VARIATIONAL CONSTRUCTIONS FOR SOME SATELLITE ORBITS IN PERIODIC GRAVITATIONAL FORCE FIELDS

KUO-CHANG CHEN

Abstract. In a gravitational force field, a bounded orbit of an infinitesimal point mass is called a satellite orbit. The purpose of this paper is to establish a variational theory for the existence of some relative periodic satellite orbits in periodic gravitational force fields. The gravitation field can be generated by a relative equilibrium or a uniformly rotating asteroid. Our approach is to regard the aggregate of primaries together with the small satellite as a restricted full two-body problem, and then look for direct and retrograde relative periodic satellite orbits by direct methods of calculus of variations. Regularity of the action-minimizing satellite orbit is obtained by providing satisfactory lower bound estimates for the distance between the satellite and the mass center. An upper bound estimate for this distance is also provided as a contrast to classical existence proofs by continuation from infinity.

1. Introduction

In a gravitational force field, we call a bounded orbit of an infinitesimal point mass a *satellite orbit.* A satellite orbit may represent the motion of a small planet circling around the outskirts of a multiple star system, an artificial satellite around an asteroid, a satellite around a planet in the solar system, or a planet of one primary mass in a multiple star system. There are satellite orbits not within these classes but in this paper we will focus on relative periodic orbits belonging to the first two classes.

The restricted n-body problem concerns the motion of a zero mass in a gravitation field generated by $n-1$ celestial bodies (primaries) whose motions are governed by Newton's law of universal gravitation. The zero mass is moving under the influence of a periodic gravitational force field if the primaries are in periodic motion. Classical methods for proving existence of (relative) periodic satellite orbits include the analytic continuation method, power series method, equating the Fourier coefficients, and the fixed point method [27, §8.4]. Under various conditions, satellite orbits that are either near a primary or far outside the territory of all primaries are known to exist. There are two typical families of such orbits, called the *direct* (or *prograde*) and the *retrograde* families according to the direction they revolve about the primaries in the non-rotating coordinate system. Detailed descriptions along with comprehensive bibliographies can be found in the books of Wintner [30], Szebehely [27], and Siegel-Moser [26]. See also Meyer [17] for relatively recent advances.

A classical criterion for the existence of periodic orbits in a ring due to Whittaker [28], later generalized by Signorini [25] and Birkhoff [4], is based on interpreting periodic orbits as closed geodesics on a surfaces plus some boundary convexity conditions on the ring. Similar conditions also appear in Moeckel's [19] construction of some transit orbits based on Maupertuis' principle and isolating blocks. Our present work is based on direct minimization method which establishes both the direct and retrograde families inside concentric circular rings. The ring varies its size

---

Key words and phrases. restricted n-body problem, restricted full-two body problem, satellite orbit, relative equilibrium, variational method.

depending on how close the relative period T is to the period 2π of primaries. When T as a parameter approaches 2π, the inner and outer radii of the ring increases to infinity. This result relates the classical treatises of Wintner [29] and Moulton [21], one deals with the case where a mass ratio of primaries is nearly zero, the other deals with the case where a distance ratio is nearly zero. Our approach can be generalized to restricted n-body problems with one axis of symmetry, where the families of satellite orbits near infinity (see Meyer [18]) can therefore be continued to nearby satellite orbits.

As far as the restricted three-body problem is concerned, this paper can be considered a continuation of [8] where the author proves the existence of the retrograde family (on a rotating frame) with general masses. A generalization of [8] along with some direct orbits were obtained in [9]. The direct and retrograde families of satellite orbits “outside” all primaries can be viewed as limiting cases in [8] with masses m_1, $m_2 \gg m_3$. It is tempting to seek a unified variational treatment for more general restricted problems that are of astrophysical interests. Our idea that the minimizing method is a feasible approach has originally inspired by two sources: the work of Gordon [14] and Chenciner-Montgomery [11]. We refer to [10, 12] (and references in there) for some influential ideas that inaugurated the minimizing theory for symmetric periodic solutions. We shall also mention a related work by Arioli-Gazzola-Terracini [2] which characterizes some retrograde Hill’s orbits as minima of the action functional. Discussions for Hill-type satellite orbits can be also found in [17, 26, 27, 30].

Another incentive of our present work comes from the so-called full two-body problem. The classical Newtonian two-body problem, also called the Kepler problem, concerns the motion of two celestial bodies with perfect spherical symmetry, in which case each celestial body can be replaced by a point mass without altering the potential energy outside the celestial body. While in practical problems the celestial body is frequently assumed to be oblate or elliptic, the potential, nonetheless, has the Newtonian potential as its first order approximation. The error term is comparatively small when the celestial bodies are nearly spherically symmetric or if the sizes of celestial bodies are small compared to their mutual distance. Many systems of astrophysical interests are in one or both categories.

In recent years there are growing attention on the motion of spacecrafts around an irregularly shaped asteroid. There are several space missions targeting on primitive asteroids or comets as they are believed to be constituted by remains of the early universe. In such a system the error term is often not negligible and the two-body system can not be regarded as simply a small perturbation of the classical Kepler problem. For this type of system, often called the full two-body problem, proving rigorously the existence of Keplerian-like orbits becomes a nontrivial task. Numerically the dynamics near an irregular asteroid is much more complex and abundant than the classical Kepler problem. We refer to [13] and expository papers [16, 22] for recent advances and related references.

The purpose of this paper is to establish a variational theory for the existence of satellite orbits in periodic gravitational force fields. Consider for instance the planar restricted n-body problem with $n-1$ Keplerian-like primaries, the infinitesimal point mass orbiting around the $(n-1)$-body system can be viewed as moving under the influence of a periodic gravitational force field resulted from the primaries. More generally, given a periodic orbit of the $(n-1)$-body problem, the aggregate of these $n-1$ bodies generates a periodic gravitational force field under which the satellite (i.e. the n-th mass) moves. In this paper we regard the aggregate of primaries together with the small satellite as a restricted full two-body problem, and then look for periodic and quasi-periodic satellite orbits. Finding satellite orbits around an irregularly shaped asteroid is a problem of the same nature.

With the viewpoint described above, there are two issues needed to be taken care of: proving existence of satellite orbits, and proving that they are free from collisions. The existence part will be proved through variational arguments in section 3. The problem of collision avoidance is somewhat different from the classical n-body problem because we are not dealing with point masses. For this purpose, in section 4 we provide lower bound estimates for the distance between the satellite and the centroid of the system, under suitable assumptions. Upper bound estimates for this distance will be provided in section 5. Together with the estimates in section 4, we can therefore draw circular rings in which the action-minimizing satellites move. Sections 6 and 8 include some concrete examples.

2. The Restricted Full Two-Body Problem

By suitable choice of unit for the mass, the total mass of the system may assumed to be 1 and the equations of motion for the satellite orbit in the gravitational force field can be written

$$\ddot{q} \;=\; \frac{\partial}{\partial q} U(q,t), \tag{1}$$

where

$$U(q,t) \;=\; \frac{1}{|q|} + U_0(q,t) \tag{2}$$

is a time-dependent potential and $U_0(q,t)$ is the correction term of $U(q,t)$ from the point-mass potential $1/|q|$. The Lagrangian of the system is

$$L(q,\dot{q},t) \;=\; \frac{1}{2}|\dot{q}|^2 + U(q,t).$$

The action functional $\mathcal{A}_T$ is defined by

$$\mathcal{A}_T(q) \;=\; \int_0^T L(q,\dot{q},t)dt,$$

where q belongs to the Sobolev space $H = H^1([0,T],\mathbb{C})$.

Given $\rho > 0$. Define

$$H_\rho \;=\; \{x \in H^1([0,T],\mathbb{C}) : \; |x(t)| \geq \rho \;\text{ for all }\; t\}.$$

Clearly H_ρ is weakly (and strongly) closed since weak convergence in H implies uniform convergence. The punctured complex plane $\mathbb{C} \setminus \{0\}$ is denoted by $\mathbb{C}^\times$.

In this paper we are concerned with the restricted full two-body problem, the asteroid or aggregate of primaries being regarded as one celestial body which generates a time-dependent potential of the form (2) with U_0 satisfying the following assumptions. Let D_r denotes the closed disk centered at the origin with radius r.

- (A_1) (Regularity) For some $\rho_0 \in (0,\rho)$, $U_0 \in C^0((\mathbb{C} \setminus D_{\rho_0}) \times \mathbb{R})$ is locally Lipschitz in q and $|U_0(q,t)|\,|q| \to 0$ as $|q| \to \infty$.
- (A_2) (Periodicity) For any $q \in \mathbb{C}^\times$, $U_0(e^{it}q,t)$ is independent of t.

The assumption (A_1) is a natural one since U_0 is in practice the sum of higher order terms in the series expansion of U in terms of $1/|q|$. The second assumption (A_2) says that the potential field is spinning with a constant angular velocity $\omega = 1$. These two assumptions combined implies that $|U_0(q,t)|\,|q| \to 0$ uniformly as $|q| \to \infty$. Furthermore, the quantity

$$R_\rho \;=\; \sup_{|q|\geq\rho,t\in\mathbb{R}} |U_0(q,t)| \tag{3}$$

decreases to zero as $\rho \to \infty$. This quantity measures how far U is away from the point-mass potential when $|q| = \rho$.

For example, let $q = q_1 + iq_2$ and

$$U_0(q,t) = \frac{(q_1^2 - q_2^2)\cos 2t + 2q_1q_2 \sin 2t}{(q_1^2 + q_2^2)^{5/2}}.$$

Then $R_\rho = 1/\rho^3$ and

$$U_0(e^{it}q,t) = \frac{q_1^2 - q_2^2}{(q_1^2 + q_2^2)^{5/2}}$$

depends only on the position q. This type of U_0 arises in the study of satellite orbits near an elliptical asteroid (see [24] and section 8).

Given $T > 0$. Consider the function space

$$\Lambda_T \;=\; \{q \in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}) :\; e^{-iT}q(t+T) = q(t) \;\text{ for any }\; t\}$$

consisting of relative T-periodic loops. Given $\rho > 0$,

$$\Lambda_{T,\rho} \;=\; \{q \in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}) :\; q\,|_{[0,T]} \in H_\rho\}$$

consists of paths in Λ_T that stays at least a distance ρ away from the origin.

Consider the rotating coordinate system which rotates the inertial plane about the origin with angular velocity $\omega = 1$. Paths in Λ_T (or $\Lambda_{T,\rho}$) become closed loops in this rotating frame and, by assumption (A_2), the potential becomes time-independent.

The major result of this paper is on the existence of certain relative periodic solutions to (1) on the space $\Lambda_{T,\rho}$. Now we wish to find a reasonable topological constraint for paths in $\Lambda_{T,\rho}$. Observe that any path q in $\Lambda_{T,\rho}$ can be continuously deformed in $\mathbb{C}^\times$ to the curve $q(0)e^{it(1+\frac{2k\pi}{T})}$ on $t \in [0,T]$ with end points held fixed for some $k \in \mathbb{Z}$. We say two paths q_1, q_2 in $\Lambda_{T,\rho}$ are *homotopic* if they correspond to the same k or, equivalently, if one can be continuously deformed in $\mathbb{C}^\times$ on $[0,T]$ to the other through paths in $\Lambda_{T,\rho}$. Now we select a different notation for its homotopy classes. Let

$$\Gamma^{(k)}_{T,\rho} \;=\; \{q \in \Lambda_{T,\rho} :\; q \text{ is homotopic to } q(0)e^{it(1+\frac{2k\pi}{T})}\}.$$

The set $\Gamma^{(k)}_{T,\rho}$ consists of paths which become closed loops with winding number k around the origin on the rotating frame with angular velocity $\omega = 1$. In other words, k is the number of times q winds around the origin relative to the rotating frame which fixes the gravitation field. When $k < 0$, loops in $\Gamma^{(k)}_{-2k\pi,\rho}$ are contractible in $\mathbb{C}^\times$ to a point. Clearly $\Lambda_{T,\rho} = \bigcup_{k\in\mathbb{Z}} \Gamma^{(k)}_{T,\rho}$.

A satellite orbit in $\Gamma^{(k)}_{T,\rho}$ with $T > -2k\pi$ is said to be a *direct* or *prograde* satellite orbit because the direction it revolves about the origin (mass center) in the inertia frame is the same as the rotational direction of the gravitation field; we call it a *retrograde* satellite orbit if $T < -2k\pi$. For $k \geq 0$ all satellite orbits in $\Gamma^{(k)}_{T,\rho}$ are direct.

Suppose we have found by minimizing $\mathcal{A}_T$ a satellite orbit $q \in \Gamma^{(k)}_{T,\rho}$ with $|k| \geq 2$, it is possible, and indeed likely, that $q \in \Gamma^{(k/|k|)}_{T/|k|,\rho}$. Since there is no evidence available to us showing that a satellite orbit which minimizes $\mathcal{A}_T$ on $\Gamma^{(k)}_{T,\rho}$ may not be a path in $\Lambda_{T/|k|}$, our main results (Theorem 4, 5, 18, 19, 21) and examples (section 6, 8) are therefore confined to $k = -1, 0, 1$.

The case $k > 0$ corresponds to cases where the satellite rotates about the primaries with average angular velocity $\omega_s \approx 1 + \frac{2k\pi}{T} > 1 = \omega$. If the satellite orbit moves like Keplerian circles, then Kepler's third law of planetary motions says that the nearly circular orbit is of rather small size compared to the largest mutual distance among primaries, but that would not result in a Keplerian-like motion unless it is a small planet in direct motion around a primary and all other primaries are relatively distant from them. For similar reasonings the case $-k > \frac{T}{\pi}$ corresponds to cases where $|\omega_s| \approx -\frac{2k\pi}{T} - 1 > 1 = \omega$ and the satellite is actually a small planet in retrograde motion around one primary with other primaries staying far off. From this observation we only consider $k = 0$ or -1 for the restricted n-body problem (section 6).

3. Existence of Action Minimizers

In this section we prove the existence of action minimizers in a subset of the function space $\Lambda_{T,\rho}$. We follow a standard argument in the calculus of variations which asks for coercivity and weak lower semicontinuity of the action functional. The idea of proof for coercivity (Lemma 1) is the same as that of [7, Proposition 2.1], and the proof for weak lower semicontinuity (Proposition 2) is similar to the Newtonian case (see for instance [14, §3A] or [6, Proposition 1]).

Let

$$\Lambda^*_{T,\rho} \;=\; \left\{ q \in \Lambda_{T,\rho} : q(t) \text{ is not a contractible loop in } \mathbb{C}^\times \text{ for } t \in [0,T] \right\}.$$

According to the definition of Λ_T, on the time interval $[0,T]$ a path $q \in \Lambda_T$ is a closed loop in the inertia frame if and only if T is an integral multiple of 2π. Therefore $\Lambda_{T,\rho} = \Lambda^*_{T,\rho}$ except when $T = 2k\pi$, $k \in \mathbb{N}$. The space $\Lambda^*_{2k\pi,\rho}$ is the proper subset of $\Lambda_{2k\pi,\rho}$ consisting of $2k\pi$-periodic loops with nonzero winding numbers about the origin (on the inertia frame).

Lemma 1. *Given* T, $\rho > 0$. *Assume* (A_1) *and* (A_2) *hold. The action functional* $\mathcal{A}_T$ *restricted to* $\Lambda^*_{T,\rho}$ *is coercive; that is,* $\mathcal{A}_T(q) \to +\infty$ *as* $\|q\|_{H^1} \to +\infty$.

Proof. Let R_ρ be given by (3). Then

$$|U_0(q,t)| \le R_\rho$$

for all $|q| \ge \rho$ and $t \in \mathbb{R}$.

Associated to each q we consider the following auxiliary term

$$\delta_q \;:=\; \max_{t\in[0,T]} |q(0) - q(t)|.$$

If T is not an integral multiple of 2π, then

$$\delta_q \;\ge\; |q(0) - q(T)| \;=\; |1 - e^{iT}|\,|q(0)| \;=\; \sqrt{2(1-\cos T)}|q(0)|.$$

When $T = 2k\pi$ for some $k \in \mathbb{N}$, the assumption that q is not a contractible loop in $\mathbb{C}^\times$ implies the existence of some $\tau \in (0,T)$ such that $q(\tau)$ is a negative real multiple of $q(0)$, and thus

$$\delta_q \;\ge\; |q(0) - q(\tau)| \;\ge\; |q(0)|.$$

In either case, there exists some positive constant C independent of q such that

$$\delta_q \;\ge\; C|q(0)|.$$

For any $t \in [0, T]$,

$$|q(t)| \ \leq \ |q(0)| + \delta_q \ \leq \ \left(\frac{1}{C} + 1\right) \delta_q$$

where hence

$$\int_0^T |q|^2 dt \leq \left(\frac{1}{C} + 1\right)^2 \delta_q^2 \, T$$

By the Cauchy-Schwartz inequality,

$$\delta_q^2 \leq \left(\int_0^T |\dot{q}| dt\right)^2 \leq T \int_0^T |\dot{q}|^2 dt.$$

The H^1-norm of q is therefore controlled by the value of its action:

$$\begin{aligned} \|q\|_{H^1}^2 &= \int_0^T |q|^2 + |\dot{q}|^2 dt \\ &\leq \left((\tfrac{1}{C} + 1)^2 \, T^2 + 1\right) \int_0^T |\dot{q}|^2 dt \\ &< 2\left((\tfrac{1}{C} + 1)^2 \, T^2 + 1\right) (\mathcal{A}_T(q) + R_\rho T) \,. \end{aligned}$$

This implies $\mathcal{A}_T$ restricted to $\Lambda_{T,\rho}^*$ is coercive. □

Proposition 2. *Given* T, $\rho > 0$. *Assume* (A_1) *and* (A_2) *hold. The action functional* $\mathcal{A}_T$ *restricted to* $\Lambda_{T,\rho}^*$ *attains its infimum.*

Proof. Using the fact that weakly convergent sequences in $H^1([0, T], \mathbb{C})$ converge uniformly, the weak limit of any weakly convergent sequence in $\Lambda_{T,\rho}^*$ satisfies the boundary constraints required in H_ρ, Λ_T, as well as the topological constraint required in $\Lambda_{T,\rho}^*$. Therefore $\Lambda_{T,\rho}^*$ is a weakly (and strongly) closed subset of Λ_T.

Let $q^{(k)}$ be a sequence in $\Lambda_{T,\rho}^*$ which converges weakly to $\tilde{q} \in \Lambda_{T,\rho}^*$. Note that $U_0(q, t)$ is locally Lipschitz continuous with respect to q and the Lipschitz constant can be independent of t, due to the assumption (A_2). Since $|q^{(k)}|$, $|\tilde{q}| \geq \rho$, it follows that the sequence $\frac{1}{|q^{(k)}|} + U_0(q^{(k)}, t)$ converges uniformly to $\frac{1}{|\tilde{q}|} + U_0(\tilde{q}, t)$. By uniform convergence the sequence $q^{(k)}$ converges strongly to $\tilde{q}$ in $L^2([0, T], \mathbb{C})$, then by weak lower semicontinuity of norms,

$$\|\dot{\tilde{q}}\|_{L^2}^2 \ = \ \|\tilde{q}\|_{H^1}^2 - \|\tilde{q}\|_{L^2}^2 \ \leq \ \liminf_{k\to\infty} \|q^{(k)}\|_{H^1}^2 - \|\tilde{q}\|_{L^2}^2 = \liminf_{k\to\infty} \|\dot{q}^{(k)}\|_{L^2}^2.$$

Consequently,

$$\begin{aligned} \mathcal{A}_T(\tilde{q}) \ &= \ \frac{1}{2}\|\dot{\tilde{q}}\|_{L^2}^2 + \int_0^T \frac{1}{|\tilde{q}|} + U_0(\tilde{q}, t) \, dt \\ &\leq \ \frac{1}{2} \liminf_{k\to\infty} \|\dot{q}^{(k)}\|_{L^2}^2 + \lim_{k\to\infty} \int_0^T \frac{1}{|q^{(k)}|} + U_0(q^{(k)}, t) \, dt \\ &= \ \liminf_{k\to\infty} \mathcal{A}(q^{(k)}). \end{aligned}$$

This shows that $\mathcal{A}_T$ is weakly sequentially lower semicontinuous on $\Lambda_{T,\rho}^*$.

Now suppose $q^{(k)}$ is a minimizing sequence of $\mathcal{A}_T$ on $\Lambda_{T,\rho}^*$. It is H^1-bounded, by Lemma 1, and hence has a weakly convergent subsequence. Without loss of generality, assume $q^{(k)}$ converges weakly to $\tilde{q}$. Then $\tilde{q} \in \Lambda_{T,\rho}^*$ and

$$\mathcal{A}_T(\tilde{q}) \ \leq \ \liminf_{k\to\infty} \mathcal{A}(q^{(k)}) \ = \ \inf\{\mathcal{A}_T(q) : q \in \Lambda_{T,\rho}^*\}.$$

This proves that $\tilde{q}$ is indeed a minimizer of $\mathcal{A}_T$ on $\Lambda^*_{T,\rho}$. $\square$

Using again the fact that weakly convergent sequences in $H^1([0,T],\mathbb{C})$ converge uniformly, the spaces $\Lambda_{T,\rho}$ and $\Gamma^{(k)}_{T,\rho}$ are weakly closed. Note that $\Gamma^{(k)}_{T,\rho} \subset \Lambda^*_{T,\rho}$ whenever $T \neq -2k\pi$. It follows immediately from Proposition 2 that

Corollary 3. *Given T, $\rho > 0$ and $k \in \mathbb{Z}$. Assume (A_1) and (A_2) hold. The action functional $\mathcal{A}_T$ restricted to $\Gamma^{(k)}_{T,\rho}$ attains its infimum provided $T \neq -2k\pi$.*

4. Satellite Orbits without Collisions

The gravitational force field is in general generated by celestial bodies (primaries) which occupy a closed region $\Omega(t)$ with positive measure. If the system moves rigidly with the mass center fixed at the origin, then the region $\Omega(t)$ rotates with a constant angular velocity so that the assumption (A_2) is fulfilled by choosing suitable unit for time. A solution q of (1) is said to has *collision* if $q(t) \in \Omega(t)$ for some t. A non-collision C^2 solution q of (1) is said to be a *regular* or *classical* solution.

In this section we provide a criterion for an action minimizer q obtained in the previous section to stay a prescribed distance away from the origin, and show that q is indeed a classical solution of (1) under certain conditions.

In order to shorten the statements of our main results, we introduce here an auxiliary function $\kappa : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\kappa(T,\rho) \;=\; \min_{s\in(0,\infty)} \left\{ \frac{1}{2T^2}(s-\rho)^2 + \frac{1}{s} \right\}.$$

The value of $\kappa(T,\rho)$ has a closed form. Let

$$\alpha \;=\; \alpha(T,\rho) \;=\; 2^{-\frac{1}{3}} \left(2\rho^3 + 27T^2 + 3\sqrt{3}T\sqrt{4\rho^3+27T^2} \right)^{\frac{1}{3}}.$$

Then the function $\frac{1}{2T^2}(s-\rho)^2 + \frac{1}{s}$ in s reaches its minimum value on $(0,\infty)$ at

$$s_{\min} \;=\; \frac{1}{3}\left(\rho + \frac{\rho^2}{\alpha} + \alpha \right).$$

Therefore

$$\kappa(T,\rho) \;=\; \frac{1}{2T^2}(s_{\min}-\rho)^2 + \frac{1}{s_{\min}}.$$

Our main results are

Theorem 4 (Existence of Direct and Retrograde Satellite Orbits). *Given T, $\rho > 0$, $T \neq 2\pi$. Assume (A_1) and (A_2) hold. Suppose $\rho \le \left|1 - \frac{2\pi}{T}\right|^{-\frac{2}{3}}$, R_ρ is given by (3), and $U_0(q,t)$ is of class C^1 whenever $|q| > \rho$. If*

$$\frac{3}{2}\left|1-\frac{2\pi}{T}\right|^{\frac{2}{3}} + 2R_\rho \;<\; \kappa(T,\rho) + \frac{\rho^2}{2}\left|1-\frac{2\pi}{T}\right|^2, \tag{4}$$

then there exists a classical solution q of (1) which minimizes $\mathcal{A}_T$ on $\Gamma^{(-1)}_{T,\rho}$ and $|q(t)| > \rho$ for any $t \in \mathbb{R}$. The satellite orbit q is retrograde if $T \in (0,2\pi)$; it is direct if $T > 2\pi$.

Theorem 5 (**Existence of Satellite Orbits near Relative Equilibria).** *Given* T, $\rho > 0$. *Assume* (A_1) *and* (A_2) *hold. Suppose* $\rho \leq 1$, R_ρ *is given by (3), and* $U_0(q,t)$ *is of class* C^1 *whenever* $|q| > \rho$. *If*

$$\frac{3}{2} + 2R_\rho \ < \ \kappa(T,\rho) + \frac{\rho^2}{2}, \tag{5}$$

then there exists a classical direct solution q *of (1) which minimizes* $\mathcal{A}_T$ *on* $\Gamma^{(0)}_{T,\rho}$ *and* $|q(t)| > \rho$ *for any* $t \in \mathbb{R}$.

Theorem 4-5 are special cases of the following theorem; we have split the result into two separate theorems due to the very different nature of these satellite orbits when k's are of different signs. Theorem 4 can be applied to the problem of satellite orbits near infinity as in [21, 17]; it is also applicable to satellites for asteroids. Theorem 5 applies to satellite orbits that rotate with the same frequency as the primaries.

Theorem 6. *Given* T, $\rho > 0$, $k \in \mathbb{Z}$. *Assume* (A_1) *and* (A_2) *hold. Suppose* $T \neq -2k\pi$, $\rho \leq \left|1 + \frac{2k\pi}{T}\right|^{-\frac{2}{3}}$, R_ρ *is given by (3), and* $U_0(q,t)$ *is of class* C^1 *whenever* $|q| > \rho$. *If*

$$\frac{3}{2}\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} + 2R_\rho \ < \ \kappa(T,\rho) + \frac{\rho^2}{2}\left|1 + \frac{2k\pi}{T}\right|^2, \tag{6}$$

then there exists a relative periodic solution q *of (1) which minimizes* $\mathcal{A}_T$ *on* $\Gamma^{(k)}_{T,\rho}$ *and* $|q(t)| > \rho$ *for any* $t \in \mathbb{R}$.

The region of (T,ρ) satisfying (6) with $k = 1$ is empty; that's why we specified our results for the cases $k = -1$ and $k = 0$. We will see later (section 7) how this can be overcome when there are additional symmetries.

In practical problems, the region $\Omega(t)$ is contained in some closed disk D_ρ with radius ρ and with center at the origin. Let ρ_0 be the smallest possible radius with this property. When the potential generator is far from being spherically symmetric, the term R_{ρ_0} is sometimes too large to conclude (6). Nonetheless, Theorem 6 may still be applicable because what we truly need is the existence of some $\rho > \rho_0$ such that (6) holds, and R_ρ is often negligible when ρ is large. Providing existence of such ρ and by considering

$$\eta \ = \ \sup\{\rho : \rho_0 \leq \rho \leq \left|1 + \frac{2k\pi}{T}\right|^{-\frac{2}{3}} \text{ and (6) holds}\}, \tag{7}$$

we obtain a sharper lower bound estimate η for the distance between the action minimizer and the origin.

By considering the rotating coordinate system $z = x + iy = e^{-it}q$, the equations of motion (1) become

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial V}{\partial y}, \end{aligned}$$

where the amended potential

$$\begin{aligned} V(x,y) &= \frac{1}{2}(x^2+y^2) + \frac{1}{\sqrt{x^2+y^2}} + U_0(e^{it}(x+iy),t) \\ &= \frac{1}{2}(x^2+y^2) + \frac{1}{\sqrt{x^2+y^2}} + U_0(x+iy,0) \end{aligned}$$

becomes time-independent, according to the assumption (A_2). In terms of z or (x, y), the Lagrangian of the system becomes

$$L(x, y, \dot{x}, \dot{y}) \;=\; \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x\dot{y} - \dot{x}y) + V(x, y).$$

Lemma 7. *Given T, $\rho > 0$, $k \in \mathbb{Z}$. Assume (A_1) and (A_2) hold. Suppose $T \neq -2k\pi$ and $\rho \leq \left|1 + \frac{2k\pi}{T}\right|^{-\frac{2}{3}}$. Then*

$$\begin{aligned}
\frac{1}{T} \inf_{\Gamma^{(k)}_{T,\rho}} \mathcal{A}_T &\leq \frac{3}{2}\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} + R_\rho, \\
\frac{1}{T} \inf_{\Lambda_{T,\rho}} \mathcal{A}_T &\leq \frac{3}{2}\left(\min_{k\in\mathbb{Z}}\left|1 + \frac{2k\pi}{T}\right|\right)^{\frac{2}{3}} + R_\rho.
\end{aligned}$$

Proof. The second inequality follows immediately from the first since $\Lambda_{T,\rho} = \bigcup_{k\in\mathbb{Z}} \Gamma^{(k)}_{T,\rho}$.

Taking polar coordinates for the rotating coordinate system, $(x, y) = (r\cos\theta, r\sin\theta)$, the action functional can be written

$$\mathcal{A}_T(q) \;=\; \int_0^T \frac{1}{2}\left[r^2(1 + \dot{\theta})^2 + \dot{r}^2\right] + \frac{1}{r} + U_0(r\cos\theta + ir\sin\theta, 0)\, dt.$$

On $\Gamma^{(k)}_{T,\rho}$, r and θ satisfy

$$r(T) = r(0),\; r(t) \geq \rho \quad \text{for all} \quad t, \quad \theta(T) - \theta(0) = 2k\pi.$$

The infimum of $\mathcal{A}_T$ on $\Gamma^{(k)}_{T,\rho}$ is bounded from above by the action value of the special path in $\Gamma^{(k)}_{T,\rho}$ given by $r(t) = \left|1 + \frac{2k\pi}{T}\right|^{-\frac{2}{3}}$ and $\theta(t) = \frac{2k\pi t}{T}$; that is,

$$\begin{aligned}
\inf_{\Gamma^{(k)}_{T,\rho}} \mathcal{A}_T &\leq \int_0^T \frac{1}{2}\left[r^2(1 + \dot{\theta})^2 + \dot{r}^2\right] + \frac{1}{r} dt + R_\rho T \\
&= \frac{3}{2}\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} T + R_\rho T.
\end{aligned}$$

□

Lemma 8. *Given T, $\rho > 0$, $k \in \mathbb{Z}$. Assume (A_1) and (A_2) hold. If $q \in \Gamma^{(k)}_{T,\rho}$, $\min_{[0,T]} |q(t)| = \underline{r} \geq \rho$, then*

$$\frac{1}{T}\mathcal{A}_T(q) \;\geq\; \frac{\underline{r}^2}{2}\left|1 + \frac{2k\pi}{T}\right|^2 + \kappa(T, \underline{r}) - R_{\underline{r}}.$$

Proof. Using polar form $(x, y) = (r\cos\theta, r\sin\theta)$, as in the previous lemma. Let

$$\overline{r} = \max_{t\in[0,T]} r(t) = \max_{t\in[0,T]} |q(t)|.$$

By the symmetry and topological assumptions imposed to $\Gamma^{(k)}_{T,\rho}$, $t+\theta(t)$ equals $\theta(0)$ at $t=0$ and equals $T+2k\pi+\theta(0)$ at $t=T$. Therefore,

$$\begin{aligned}
\mathcal{A}_T(q) &= \frac{1}{2}\int_0^T r^2(1+\dot\theta)^2\,dt+\frac{1}{2}\int_0^T \dot r^2\,dt+\int_0^T \frac{1}{r}+U_0(r\cos\theta+ir\sin\theta,0)\,dt\\
&\geq \frac{\underline{r}^2}{2}\int_0^T(1+\dot\theta)^2dt+\frac{1}{2T}\left(\int_0^T|\dot r|dt\right)^2+\frac{T}{\overline{r}}-R_{\underline{r}}T\\
&\geq \frac{\underline{r}^2}{2T}\left(\int_0^T|1+\dot\theta|dt\right)^2+\frac{1}{2T}\left(\overline{r}-\underline{r}\right)^2+\frac{T}{\overline{r}}-R_{\underline{r}}T\\
&\geq \frac{\underline{r}^2}{2T}\left(T+2k\pi\right)^2+T\kappa(T,\underline{r})-R_{\underline{r}}T.
\end{aligned}$$

This implies the lower bound estimate we claimed. □

Proof. (of Theorem 6) As observed earlier, $T\neq-2k\pi$ ensures that $\Gamma^{(k)}_{T,\rho}\subset\Lambda^*_{T,\rho}$. By Corollary 3, $\mathcal{A}_T$ attains its infimum in it.

Let q be a minimizer of $\mathcal{A}_T$ on $\Gamma^{(k)}_{T,\rho}$. $\underline{r}=\min\limits_{[0,T]}|q(t)|$. Then $\underline{r}\geq\rho$ and the strict inequality implies that q is in the interior of $\Gamma^{(k)}_{T,\rho}$.

Combining estimates in Lemma 7 and Lemma 8, by (6),

$$\frac{1}{T}\mathcal{A}_T(q)\ \leq\ \frac{3}{2}\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}}+R_\rho\ <\ \frac{\rho^2}{2}\left|1+\frac{2k\pi}{T}\right|^2+\kappa(T,\rho)-R_\rho$$

implying that the minimizer q of $\mathcal{A}_T$ on $\Gamma^{(k)}_{T,\rho}$ is actually in the interior of $\Gamma^{(k)}_{T,\rho}$, for otherwise $\underline{r}=\rho$ and Lemma 8 would fail. Such an action minimizer is an interior critical point of $\mathcal{A}_T$ on $\Lambda_{T,\rho}$ since $\Gamma^{(k)}_{T,\rho}$ is relatively open in $\Lambda_{T,\rho}$. Therefore q solves the Euler-Lagrange equation (1) of $\mathcal{A}_T$. The smoothness assumption on $U_0(q,t)$ for $|q|>\rho$ ensures smoothness of q. This completes the proof of Theorem 6. □

In figure 1, the gray regions represent the sets of (T,ρ) satisfying the inequality

$$Q(T,\rho,k)\ =\ \kappa(T,\rho)+\frac{\rho^2}{2}\left|1+\frac{2k\pi}{T}\right|^2-\frac{3}{2}\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}}\ \geq\ 0$$

with $k=-1$ and $k=0$. The dashed curves are the graphs of $\rho=\left|1+\frac{2k\pi}{T}\right|^{-\frac{2}{3}}$. The sets of admissible (T,ρ) in Theorem 6 given by (4) are the gray regions beneath these dashed curves. Their shapes depends on how R_ρ varies with ρ. We will see in section 6 some concrete examples.

When $k=-1$, we see from the figure that the region is approximately within the strip $T\in[5.3,7.2]$. When T is close to 2π, there is plentiful room for admissible ρ, and consequently the action minimizer obtained in Theorem 6 stays far away from the origin. More precisely, given a compact interval $[T_-,T_+]$ with $T_->0$, one can verify that

$$s_{\min}=\rho+f(T,\rho)\quad\text{and}\quad\kappa(T,\rho)=\frac{1}{\rho}+g(T,\rho)$$

where $f(T,\rho)=O(\rho^{-2})$ and $g(T,\rho)=O(\rho^{-2})$ as $\rho\to\infty$, uniformly for $T\in[T_-,T_+]$. This can be easily seen by applying implicit function theorem to $(s_{\min}-\rho)/T^2-s_{\min}^{-2}=0$ or, equivalently, to the identity

$$\rho^2(s_{\min}-\rho)+2\rho(s_{\min}-\rho)^2+(s_{\min}-\rho)^3=T^2.$$

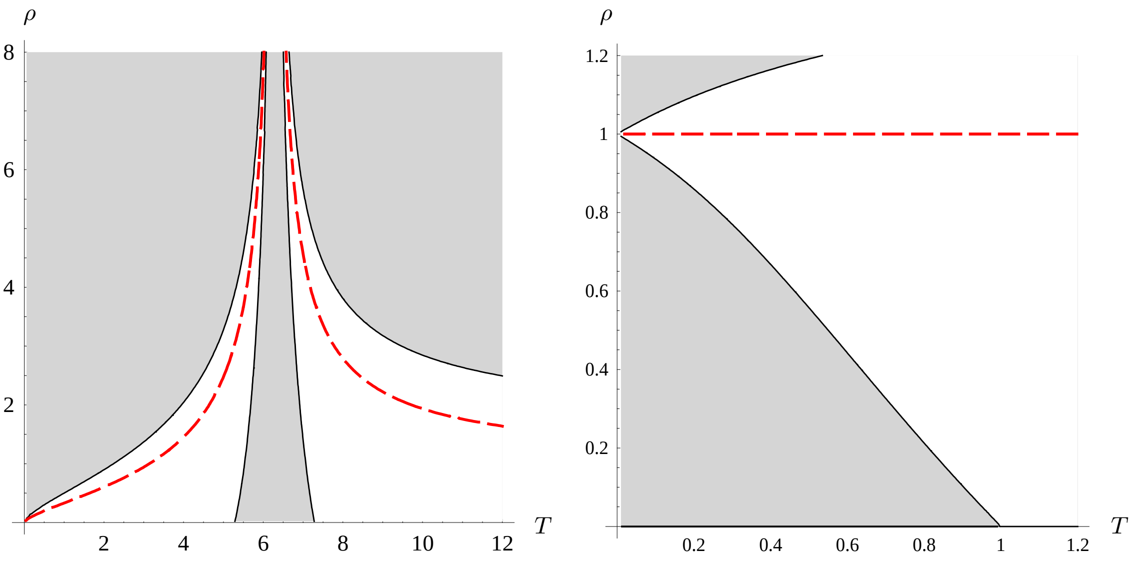

Figure 1. Admissible T and ρ with $k = -1$ and $k = 0$.

By assumption (A_1) we have $R_\rho = o(\rho^{-1})$ as $\rho \to \infty$, and thus one may choose $\rho > 0$ and $\epsilon = \epsilon(\rho)$ such that (4) holds for any $T \in [2\pi - \epsilon, 2\pi + \epsilon] \setminus \{2\pi\}$.

Remark 9. The right-hand side of the inequality in Lemma 8 can be replaced by a larger one, and inequalities (4), (5), (6) can therefore be replaced by larger ones. Indeed, the equality in Lemma 8 holds if and only if

(i) $q(t) = q(0)e^{it(1+2k\pi/T)}$ (and in particular $\underline{r} = \overline{r}$),
(ii) $U_0(re^{i\theta}, 0) = -R_{\underline{r}}$ for any $\theta \in \mathbb{R}$,
(iii) $\kappa(T, \underline{r}) = 1/\underline{r}$.

The last condition is never fulfilled. By definition $\kappa(T, \underline{r})$ is the minimum value of

$$f(s, T, \underline{r}) = \frac{1}{2T^2}(s - \underline{r})^2 + \frac{1}{s}$$

over $s > 0$. But

$$f(\underline{r}, T, \underline{r}) = \frac{1}{\underline{r}}, \quad \frac{\partial f}{\partial s}(\underline{r}, T, \underline{r}) = -\frac{1}{\underline{r}^2} < 0.$$

This proves that $\kappa(T, \underline{r}) < 1/\underline{r}$.

5. An Upper Bound Estimate for Satellite Orbits

Suppose $q \in \Gamma^{(k)}_{T,\rho}$ is an action-minimizing satellite orbit as obtained in Theorem 6. In the previous section (Lemma 8) we have used the quantities

$$\underline{r} \;=\; \min_{[0,T]} |q(t)|, \quad \overline{r} \;=\; \max_{[0,T]} |q(t)|$$

to deduce a lower bound estimate for $\mathcal{A}_T(q)$ which implicitly gives a lower and upper bound for $\underline{r}$. The $\eta = \eta(T)$ in (7) provides a lower bound for the $\underline{r}$, with which we obtain a criterion for q to be a regular orbit provided that the realm of primaries is within distance η from the origin. The purpose of this section is to provide a simple upper bound estimate for the maximum distance $\overline{r}$. Our intention is to deduce a simple upper bound for $\overline{r}$, the estimate is rather crude for the sake of brevity. Better estimates can be easily achieved by refining the calculations in this section.

Before moving further, let us remark here some ideas behind the inequality (6). The inequality fails to hold without the term $\kappa(T,\rho)$. This auxiliary function is introduced for two reasons: on the one hand it is a "penalty" for q to move along radial directions and stay close to the origin, so that (6) can be valid for certain T and ρ when $\underline{r}$ is relatively small; on the other hand it removes the explicit dependence of $\overline{r}$ from the lower bound estimate for $\mathcal{A}_T(q)$ in Lemma 8. The calculations in this section are variants and extensions of some calculations in the previous section; here we bring back the terms involving $\overline{r}$ and use them to find an upper bound for $\overline{r}$.

By Lemma 7 and Lemma 8, and the simple fact that R_ρ is decreasing in ρ, we have

$$\frac{\underline{r}^2}{2}\left|1+\frac{2k\pi}{T}\right|^2 \;\leq\; \frac{3}{2}\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}}+2R_\eta,$$

yielding an upper bound for $\underline{r}$:

$$\underline{r} \;\leq\; \left(3\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}}+4R_\eta\right)^{\frac{1}{2}}\left|1+\frac{2k\pi}{T}\right|^{-1} \;=:\; g(T,\eta,k). \tag{8}$$

This, of course, depends on the error term U_0 from the point-mass potential.

Following the proof of Lemma 8 but without using $\kappa(T,\rho)$, one easily obtains

$$\begin{aligned}\frac{1}{T}\mathcal{A}_T(q) \;&\geq\; \frac{\underline{r}^2}{2}\left|1+\frac{2k\pi}{T}\right|^2+\frac{1}{2T^2}(\overline{r}-\underline{r})^2-R_{\underline{r}}\\ &\geq\; \frac{\eta^2}{2}\left|1+\frac{2k\pi}{T}\right|^2+\frac{1}{2T^2}(\overline{r}-\underline{r})^2-R_\eta.\end{aligned}$$

Applying Lemma 7 again, then

$$\overline{r}-\underline{r} \;\leq\; T\left(3\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}}-\eta^2\left|1+\frac{2k\pi}{T}\right|^2+4R_\eta\right)^{\frac{1}{2}} \;=:\; h(T,\eta,k). \tag{9}$$

As an addendum to Theorem 6, from inequalities (8) and (9) we conclude that:

Proposition 10. *Let $q\in\Gamma^{(k)}_{T,\rho}$ be an action-minimizing satellite orbit obtained in Theorem 6. Let $\rho=\eta$ and let functions $g(T,k,\eta)$ and $h(T,k,\eta)$ be as in (8), (9). Then*

$$\overline{r} \;\leq\; g(T,\eta,k)+h(T,\eta,k).$$

We remark here that $g(T,\eta,k)+h(T,\eta,k)$ is decreasing in η. This relates the simple fact that better upper bound estimates for $\underline{r}$ lead to better upper bound estimates for $\overline{r}$. In practice we don't have explicit formulation for η but we may find some approximated $\eta^*<\eta$. Replacing ρ by η^* instead of η, then the inequality

$$\overline{r} \;\leq\; g(T,\eta^*,k)+h(T,\eta^*,k)$$

provides a less sharp upper bound estimate for $\overline{r}$.

6. THE CIRCULAR RESTRICTED n-BODY PROBLEM

Consider the circular restricted three-body problem with primaries $m_1=1-\mu$, $m_2=\mu$ orbiting on $\mathbb{C}$ around the origin with angular velocity $\omega=1$. Let x_1 and x_2 be their positions. Equations of motion for this two-body system is

$$\ddot{x}_i \;=\; \frac{m_j(x_j-x_i)}{|x_j-x_i|^3},\quad \{i,j\}=\{1,2\}.$$

We now fix a solution for this problem

$$x_1(t) = \mu e^{it}, \;\; x_2(t) = -(1-\mu)e^{it}$$

and look for satellite orbits around this system.

The equation of motion for the satellite q is

$$\ddot{q} \;=\; \frac{m_1(x_1-q)}{|x_1-q|^3} + \frac{m_2(x_2-q)}{|x_2-q|^3} \;=\; \frac{\partial}{\partial q} U(q,t),$$

where

$$U(q,t) \;=\; \frac{1-\mu}{\sqrt{|q|^2+\mu^2-2\mu|q|\cos(\theta-t)}} + \frac{\mu}{\sqrt{|q|^2+(1-\mu)^2+2(1-\mu)|q|\cos(\theta-t)}}.$$

Here θ is the argument of q. Let

$$\begin{aligned} \Delta_1 &= \left(\frac{\mu}{|q|}\right)^2 - \frac{2\mu}{|q|}\cos(\theta-t), \\ \Delta_2 &= \left(\frac{1-\mu}{|q|}\right)^2 + \frac{2(1-\mu)}{|q|}\cos(\theta-t). \end{aligned}$$

Then

$$U(q,t) \;=\; \frac{(1-\mu)}{|q|}(1+\Delta_1)^{-\frac{1}{2}} + \frac{\mu}{|q|}(1+\Delta_2)^{-\frac{1}{2}} \;=\; \frac{1}{|q|} + U_0(q,t).$$

The error term $U_0(q,t)$ can be written

$$\begin{aligned} U_0(q,t) \;=\;& \frac{(1-\mu)}{|q|}\left[-1+\frac{\Delta_1}{2}+(1+\Delta_1)^{-\frac{1}{2}}\right] + \frac{\mu}{|q|}\left[-1+\frac{\Delta_2}{2}+(1+\Delta_2)^{-\frac{1}{2}}\right] \\ & -\frac{1}{2|q|}\left[(1-\mu)\Delta_1+\mu\Delta_2\right]. \end{aligned}$$

Take $\rho_1 = 2+\sqrt{6}$. Observe that

$$\frac{1-\mu}{|q|}, \; \frac{\mu}{|q|} \;\le\; \frac{1}{|q|} \;\le\; \sqrt{\frac{3}{2}} - 1 \;\approx\; 0.2247 \quad \text{provided } \; |q| \ge \rho_1.$$

For each i, we have

$$\begin{aligned} |\Delta_i| \;&=\; \left|\left(\frac{|x_i|}{|q|}\right)^2 \pm \frac{2|x_i|}{|q|}\cos(\theta-t)\right| \;\le\; \left(\frac{1}{|q|}\right)\left(\frac{1}{|q|}+2\right) \\ &\le\; \frac{1}{|q|}\left(\sqrt{\frac{3}{2}}+1\right) \;\le\; \frac{1}{2} \quad \text{provided } \; |q| \ge \rho_1. \end{aligned} \tag{10}$$

Using this upper bound estimate for $|\Delta_i|$ together with the elementary inequality

$$\left|1-\frac{s}{2}-(1+s)^{-\frac{1}{2}}\right| \;<\; s^2 \quad \text{for any } s \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

for $|q| \geq \rho_1$ we have

$$\begin{aligned} |U_0(q,t)| &\leq \frac{(1-\mu)}{|q|}\Delta_1^2 + \frac{\mu}{|q|}\Delta_2^2 + \frac{1}{2|q|}\,|(1-\mu)\Delta_1 + \mu\Delta_2| \\ &\leq \frac{1}{|q|^3}\left(\sqrt{\frac{3}{2}}+1\right)^2 + \frac{\mu(1-\mu)}{2|q|^3} \\ &\leq \frac{1}{|q|^3}\left(\frac{21}{8}+\sqrt{6}\right). \end{aligned}$$

Thus

$$R_\rho \;\leq\; \frac{1}{\rho^3}\left(\frac{21}{8}+\sqrt{6}\right) \;\approx\; \frac{5.0745}{\rho^3} \quad \text{for } \rho \geq 2+\sqrt{6} \approx 4.4495.$$

In the first graph of figure 2, the thin lobe between the two dashed curves is contained in the region of (T,ρ) satisfying (4). Its boundary is the lower branch of the zero level of

$$\kappa(T,\rho) + \frac{\rho^2}{2}\left|1+\frac{2k\pi}{T}\right|^2 - \frac{3}{2}\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}} - \frac{2}{\rho^3}\left(\frac{21}{8}+\sqrt{6}\right).$$

The portion of the lobe above the line $\rho = 2+\sqrt{6}$ is a set of admissible (T,ρ) satisfying all required conditions in Theorem 4. The projection $[T_1, T_2]$ of this region on the T-axis is roughly $[6.14, 6.43]$, a (not too) small neighborhood of 2π. Theorem 4 ensures that, whenever $T \in [T_1, 2\pi)$ (resp. $(2\pi, T_2]$), there exists a direct (resp. retrograde) satellite orbit with relative period T which winds around the binary and its distance from the mass center is at least $2+\sqrt{6}$.

In the second graph of figure 2, where the time interval is scaled to $[T_1, T_2]$, the solid curve represents the graph of a function $\eta^* \leq \eta$, where η as a function of T on $[T_1, T_2]$ is implicitly defined by (7), and η^* is similarly defined except the R_ρ in (6) is replaced by the larger term $\frac{1}{\rho^3}\left(\frac{21}{8}+\sqrt{6}\right)$. As observed at the end of section 5, the dashed curve given by

$$m^*(T) := g(T,\eta^*(T),-1) + h(T,\eta^*(T),-1), \quad T \in [T_1, T_2]$$

provides upper bound estimates for $\overline{r}$. From this figure we see that the action-minimizing satellites are extremely far from the primaries only if T is extremely close to 2π. For instance, when $T = 6.2$, from the figure we have $\eta^* \approx 14.2$ as a lower bound for $\underline{r}$ and $m^*(6.2) \approx 33.5$ as an upper bound for $\overline{r}$. The motion of the action-minimizing satellite orbit is confined to the circular ring with inner radius η^* and outer radius $m^*(6.2)$.

Remark 11. Take a sequence T_j of relative periods converging to 2π such that

$$m^*(T_j) \;<\; \eta(T_j) \quad \text{for each } j.$$

Let $q^{(j)}$ be an action-minimizing satellite orbit with relative period T_j. The satellite orbits $q^{(j)}$ can coexist without collisions since the circular rings they correspond are pairwise disjoint.

Following the same idea, we can show the existence of direct and retrograde satellite orbits around any other kind of relative equilibria. Now we consider another example; the arguments and calculations for other examples are similar.

Consider the circular restricted four-body problem with primaries $m_1 = 1-2\mu$, $m_2 = m_3 = \mu \in [0, 1/2]$ orbiting on $\mathbb{C}$ around the origin with angular velocity $\omega = 1$. Let x_1, x_2, x_3 be their positions. Equations of motion for this three-body system is

$$\ddot{x}_i \;=\; \frac{m_j(x_j - x_i)}{|x_j - x_i|^3} + \frac{m_k(x_k - x_i)}{|x_k - x_i|^3}, \quad \{i,j,k\} = \{1,2,3\}. \tag{11}$$

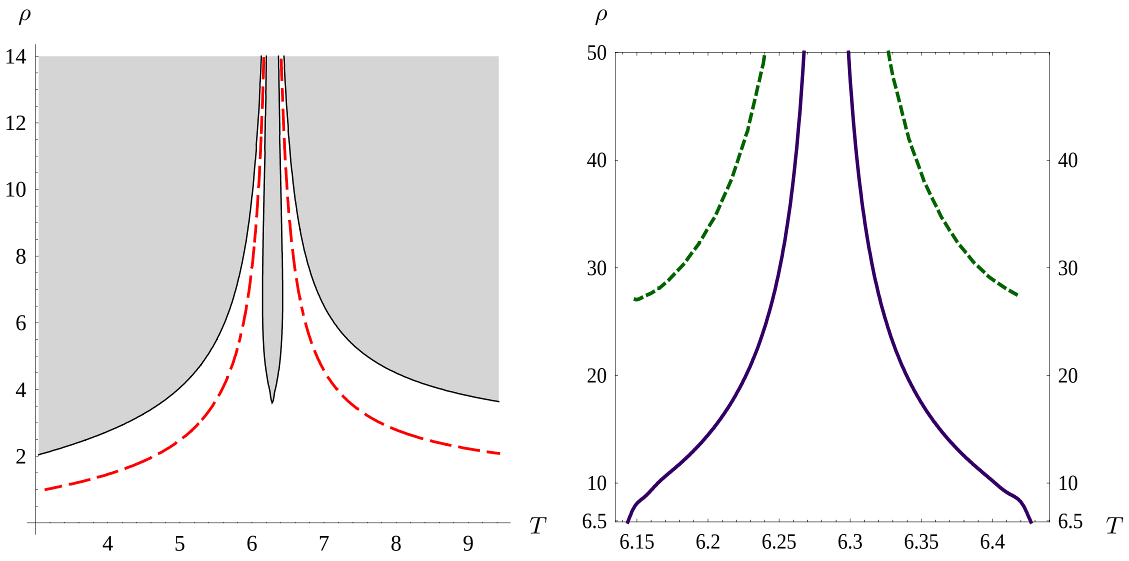

FIGURE 2. The first graph: admissible T and ρ with $k=-1$. The second graph: lower bounds for $\underline{r}$ and upper bounds for $\overline{r}$

The following circular equilateral solution, whose mutual distances are all equal to one, generates a gravitation field which satisfies assumptions (A_1) and (A_2).

$$x_1(t)=\sqrt{3}\mu e^{it},\ \ x_2(t)=(\sqrt{3}\mu-\frac{\sqrt{3}}{2}+\frac{i}{2})e^{it},\ \ x_3(t)=(\sqrt{3}\mu-\frac{\sqrt{3}}{2}-\frac{i}{2})e^{it}.$$

Now we look for direct and retrograde satellite orbits around this solution.

The equation of motion for the satellite q is

$$\ddot{q} \quad = \quad \frac{m_1(x_1-q)}{|x_1-q|^3}+\frac{m_2(x_2-q)}{|x_2-q|^3}+\frac{m_3(x_3-q)}{|x_3-q|^3} \quad = \quad \frac{\partial}{\partial q}U(q,t),$$

where

$$U(q,t) \quad = \quad \frac{1-2\mu}{|x_1-q|}+\frac{\mu}{|x_2-q|}+\frac{\mu}{|x_3-q|}.$$

Let $\theta_i(t)$ be the argument of $x_i(t)$ and $\theta(t)$ be the argument of $q(t)$. As in the case of two bodies, let

$$\Delta_i \quad = \quad \left(\frac{|x_i|}{|q|}\right)^2-\frac{2|x_i|}{|q|}\cos(\theta-\theta_i), \quad i=1,2,3.$$

Then

$$U(q,t) \quad = \quad \frac{(1-2\mu)}{|q|}(1+\Delta_1)^{-\frac{1}{2}}+\frac{\mu}{|q|}(1+\Delta_2)^{-\frac{1}{2}}+\frac{\mu}{|q|}(1+\Delta_3)^{-\frac{1}{2}} \quad = \quad \frac{1}{|q|}+U_0(q,t).$$

The error term $U_0(q,t)$ can be written

$$\begin{aligned} U_0(q,t) \quad = \quad & \frac{(1-2\mu)}{|q|}\left[-1+\frac{\Delta_1}{2}+(1+\Delta_1)^{-\frac{1}{2}}\right]+\frac{\mu}{|q|}\left[-1+\frac{\Delta_2}{2}+(1+\Delta_2)^{-\frac{1}{2}}\right] \\ & +\frac{\mu}{|q|}\left[-1+\frac{\Delta_3}{2}+(1+\Delta_3)^{-\frac{1}{2}}\right]-\frac{1}{2|q|}\left[(1-2\mu)\Delta_1+\mu\Delta_2+\mu\Delta_3\right]. \end{aligned}$$

We have seen that for any q with $|q| \geq \rho_1 = 2 + \sqrt{6}$, the inequality (10) supplies upper bounds for each Δ_i. Proceed as before, for $|q| \geq \rho_1$ we have

$$\begin{aligned} |U_0(q,t)| &\leq \frac{(1-2\mu)}{|q|}\Delta_1^2 + \frac{\mu}{|q|}\Delta_2^2 + \frac{\mu}{|q|}\Delta_3^2 + \frac{1}{2|q|}\left|(1-2\mu)\Delta_1 + \mu\Delta_2 + \mu\Delta_3\right| \\ &\leq \frac{1}{|q|^3}\left(\sqrt{\frac{3}{2}} + 1\right)^2 + \frac{\mu(2-3\mu)}{2|q|^3} \\ &\leq \frac{1}{|q|^3}\left(\frac{8}{3} + \sqrt{6}\right). \end{aligned}$$

Note that in above, using the fact that the mass center is the origin, the term $(1-2\mu)\Delta_1 + \mu\Delta_2 + \mu\Delta_3$ is exactly the moment of inertia of the system devided by $|q|^2$. Now an estimate similar to the previous case is obtained:

$$R_\rho \leq \frac{1}{\rho^3}\left(\frac{8}{3} + \sqrt{6}\right) \approx \frac{5.1162}{\rho^3} \quad \text{for } \rho \geq 2 + \sqrt{6} \approx 4.4495.$$

Here we do not produce the figure for the region of admissible (T, ρ) resulted from this estimate because it looks almost identical to the one in figure 2. Its projection $[T_1, T_2]$ on the T-axis is also roughly $[6.14, 6.43]$. As the restricted three-body problem, we now draw the same conclusion from Theorem 4 with the binary replaced by the Lagrange equilateral solution.

Remark 12. In a system with one primary circling in a small neighborhood of the origin, direct and retrograde satellite orbits near this primary also exist since the potential field due to other remote primaries can be regarded as small perturbations of the point-mass potential due to this primary. Consider for instance the Eulerian relative equilibrium with primaries m_1, $m_3 \ll m_2$, initially aligned on the x-axis in the order $x_3 < x_2 < x_1$. The potential field near m_2 is close to the point-mass potential due to m_2. In addition to the lower bound estimate ρ for action-minimizing satellite orbits from Theorem 6, in this case a satisfactory upper bound estimate, such as the one given by Proposition 10, for the distance between action-minimizing satellite orbit and m_2 is also needed. The measurement R_ρ of the error term should be modified to the supremum of $|U_0(q,t)|$ over some annulus.

Remark 13. It would be interesting to extend our main theorems to cases where the potentials satisfy a periodic condition weaker than (A_2). For instance, satellite orbits around elliptical Keplerian orbits, which were obtained by power series methods when a mass ratio of primaries is nearly zero [1, 3], can probably be characterized as action-minimizers in suitable loop spaces. The main difficulty in the generalization will be due to the fact that action-minimizing periodic satellite orbits, if exist, are probably those with very long periods obtained by the fixed point method [5, 20].

7. Potentials with Additional Symmetries

When the gravitational force field has additional symmetries, the estimates in Theorem 6 can be improved, as we shall see in this section.

Fix $m \in \mathbb{N}$, assume the following dihedral symmetry assumption:

(D_m) (D_m-Symmetry) For any $q \in \mathbb{C}^\times$, $U_0(q,0) = U_0(\bar{q},0) = U_0(e^{\frac{2\pi i}{m}}q, 0)$.

For example, if the periodic potential is generated by a Lagrange's equilateral relative equilibrium with equal masses, then the time variable or coordinates can be chosen properly so that the potential satisfies the D_3-symmetry (the special solution (11) with $\mu = 1/3$); if the periodic potential is generated by some Euler's collinear relative equilibrium with two equal masses on two sides, then D_2-symmetry is fulfilled. The potential illustrated in section 2,

$$U_0(q,t) = \frac{(q_1^2 - q_2^2)\cos 2t + 2q_1q_2\sin 2t}{(q_1^2+q_2^2)^{5/2}},$$

is another example of potential satisfying D_2-symmetry.

Given $T>0$, $m\in\mathbb{N}$, and $k\in\mathbb{Z}$. Consider the function space

$$\Sigma_T^{m,k} \;=\; \{q\in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}) :\; e^{-i(\frac{T}{m}+\frac{2k\pi}{m})}q(t+\frac{T}{m}) = q(t) \;\text{ for any }\; t\}.$$

This is a weakly closed subspace of Λ_T and $\Sigma_T^1 = \Lambda_T$. As Corollary 3, it follows immediately from Proposition 2 that

Corollary 14. *Given T, $\rho>0$, $k\in\mathbb{Z}$, and $m\in\mathbb{N}$. Assume (A_1) and (A_2) hold. The action functional $\mathcal{A}_T$ restricted to $\Gamma^{(k)}_{T,\rho}\cap\Sigma_T^{m,k}$ attains its infimum provided $T\neq -2k\pi$.*

The special path selected in the proof of Lemma 7 is a circular path which clearly belongs to Σ_T^m for any $m\in\mathbb{N}$. Therefore Lemma 7 can be also written as follows.

Lemma 15. *Given T, $\rho>0$, $k\in\mathbb{Z}$, $m\in\mathbb{N}$. Assume (A_1) and (A_2) hold. Suppose $T\neq-2k\pi$ and $\rho\le\left|1+\frac{2k\pi}{T}\right|^{-\frac{2}{3}}$. Then*

$$\begin{aligned}\frac{1}{T}\inf_{\Gamma^{(k)}_{T,\rho}\cap\Sigma_T^{m,k}}\mathcal{A}_T &\le \frac{3}{2}\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}} + R_\rho,\\ \frac{1}{T}\inf_{\Gamma_{T,\rho}\cap\Sigma_T^{m,k}}\mathcal{A}_T &\le \frac{3}{2}\left(\min_{k\in\mathbb{Z}}\left|1+\frac{2k\pi}{T}\right|\right)^{\frac{2}{3}} + R_\rho.\end{aligned}$$

The next lemma is similar to Lemma 8.

Lemma 16. *Given T, $\rho>0$, $k\in\mathbb{Z}$, $m\in\mathbb{N}$. Assume (A_1) and (A_2) hold. If $q\in\Gamma^{(k)}_{T,\rho}\cap\Sigma_T^{m,k}$, $\min\limits_{[0,T]}|q(t)| = \underline{r}\ge\rho$, then*

$$\frac{1}{T}\mathcal{A}_T(q) \;\ge\; \frac{\underline{r}^2}{2}\left|1+\frac{2k\pi}{T}\right|^2 + \kappa(\frac{T}{m},\underline{r}) - R_\rho.$$

Proof. Writing $(x,y) = (r\cos\theta, r\sin\theta)$, $\overline{r} = \max\limits_{t\in[0,T]}|q(t)|$, as in the proof of Lemma 8. Let $\overline{r} = \max\limits_{t\in[0,T]}|r(t)|$. Since $q\in\Sigma_T^{m,k}$,

$$\begin{aligned}\mathcal{A}_T(q) &= \frac{1}{2}\int_0^T r^2(1+\dot\theta)^2dt + \frac{m}{2}\int_0^{\frac{T}{m}}\dot r^2dt + \int_0^T\frac{1}{r} + U_0(r\cos\theta + ir\sin\theta, 0)\,dt\\ &\ge \frac{\underline{r}^2}{2}\int_0^T(1+\dot\theta)^2dt + \frac{m^2}{2T}\left(\int_0^{\frac{T}{m}}|\dot r|dt\right)^2 + \frac{T}{\overline{r}} - R_{\underline{r}}T\\ &\ge \frac{\underline{r}^2}{2T}\left(\int_0^T|1+\dot\theta|dt\right)^2 + \frac{m^2}{2T}\left(\overline{r}-\underline{r}\right)^2 + \frac{T}{\overline{r}} - R_{\underline{r}}T\\ &\ge \frac{\underline{r}^2}{2T}\left(T+2k\pi\right)^2 + T\kappa(\frac{T}{m},\underline{r}) - R_{\underline{r}}T.\end{aligned}$$

This implies the lower bound estimate we claimed. $\square$

Combining estimates in Lemma 15 and Lemma 16, and following the proof of Theorem 6, we obtain a generalization of Theorem 6:

Theorem 17. *Given* T, $\rho > 0$, $k \in \mathbb{Z}$, $m \in \mathbb{N}$. *Assume* (A_1), (A_2), (D_m) *hold. Suppose* $T \neq -2k\pi$, $\rho \leq \left|1 + \frac{2k\pi}{T}\right|^{-\frac{2}{3}}$, $U_0(q,t)$ *is of class* C^1 *whenever* $|q| > \rho$. *If*

$$\frac{3}{2}\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} + 2R_\rho \quad < \quad \kappa(\frac{T}{m}, \rho) + \frac{\rho^2}{2}\left|1 + \frac{2k\pi}{T}\right|^2, \tag{12}$$

then there exists a relative periodic solution q *to (1) which minimizes* $\mathcal{A}_T$ *on* $\Gamma^{(k)}_{T,\rho} \cap \Sigma^{m,k}_T$ *and* $|q(t)| > \rho$ *for any* $t \in \mathbb{R}$.

The special cases $k = -1$ and $k = 0$ generalize Theorems 4 and 5:

Theorem 18 (**Existence of Direct and Retrograde Satellite Orbits**)**.** *Given* T, $\rho > 0$, $m \in \mathbb{N}$, $T \neq 2\pi$. *Assume* (A_1), (A_2), (D_m) *hold. Suppose* $\rho \leq \left|1 - \frac{2\pi}{T}\right|^{-\frac{2}{3}}$ *and* $U_0(q,t)$ *is of class* C^1 *whenever* $|q| > \rho$. *If*

$$\frac{3}{2}\left|1 - \frac{2\pi}{T}\right|^{\frac{2}{3}} + 2R_\rho \quad < \quad \kappa(\frac{T}{m}, \rho) + \frac{\rho^2}{2}\left|1 - \frac{2\pi}{T}\right|^2, \tag{13}$$

then there exists a classical solution q *of (1) which minimizes* $\mathcal{A}_T$ *on* $\Gamma^{(-1)}_{T,\rho} \cap \Sigma^{m,-1}_T$ *and* $|q(t)| > \rho$ *for any* $t \in \mathbb{R}$. *The satellite orbit* q *is retrograde if* $T \in (0, 2\pi)$*; it is direct if* $T > 2\pi$.

Theorem 19 (**Existence of Satellite Orbits near Relative Equilibria**)**.** *Given* T, $\rho > 0$, $m \in \mathbb{N}$. *Assume* (A_1), (A_2), (D_m) *hold. Suppose* $\rho \leq 1$ *and* $U_0(q,t)$ *is of class* C^1 *whenever* $|q| > \rho$. *If*

$$\frac{3}{2} + 2R_\rho \quad < \quad \kappa(\frac{T}{m}, \rho) + \frac{\rho^2}{2}, \tag{14}$$

then there exists a classical direct solution q *of (1) which minimizes* $\mathcal{A}_T$ *on* $\Gamma^{(0)}_{T,\rho} \cap \Sigma^{m,0}_T$ *and* $|q(t)| > \rho$ *for any* $t \in \mathbb{R}$.

Following the arguments in section 5, the inequality (9) can be easily modified to

$$\overline{r} - \underline{r} \quad \leq \quad \frac{T}{m}\left(3\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} - \eta^2\left|1 + \frac{2k\pi}{T}\right|^2 + 4R_\eta\right)^{\frac{1}{2}} \quad =: \quad h_m(T, \eta, k). \tag{15}$$

Combining with inequalities (8), we have the following refined upper bound estimate for $\overline{r}$:

Proposition 20. *Let* $q \in \Gamma^{(k)}_{T,\rho} \cap \Sigma^{m,k}_T$ *be an action-minimizing satellite orbit obtained in Theorem 17. Let* η *be as in (7), set* $\rho = \eta$, *and let functions* $g(T, k, \eta)$ *and* $h_m(T, k, \eta)$ *be as in (8), (15). Then*

$$\overline{r} \quad \leq \quad g(T, \eta, k) + h_m(T, \eta, k).$$

As remarked at the end of section 5, the function $g(T, \eta, k) + h_m(T, \eta, k)$ is decreasing in η. Clearly a sharper upper bound estimate $g(T, \eta^*, k)$ for $\underline{r}$ yields a sharper upper bound estimate for $\overline{r}$. In practice we don't have explicit formulation for η but we may find some approximated $\eta^* < \eta$. By taking $\rho = \eta^*$ instead of η, the inequality

$$\overline{r} \quad \leq \quad g(T, \eta^*, k) + h_m(T, \eta^*, k)$$

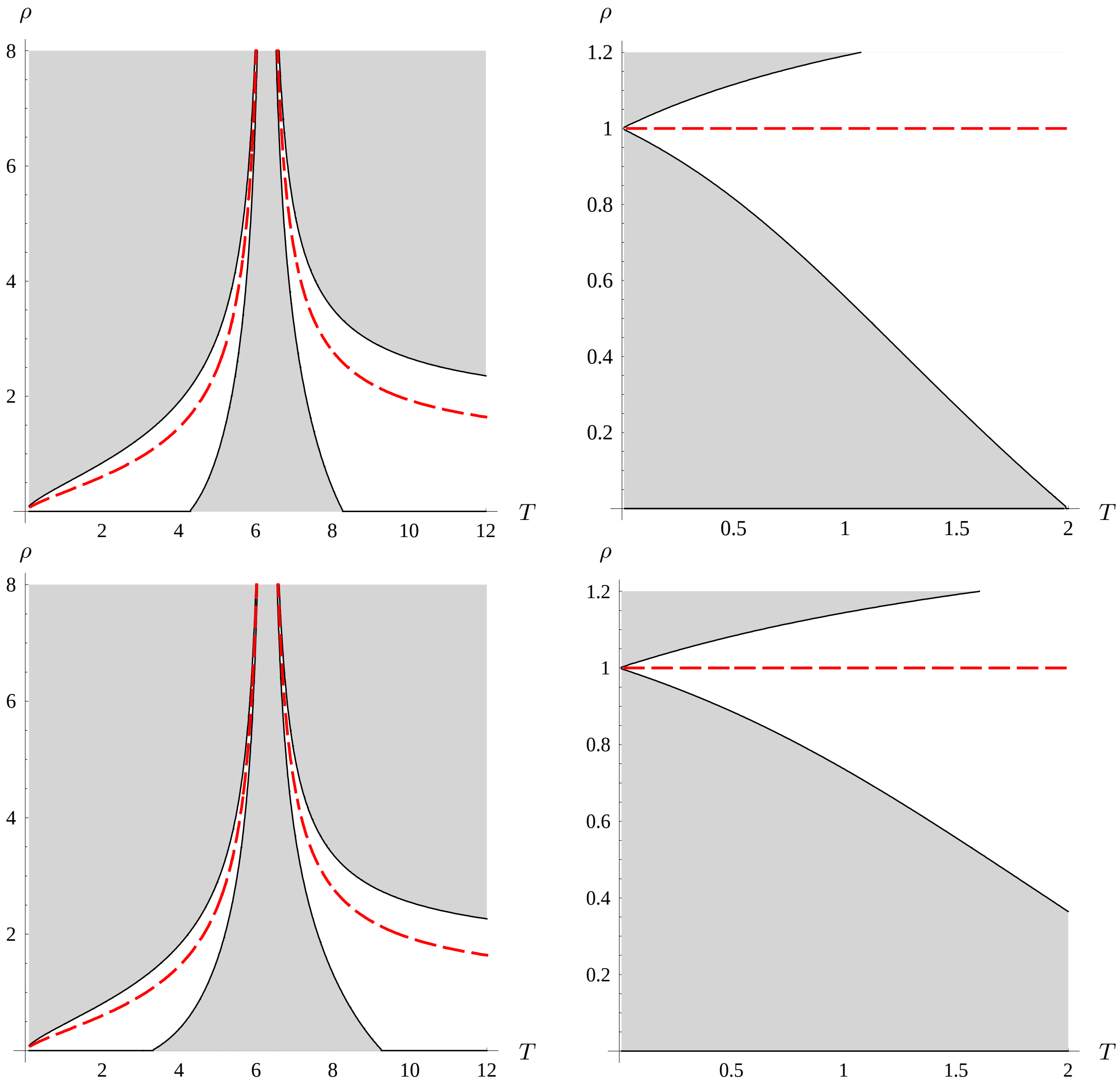

FIGURE 3. Admissible T and ρ with $k=-1$ and $k=0$, $m=2$ and $m=3$.

provides a less sharp upper bound estimate for $\bar{r}$.

In figure 3, the gray regions are the sets of (T,ρ) satisfying the inequality

$$Q_m(T,\rho,k) \;=\; \kappa(\frac{T}{m},\rho)+\frac{\rho^2}{2}\left|1+\frac{2k\pi}{T}\right|^2-\frac{3}{2}\left|1+\frac{2k\pi}{T}\right|^{\frac{2}{3}} \;\geq\; 0$$

with $k=-1$ and $k=0$. The top two graphics are for $m=2$ and the other two are for $m=3$. As in figure 1, the dashed curves are the graphs of $\rho=\left|1+\frac{2k\pi}{T}\right|^{-\frac{2}{3}}$, and the branches of gray regions below the dashed curves are sets of (T,ρ) for which the inequality of Theorem 17 is valid for some $R_\rho>0$. Figure 3, in comparison with figure 1, shows the improvements of Theorem 17 from Theorem 6 when there are additional symmetries.

The sets of admissible (T,ρ) in Theorem 17 given by (12) are subsets of the gray regions beneath these dashed curves. Their shapes depends on how R_ρ varies with ρ. Theorem 18 applies to the example of restricted three-body problem in section 6 with $m=2$ if $\mu=1/2$ (the Strömgren problem), and the restricted four-body problem in section 6 with $m = 3$ if $\mu = 1/3$. The resulting regions of admissible (T, ρ) is slightly larger than the one in figure 2.

8. SATELLITE ORBITS NEAR AN ASTEROID

A model for spacecraft motion about an elliptical asteroid rotating with angular velocity $\omega = 1$ can be written, in the body-fixed frame $z = x + iy = e^{-it}q$,

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial V}{\partial y}, \end{aligned}$$

where the potential V takes the form

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\sqrt{x^2 + y^2}} + V_{20}(x, y) + V_{22}(x, y).$$

The terms V_{20} and V_{22} are some order $|q|^{-3}$ terms multiplied by some small gravity field coefficients C_{20} and C_{22}. See [15, 23] and references therein.

Discussions in section 6 clearly applies to this case. In practical problems, the error term R_ρ for an asteroid is often much smaller than those for circular restricted n-body problems, thus resulting in a much larger region of admissible (T, ρ). In this section, as applications of Theorem 17, we consider only cases with additional symmetries.

The error term U_0 in (2) is uniquely determined by (A_2) and

$$U_0(x + iy, 0) = V_{20}(x, y) + V_{22}(x, y).$$

The simplest model for the restricted full two-body problem is

$$V_{20} = 0, \quad V_{22} = \frac{3C_{22}(x^2 - y^2)}{(x^2 + y^2)^{5/2}}.$$

The "ellipticity" coefficient C_{22} varies between 0 and 0.05 for physical systems [16]. We shall therefore assume the estimate

$$R_\rho \leq \frac{0.15}{\rho^3}.$$

We also assume that the asteroid is small enough, for instance it is assumed to be contained in the disc D_{ρ_0} centered at the origin with radius $\rho \leq 0.5$. The above estimate is assumed to hold for $\rho \geq \rho_0$.

Applying Theorems 18 and 19, the regions of (T, ρ) satisfying (13) and (14) contains the gray areas below the dashed curves as shown in in figure 4. The boundary of the gray region is the zero level of

$$\kappa(\frac{T}{m}, \rho) + \frac{\rho^2}{2}\left|1 + \frac{2k\pi}{T}\right|^2 - \frac{3}{2}\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} - \frac{0.3}{\rho^3}.$$

In the upper left graph ($k = -1$) of figure 4, the set of admissible (T, ρ) projected to the T-axis is roughly $[T_1, T_2] \approx [5.51, 7.12]$, a large neighborhood of 2π. This shows that direct and retrograde satellite orbits exist for a long range of T. The upper right graph shows the graph of η^* (solid curve), a lower bound for the $\eta \leq \underline{r}$ given by (7), and an upper bound m^* (dashed curve) for $\overline{r}$ given by

$$m^*(T) := g(T, \eta^*(T), -1) + h_m(T, \eta^*(T), -1), \quad T \in [T_1, T_2].$$

For each $T \in [T_1, T_2]$, the heights of the solid and dashed curves provide lower and upper bound estimates for $\underline{r}$, $\overline{r}$.

The lower two graphs ($k = 0$) corresponds to the relative equilibria or synchronous motions of the system with small $C_{22} \leq 0.005$, in which case the relative period T can be any small positive number. It is probable that action minimizers are some relative equilibria. There are at least two relative equilibria in our case, one is on the long axis and the other on the short axis of the elliptical asteroid. Their action values are in general different.

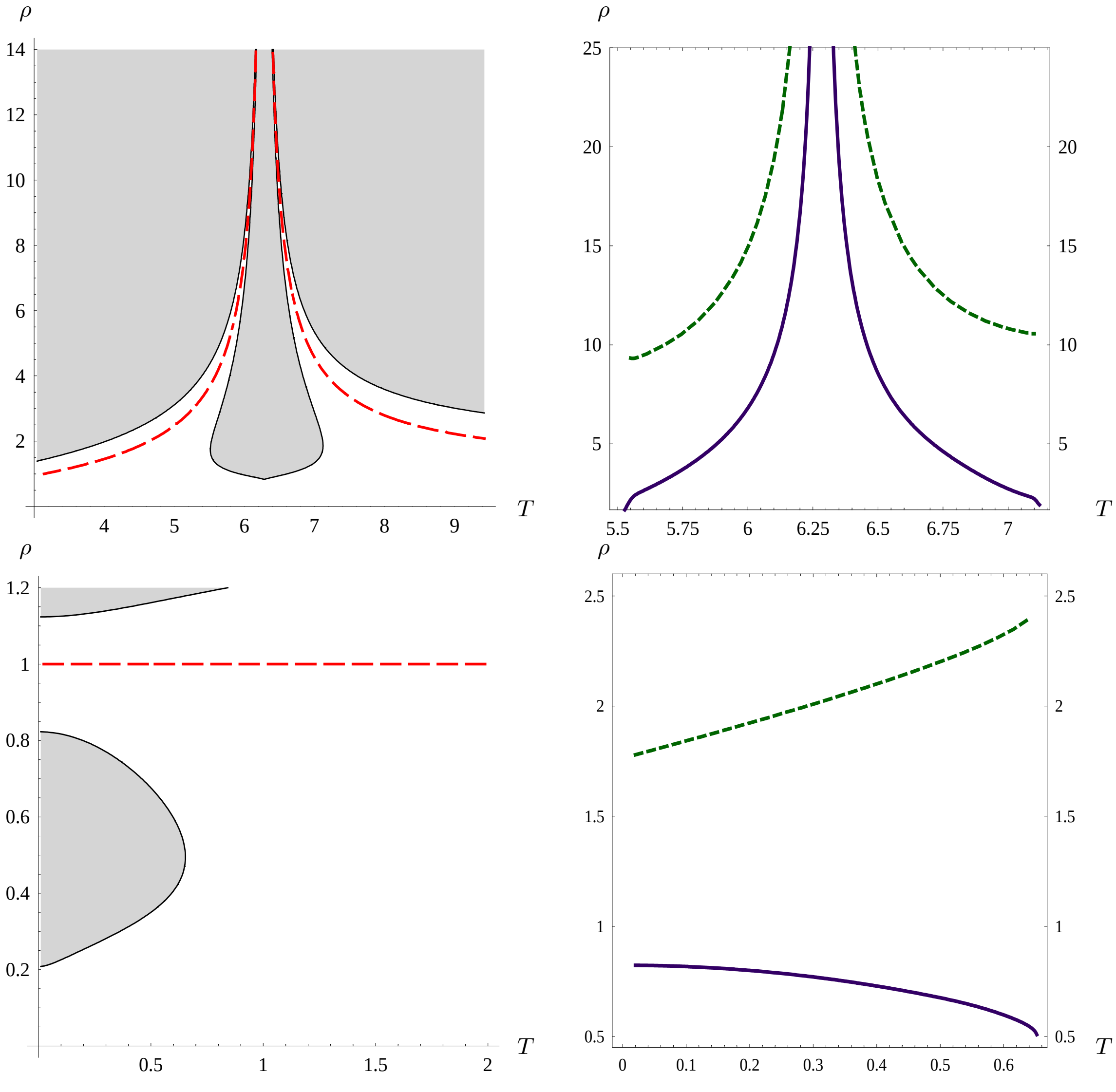

FIGURE 4. Admissible T and ρ with $k = -1$ and $k = 0$, $m = 2$.

As a corollary of Theorem 17, when $k = 1$ we have

Theorem 21 (**Existence of Fast Direct Satellite Orbits Around Small Asteroids**). *Given* T, $\rho > 0$, $m \in \mathbb{N}$. *Assume* (A_1), (A_2), (D_m) *hold. Suppose* $\rho \leq \left|1 + \frac{2\pi}{T}\right|^{-\frac{2}{3}}$ *and* $U_0(q, t)$ *is of class*

C^1 whenever $|q| > \rho$. If

$$\frac{3}{2}\left|1+\frac{2\pi}{T}\right|^{\frac{2}{3}} + 2R_\rho \quad < \quad \kappa(\frac{T}{m},\rho) + \frac{\rho^2}{2}\left|1+\frac{2\pi}{T}\right|^2, \tag{16}$$

then there exists a classical direct solution q of (1) which minimizes $\mathcal{A}_T$ on $\Gamma^{(1)}_{T,\rho} \cap \Sigma^{m,1}_T$ and $|q(t)| > \rho$ for any $t \in \mathbb{R}$.

This region of admissible (T,ρ) is nonempty for m sufficiently large and it is rather small in size. The lower branch of the gray areas in figure 5 shows the region of (T,ρ) which satisfies (16) for sufficiently small R_ρ. Here $m = 8$, a highly symmetric gravitation field is required. The satellite orbit is direct in both the inertia and the rotating frame.

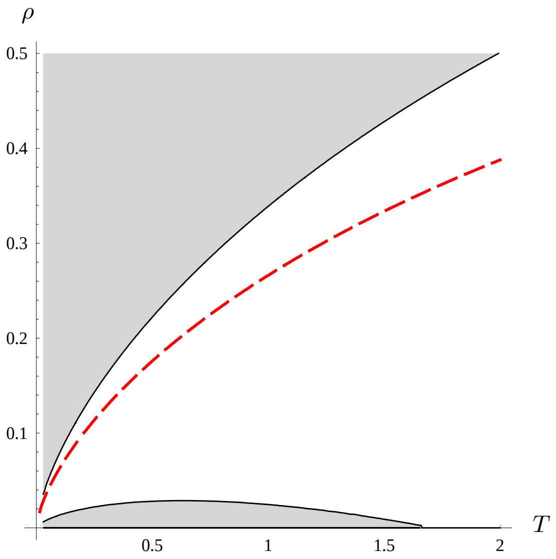

FIGURE 5. Admissible T and ρ with $k = 1$, $m = 8$.

Acknowledgement.
The author thanks the anonymous referee for making several valuable corrections and suggestions, especially on refining section 2 and on the remarks at the end of section 4. The research work is partially supported by the National Science Council and the National Center for Theoretical Sciences in Taiwan.

REFERENCES

[1] ARENSTORF, R. F., Periodic solutions of the restricted three-body problem representing analytic continuations of Keplerian elliptic motions. *Amer. J. Math.* 85 (1963), 27–35.

[2] ARIOLI, G.; GAZZOLA, F.; TERRACINI, S., Minimization properties of Hill's orbits and applications to some N-body problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17** (2000), 617–650.

[3] BARRAR, R. B., Existence of periodic orbits of the second kind in the restricted problem of three bodies. *Astronom. J.* **70** (1965), 3–4.

[4] BIRKHOFF, G. D., Dynamical systems with two degrees of freedom. *Trans. Amer. Math. Soc.* **18** (1917), 199–300.

[5] BIRKHOFF, G. D., Surface transformations and their dynamical applications. *Acta Math.* **43** (1922), 1–119.

[6] CHEN, K.-C., Action-minimizing orbits in the parallelogram four-body problem with equal masses. *Arch. Ration. Mech. Anal.* **158** (2001), 293–318.

[7] Chen, K.-C., Variational methods on periodic and quasi-periodic solutions for the N-body problem. *Ergodic Theory Dynam. Systems* **23** (2003), 1691-1715.

[8] Chen, K.-C., Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses. *Annals of Math.*, **167** (2008), 325–348.

[9] Chen, K.-C.; Lin, Y.-C., On action-minimizing retrograde and prograde orbits of the three-body problem. *Comm. Math. Phys.*, to appear.

[10] Chenciner, A., Action minimizing solutions in the Newtonian n-body problem: from homology to symmetry. *Proceedings of the International Congress of Mathematicians* (Beijing, 2002). Vol III, 279–294.

[11] Chenciner, A.; Montgomery, R., A remarkable periodic solution of the three-body problem in the case of equal masses. *Annals of Math.* **152** (2000), 881–901.

[12] Ferrario, D.; Terracini, S., On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, **155** (2004), 305–362.

[13] Gabern, F.; Koon, W.S.; Marsden, J.E.; Scheeres, D.J., Multiple gravity assists, capture, and escape in the restricted three-body problem. *SIAM J. Applied Dynamical Systems* **5** (2006), 252–279.

[14] Gordon, W., A minimizing property of Keplerian orbits, *Amer. J. Math.* **99** (1977), 961–971.

[15] Hu, W.; Scheeres, D. J., Spacecraft motion about slowly rotating asteroids. *J. Guidance, Control, and Dynamics.* **25** (2002), 765–775.

[16] Koon, W. S.; J. E. Marsden; S. Ross, M. Lo; D. J. Scheeres, Geometric mechanics and the dynamics of asteroid pairs, *Ann. N.Y. Acad. Sci.*, **1017** (2004), 11-38.

[17] Meyer, K. R., *Periodic solutions of the N-body problem.* Lecture Notes in Mathematics, Vol. 1719, Springer-Verlag, Berlin, 1999.

[18] Meyer, K. R., Periodic solutions of the N-body problem. *J. Differential Equations* **39** (1981), 2–38.

[19] Moeckel, R., A variational proof of existence of transit orbits in the restricted three-body problem. *Dyn. Syst.* **20** (2005), 45–58.

[20] Moser, J. Periodische Lösungen des restringierten Dreikörperproblems, die sich erst nach vielen Umlaufen schliessen. *Math. Ann.* **126** (1953), 325–335.

[21] Moulton, F. R., A class of periodic orbits of superior planets. *Trans. Amer. Math. Soc.* **13** (1912), 96–108.

[22] Marsden, J. E.; Ross, S. D., New methods in celestial mechanics and mission design. *Bull. Amer. Math. Soc.* **43** (2006), 43–73

[23] Scheeres, D.J., Dynamics about uniformly rotating tri-axial ellipsoids. Applications to asteroids. *Icarus* **110** (1994), 225–38.

[24] Seeber, G., *Satellite geodesy: foundations, methods, and applications.* Walter de Gruyter, 2003.

[25] Signorini, A., Esistenza di un'estremale chiusa dentro un contorno di Whittaker. *Rend. Circ. Math. Palermo* **33** (1912), 187–193.

[26] Siegel, C. L.; Moser, J. K., *Lectures on celestial mechanics.* Die Grundlehren der mathematischen Wissenschaften, Band 187. Springer-Verlag, New York-Heidelberg, 1971.

[27] Szebehely, V., *Theory of orbits – The restricted problem of three bodies.* Academic Press, New York, 1967.

[28] Whittaker, E. T., *Analytical dynamics.* Cambridge Univ. Press, 1904. Reprinted by Dover, New York, 1944.

[29] Wintner, A., Grundlagen einer Genealogie der periodischen Bahnen im restringierten Dreikörperproblem. *Math. Z.* **34** (1932), 321–402.

[30] Wintner, A., *The analytical foundation of celestial mechanics.* Princeton University Press, Princeton, NJ, 1941.

Department of Mathematics, National Tsing Hua University, Hsinchu 30013, Taiwan

E-mail address: `kchen@math.nthu.edu.tw`