1 Chromatic polynomials

Definition 1.1. Let $G$ be a graph. Denote the number of $n$-colorings of $G$ by $P(G,n)$.

Example 1.2. For $C_3$,

\[
P(C_3, 1) = 0, \quad P(C_3, 2) = 0, \quad P(C_3, 3) = 6, \\
P(C_3, 4) = 4 \cdot 3 \cdot 2, \quad P(C_3, n) = n(n-1)(n-2) \text{ for } n \in \mathbb{N}
\]

Example 1.3. $P(K_m, n) = n(n-1) \cdots (n-m+1)$

Definition 1.4. Let $G$ be a graph and $e = uv \in E(G)$. The contraction $G/e$ is the graph obtained from $G - e$ by first deleting $e$, identifying $u$ and $v$, then identifying multiple edges which have the same endpoints.

\[\text{Diagram of contraction} \]

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Theorem 1.5. (The chromatic reduction theorem) Let $G$ be a graph. If $e = uv \in E(G)$, then
\[ P(G, n) = P(G - e, n) - P(G/e, n) \]

Proof. An $n$-coloring of $G$ is an $n$-coloring of $G - e$. Hence
\[ P(G - e, n) \geq P(G, n) \]
If $\phi$ is an $n$-coloring of $G - e$ but not of $G$ then
\[ \phi(u) = \phi(v) \]
Hence $\phi$ gives an $n$-coloring of $G/e$. An $n$-coloring of $G/e$ defines an $n$-coloring of $G - e$ with
\[ \phi(u) = \phi(v), \]
which implies that
\[ P(G, n) = P(G - e, n) - P(G/e, n) \]

Example 1.6. Evaluate $P(K_4 - e, n)$.

Solution Since $P(K_4, n) = n(n - 1)(n - 2)(n - 3)$, $K_4/e = K_3$, hence $P(K_4/e, n) = n(n - 1)(n - 2)$. Therefore,
\[ P(K_4 - e, n) = P(K_4, n) + P(K_4/e, n) = n(n - 1)(n - 2)(n - 3) + n(n - 1)(n - 2) \]
\[ = n(n - 1)(n - 2)^2 \]

Remark 1.7. Let $G$ be a graph.

1. $x = \chi(G)$ is the smallest positive integer such that $P(G, x) > 0$.
2. Let $\ell = \chi(G) - 1$.
   Since $G$ has no 0-coloring, 1-coloring, ..., $\ell$-coloring, hence
   \[ P(G, 0) = P(G, 1) = \ldots = P(G, \ell) = 0 \]
   Thus
   \[ P(G, x) = x^{k_1}(x - 1)^{k_2} \cdots (x - \ell)^{k_{\ell+1}} q(x) \]
   where $q(x)$ is a polynomial with integer coefficients that has no integer roots in $[0, \ell]$. 


Example 1.8. $G = K_4 - e, \mathcal{X}(G) = 3$.

From the calculation above, we have

$$P(G, x) = x^4 - 5x^3 + 8x^2 - 4x = x(x - 1)(x - 2)^2$$

Theorem 1.9. If $T$ is a tree on $n$ vertices,

$$P(T, x) = x(x - 1)^{n-1}$$

Proof. Prove by induction on $n$. When $n = 1$, it is clear.

Suppose that the result is true for trees with less than $n$ vertices. Let $u \in V(T)$ be a vertex of degree 1 and $e = uv \in E(T)$. Then $T - e$ has two components $T_1$ and $T_2$ where $T_1$ is the trivial graph $\{u\}$, $T_2$ is a tree isomorphic to $T/e$. Hence

$$P(T - e, x) = xP(T/e, x)$$

By the chromatic reductive theorem,

$$P(T, x) = P(T - e, x) - P(T/e, x) = xP(T/e, x) - P(T/e, x) = (x - 1)P(T/e, x)$$

$$= (x - 1)x(x - 1)^{n-2} = x(x - 1)^{n-1}$$

Corollary 1.10. If $G$ is a graph on $n$ vertices, then $P(G, x)$ is a monic polynomial in $x$ of degree $n$ with integer coefficients.
Proof. Induction on the number of edges $m = |E(G)|$.

If $m = 1$ and $|V(G)| = n$,

$$P(G, x) = x^{n-2}x(x - 1) = x^{n-1}(x - 1) = x^n - x^{n-1}$$

The result is true.

Suppose that the result is true for all graphs with less than $m$ edges. Let $G$ be a graph with $m$ edges and $n$ vertices. Let $e \in E(G)$. By the chromatic reduction theorem,

$$P(G, x) = P(G - e, x) - P(G/e, x)$$

Since $G - e$ has $m - 1$ edges and $n$ vertices, $G/e$ has at most $m - 1$ edges and $n - 1$ vertices. By the induction hypothesis, $P(G - e, x)$ and $P(G/e, x)$ are monic polynomials with integer coefficients and of degree $n$ and less than $n$ respectively. Hence, $P(G, x)$ is a monic polynomial with integer coefficients and of degree $n$.

Definition 1.11. The polynomial $P(G, x)$ is called the chromatic polynomial of $G$.

Conjecture 1.12. (Birkhoff-Lewis conjecture) If $G$ is a planar graph, then

$$P(G, x) > 0$$

for all $x \in [4, \infty)$.

This implies the 4-color theorem when $x = 4$. 

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