2} \quad \text{for any } k \geq K,$$
$$\rho(x_n, x_m) < \frac{\varepsilon}{2} \quad \text{for any } n, m \geq N.$$

Taking n_k such that $k \geq K$ and $n_k \geq N$, then

$$\rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) < \varepsilon \quad \text{for any } n \geq N.$$

This shows that $\{x_n\}$ converges to x. □

THEOREM 1.2. (Uniform Convergence) *Given a sequence of functions* $\{f_n\}$ *from a nonempty set* S *to a metric space* (X, ρ). *Suppose* $\{f_n\}$ *converges uniformly to a function* $f : S \to X$.

(a) *If each* f_n *is bounded, then so is* f.
(b) *Assume* S *is a topological space,* $E \subset S$. *If each* f_n *is continuous on* E, *then so is* f.

PROOF. (a) Choose $N \in \mathbb{N}$ such that $\rho(f(s), f_n(s)) < 1$ for any $n \geq N$ and $s \in S$. Given $x \in X$, choose $M > 0$ such that $\rho(f_N(s), x) < M$ for any $s \in S$. Then f is bounded since

$$\rho(f(s), x) \leq \rho(f(s), f_N(s)) + \rho(f_N(s), x) \leq 1 + M \quad \text{for any } t \in S.$$

(b) Given $\varepsilon > 0$, $e \in E$. Choose $N \in \mathbb{N}$ such that $\rho(f(s), f_n(s)) < \varepsilon/3$ for any $n \geq N$ and $s \in S$. For this particular N, f_N is continuous at e, and so there is a neighborhood U of e such that $\rho(f_N(e), f_N(u)) < \varepsilon/3$ whenever $u \in U$. Then

$$\rho(f(e), f(u)) \;\leq\; \rho(f(e), f_N(e)) + \rho(f_N(e), f_N(u)) + \rho(f_N(u), f(u)) \;<\; \varepsilon \quad \forall u \in U.$$

Therefore f is continuous at e, and is continuous on E since $e \in E$ is arbitrary. □

Definition 1.5. A vector space V over field $\mathbb{F}$ is called a *normed vector space* (or *normed space*) if there is a real-valued function $\|\cdot\|$ on V, called the *norm*, such that for any $x, y \in V$ and any $\alpha \in \mathbb{F}$,

(a) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
(b) $\|\alpha x\| = |\alpha|\|x\|$.
(c) $\|x + y\| \leq \|x\| + \|y\|$. (Triangle inequality)

A norm $\|\cdot\|$ of V defines a metric ρ on V via $\rho(x, y) = \|x - y\|$. All concepts from metric and topological spaces are applicable to normed spaces.

There are multiple ways of choosing norms once a norm is selected. A trivial one is to multiply the original norm by a positive constant. Concepts like neighborhood, convergence, and completeness are independent of the choice of these two norms, and so we shall consider them equivalent norms. A more precise characterization of equivalent norms is as follows.

Definition 1.6. Let V be a vector space with two norms $\|\cdot\|$, $\|\cdot\|'$. We say these two norms are *equivalent* if there exists some constant $c > 0$ such that

$$\frac{1}{c}\|x\|' \;\leq\; \|x\| \;\leq\; c\|x\|' \quad \text{for any } x \in V.$$

Example 1.1. The Euclidean space $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with the standard norm $\|\cdot\|$ defined by

$$\|x\| \;=\; (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$$

is a normed space. Now consider two other norms of $\mathbb{F}^n$ defined by

$$\begin{aligned} \|x\|_\infty &= \max\{|x_1|, \cdots, |x_n|\}, \quad \text{called the } \textit{sup norm}; \\ \|x\|_1 &= |x_1| + \cdots + |x_n|, \quad \text{called the } \textit{1-norm}. \end{aligned}$$

Verifications for axioms of norms are completely straightforward.

In the case of sup norm, "balls" in $\mathbb{R}^n$ are actually cubes in $\mathbb{R}^n$ with faces parallel to coordinate axes. In the case of 1-norm, "balls" in $\mathbb{R}^n$ are cubes in $\mathbb{R}^n$ with vertices on coordinate axes. These norms are equivalent since

$$\|x\|_\infty \leq \|x\| \leq \|x\|_1 \leq n\|x\|_\infty.$$

Definition 1.7. A complete normed vector space is called a *Banach space*.

Example 1.2. Consider the Euclidean space $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with the standard norm $\|\cdot\|$. The normed space $(\mathbb{R}^n, \|\cdot\|)$ is complete since every Cauchy sequence is bounded and every bounded sequence has a convergent subsequence with limit in $\mathbb{R}^n$ (the Bolzano-Weierstrass theorem). The spaces $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_\infty)$ are also Banach spaces since these norms are equivalent.

Example 1.3. Given a nonempty set X and a normed space $(Y, \|\cdot\|)$ over field $\mathbb{F}$. The space of functions from X to Y form a vector space over $\mathbb{F}$, where addition and scalar multiplication are defined in a trivial manner: Given two functions f, g, and two scalars α, $\beta \in \mathbb{F}$, define $\alpha f + \beta g$ by

$$(\alpha f + \beta g)(x) = \alpha f(s) + \beta g(s), \quad x \in X.$$

Let $b(X, Y)$ be the subspace consisting of bounded functions from X to Y. Define a real-valued function $\|\cdot\|_\infty$ on $b(X, Y)$ by

$$\|f\|_\infty = \sup_{x \in X} \|f(x)\|.$$

It is clearly a norm on $b(X,Y)$, also called the *sup norm*. Convergence with respect to the sup norm is clearly the same as uniform convergence.

If $(Y, \|\cdot\|)$ is a Banach space, then any Cauchy sequence $\{f_n\}$ in $b(X,Y)$ converges pointwise to some function $f: X \to Y$, since $\{f_n(x)\}$ is a Cauchy sequence in Y for any fixed $x \in X$. In fact, the convergence $f_n \to f$ is uniform. To see this, let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon/2$ whenever $n, m \geq N$. For any $x \in X$, there exists some $m_x \geq N$ such that $\|f_{m_x}(x) - f(x)\| < \varepsilon/2$. Then for any $n \geq N$,

$$\|f_n(x) - f(x)\| \leq \|f_n(x) - f_{m_x}(x)\| + \|f_{m_x}(x) - f(x)\| < \varepsilon.$$

This proves that the convergence $f_n \to f$ is uniform. By Theorem 1.2(a), $f \in b(X,Y)$, and so the space $b(X,Y)$ with the sup norm is a Banach space.

Example 1.4. Let $(X, \mathcal{T})$ be a topological space and let $(Y, \|\cdot\|)$ be a Banach space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Denote by $C(X,Y)$ the space of continuous functions from X to Y. Let $C_b(X,Y) = C(X,Y) \cap b(X,Y)$, the space of bounded continuous functions from X to Y. Given any sequence $\{f_n\}$ in $C_b(X,Y)$ which converges uniformly to $f \in b(X,Y)$. By Theorem 1.2(b), $f \in C_b(X,Y)$. This shows that $C_b(X,Y)$ is a closed subspace of $b(X,Y)$, and is therefore a Banach space (see Exercise 1.1).

Definition 1.8. A series $\sum_{k=1}^\infty a_k$ in a normed space X is said to be *convergent* (or *summable*) if its partial sum $\sum_{k=1}^n a_k$ converges to some $s \in X$ as $n \to \infty$. We say $\sum_{k=1}^\infty a_k$ is *absolutely convergent* (or *absolutely summable*) if $\sum_{k=1}^\infty \|a_k\| < \infty$.

In the following we prove some useful criteria for completeness and uniform convergence of series.

Theorem 1.3. *A normed space X is complete if and only if every absolutely convergent series is convergent.*

Proof. Suppose X is complete, $\sum_{k=1}^\infty a_k$ is absolutely convergent. We need to show the convergence of $s_n = \sum_{k=1}^n a_k$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\sum_{k=N}^\infty \|a_k\| < \varepsilon$, then $\|s_n - s_m\| < \varepsilon$ whenever $n, m \geq N$. Thus $\{s_n\}_{n=1}^\infty$ is a Cauchy sequence, and so it converges.

Conversely, suppose every absolutely convergent series in X converges. Let $\{s_n\}_{n=1}^\infty$ be a Cauchy sequence in X. By Theorem 1.1 it suffices to show that $\{s_n\}_{n=1}^\infty$ has a convergent subsequence $\{s_{n_k}\}_{k=1}^\infty$. Now choose n_k such that

$$n_k < n_{k+1}, \quad \left\|s_{n_k} - s_{n_{k+1}}\right\| < \frac{1}{2^k} \quad \text{for any } k \in \mathbb{N}.$$

Then the series $s_{n_1} + \sum_{k=1}^\infty \left(s_{n_{k+1}} - s_{n_k}\right)$ converges absolutely, so that it converges to some $s \in X$. This implies that $s_{n_k} = s_{n_1} + \sum_{j=1}^{k-1} \left(s_{n_{j+1}} - s_{n_j}\right)$ converges to s as $k \to \infty$, completing the proof. □

Corollary 1.4. (Weierstrass M-test)
Let $b(X,Y)$ be the space bounded functions from a nonempty set X to a Banach space $(Y, \|\cdot\|)$. Given a sequence of functions $\{f_n\}_{n=1}^\infty$ in $b(X,Y)$. If $\|f_n\|_\infty \leq M_n$ for any $n \in \mathbb{N}$ and $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly.

Proof. The assumption says that $\sum_{n=1}^\infty f_n$ is absolutely convergent. By Theorem 1.3 (and Example 1.3), the series converges in $b(X,Y)$, implying that the convergence is uniform. □

Exercises.

1.1. Show that a subset of a complete metric space is complete if and only if it is closed.

1.2. Let $\Sigma_2 = \{0,1\}^{\mathbb{N}}$, the space of infinite sequences of $\{0,1\}$; that is,

$$\Sigma_2 = \{(a_1, a_2, ...) : \ a_k = 0 \text{ or } 1 \text{ for each } k\}.$$

Given $\lambda > 1$, $a, b \in \Sigma_2$, let

$$\rho_\lambda(a,b) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{\lambda^k}.$$

Show that (Σ_2, ρ_λ) is a complete metric space.

1.3. Consider the space of real sequences s. Let

$$\rho(a,b) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k(1 + |a_k - b_k|)}.$$

Show that (s, ρ) is a complete metric space.

1.4. Consider the space $BV[a,b]$ of functions on $[a,b]$ with bounded variations. For any $f \in BV[a,b]$, let $\|f\| = |f(a)| + V_a^b(f)$. Show that $(BV[a,b], \|\cdot\|)$ is a Banach space. Is it separable?

2. The ℓ^p Space

DEFINITION 2.1. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Given $0 < p < \infty$, define

$$\ell^p = \{a = (a_1, a_2, \cdots) : a_k \in \mathbb{F} \ \text{ for any } k, \sum_k |a_k|^p < \infty\}, \qquad \|a\|_p = \left(\sum_k |a_k|^p\right)^{\frac{1}{p}},$$
$$\ell^\infty = \{a = (a_1, a_2, \cdots) : a_k \in \mathbb{F} \ \text{ for any } k, \sup_k |a_k| < \infty\}, \qquad \|a\|_\infty = \sup_k |a_k|.$$

The space ℓ^∞ consists of bounded sequences in $\mathbb{F}$. Addition and multiplication of sequences are defined componentwise:

$$\begin{aligned}(a_1, a_2, \cdots) + (b_1, b_2, \cdots) &= (a_1 + b_1, a_2 + b_2, \cdots)\\ (a_1, a_2, \cdots) \cdot (b_1, b_2, \cdots) &= (a_1 b_1, a_2 b_2, \cdots).\end{aligned}$$

Clearly ℓ^p with any $0 < p \le \infty$ is a vector space, since $\|\alpha a\|_p = |\alpha|\|a\|_p$ for any $\alpha \in \mathbb{F}$ and

$$\begin{aligned} a, b \in \ell^p &\Rightarrow \sum_k |a_k + b_k|^p \le \sum_k (2\max\{|a_k|, |b_k|\})^p \le 2^p \sum_k (|a_k|^p + |b_k|^p),\\ a, b \in \ell^\infty &\Rightarrow \sup_k |a_k + b_k| \le \sup_k |a_k| + \sup_k |b_k|. \end{aligned}$$

.

When $1 \le p \le \infty$, the function $\|\cdot\|_p$ is a norm on ℓ^p. This follows from Theorem 2.1 below.

THEOREM 2.1. *Given $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b be sequences of complex numbers.*

(a) (YOUNG'S INEQUALITY) *If $u, v \ge 0$, $1 < p, q < \infty$, then*

$$uv \le \frac{u^p}{p} + \frac{v^q}{q}.$$

(b) (Hölder's Inequality for ℓ^p) *If* $a \in \ell^p$, $b \in \ell^q$, *then* $ab \in \ell^1$ *and*

$$\|ab\|_1 \leq \|a\|_p \|b\|_q.$$

(c) (Minkowski's Inequality for ℓ^p)

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p.$$

Proof. It would be convenient to write $q = \frac{p}{p-1}$ ($q = \infty$ if $p = 1$, $q = 1$ if $p = \infty$). The curve $y = x^{p-1}$ can be alternatively written $x = y^{q-1}$. Part (a) follows easily by observing

$$U = \int_0^u x^{p-1} dx = \frac{u^p}{p}, \quad V = \int_0^v y^{q-1} dy = \frac{v^q}{q}, \quad U + V \geq uv.$$

For part (b), the cases $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$ are obvious. Consider $1 < p, q < \infty$. The cases $\|a\|_p = 0$ or $\|b\|_q = 0$ are also obvious, so we assume $0 < \|a\|_p$, $\|b\|_q < \infty$.

Let $A = \frac{a}{\|a\|_p}$, $B = \frac{b}{\|b\|_q}$. By Young's inequality,

$$\begin{aligned} \sum_k |A_k B_k| &\leq \sum_k \left(\frac{|A_k|^p}{p} + \frac{|B_k|^q}{q} \right) = \frac{\|A\|_p^p}{p} + \frac{\|B\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1, \\ \|ab\|_1 &= \sum_k |a_k b_k| = \|a\|_p \|b\|_q \sum_k |A_k B_k| \leq \|a\|_p \|b\|_q. \end{aligned}$$

The cases $p = 1$ and $p = \infty$ for (c) are obvious. For $1 < p < \infty$, (c) follows easily from (b):

$$\begin{aligned} \|a+b\|_p^p &= \sum_k |a_k + b_k|^p \\ &\leq \sum_k |a_k + b_k|^{p-1} |a_k| + \sum_k |a_k + b_k|^{p-1} |b_k| \\ &\leq \left(\sum_k |a_k + b_k|^p \right)^{\frac{p-1}{p}} \left(\sum_k |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_k |a_k + b_k|^p \right)^{\frac{p-1}{p}} \left(\sum_k |b_k|^p \right)^{\frac{1}{p}} \\ &= \|a+b\|_p^{p-1} (\|a\|_p + \|b\|_p). \end{aligned}$$

□

Theorem 2.2. *For any* $1 \leq p \leq \infty$, $(\ell^p, \|\cdot\|_p)$ *is a Banach space.*

Proof. Completeness of ℓ^∞ is a special case of Example 1.3. Consider $1 \leq p < \infty$. Let $\{a^{(n)}\}_{i=1}^\infty$ be a Cauchy sequence in ℓ^p. For each k, $\{a_k^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence of real numbers since

$$\left| a_k^{(n)} - a_k^{(m)} \right| \leq \left(\sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p \right)^{\frac{1}{p}} = \left\| a^{(n)} - a^{(m)} \right\|_p.$$

Then there is a sequence $a = (a_1, a_2, \cdots)$ such that, for each k,

$$a_k^{(n)} \to a_k \in \mathbb{R} \quad \text{as } n \to \infty.$$

Given $\varepsilon > 0$, $M \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that

$$\left(\sum_{k=1}^{M}\left|a_k^{(n)} - a_k^{(m)}\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty}\left|a_k^{(n)} - a_k^{(m)}\right|^p\right)^{\frac{1}{p}} < \varepsilon \quad \forall\, n > m \ge N.$$

Let $m \to \infty$, then let $M \to \infty$, we find

$$\|a^{(n)} - a\|_p = \left(\sum_{k=1}^{\infty}\left|a_k^{(n)} - a_k\right|^p\right)^{\frac{1}{p}} \le \varepsilon \quad \text{for any } n \ge N.$$

Thus $\|a^{(n)} - a\|_p \to 0$ as $n \to \infty$, and so $\|a\|_p \le \|a - a^{(n)}\|_p + \|a^{(n)}\|_p < \infty$, $a \in \ell^p$. This verifies completeness of ℓ^p. $\square$

THEOREM 2.3. *The space ℓ^p is separable if $1 \le p < \infty$, and the space ℓ^∞ is not separable.*

PROOF. The space ℓ^∞ is not separable because it has an uncountable subset $s = \{a = (a_1, a_2, \ldots) \in \ell^\infty : a_n = 0 \text{ or } 1\ \forall\, n\}$ and $\|a - b\|_\infty = 1$ for any $a \ne b \in s$.

Consider $1 \le p < \infty$. Let $\mathcal{D}$ be the set of finite sequences with rational coordinates. Clearly $\mathcal{D}$ is countable. Given $a \in \ell^p$ and any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} |a_k|^p < \varepsilon^p/2$. Now choose $b_1, \cdots, b_N \in \mathbb{Q}$ such that $\sum_{k=1}^{N} |a_k - b_k|^p < \varepsilon^p/2$. Let $b = (b_1, \cdots, b_N, 0, 0, \cdots) \in \mathcal{D}$. Then

$$\|a - b\|_p^p \;=\; \sum_{k=1}^{\infty} |a_k - b_k|^p \;=\; \sum_{k=1}^{N} |a_k - b_k|^p + \sum_{k=N+1}^{\infty} |a_k|^p \;<\; \varepsilon^p.$$

Thus $\|a - b\|_p < \varepsilon$. This shows that $\mathcal{D}$ is dense since $\varepsilon > 0$ is arbitrary. $\square$

Exercises.

2.1. Consider the ℓ^p space with $0 < p < 1$. Verify that $\rho_p(a, b) = \sum_{k=1}^{\infty} |a_k - b_k|^p$ is a metric on ℓ^p. Prove that (ℓ^p, ρ_p) is a complete separable metric space.

2.2. Prove the following generalization of Young's inequality: Given $1 < p_1, \cdots, p_n < \infty$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$. If $u_1, \cdots, u_n \ge 0$, then

$$u_1 \cdots u_n \le \frac{u_1^{p_1}}{p_1} + \cdots + \frac{u_n^{p_n}}{p_n}.$$

Use it to formulate a generalization of Hölder's inequality for ℓ^p.

2.3. Consider sequences of real numbers. Show that the space c_0 of sequences converging to zero with sup norm is a Banach space, and for any $1 \le p < q \le \infty$, $a \in \ell^p$,

$$\ell^p \subsetneq \ell^q \subsetneq c_0, \quad \|a\|_\infty \le \|a\|_q \le \|a\|_p.$$

Are these norms on ℓ^p equivalent?

2.4. Explain why the set s in the proof of Theorem 2.3 is uncountable.