Motivic integration and projective bundle theorem in morphic cohomology

By Jyh-Haur Teh

Department of Mathematics,
National Tsing Hua University,
No. 101, Sec 2, Kuang Fu Road, Hsinchu 30043, Taiwan.

(*Received* ; *revised*)

Abstract

We reformulate the construction of Kontsevich's completion and use Lawson homology to define many new motivic invariants. We show that the dimensions of subspaces generated by algebraic cycles of the cohomology groups of two K-equivalent varieties are the same, which implies that several conjectures of algebraic cycles are K-statements. We define stringy functions which enable us to ask stringy Grothendieck standard conjecture and stringy Hodge conjecture. We prove a projective bundle theorem in morphic cohomology for trivial bundles over any normal quasi-projective varieties.

1. *Introduction*

With the insight from string theory, Batyrev first showed that two birational Calabi-Yau manifolds have the same Betti numbers by using Weil's conjecture (see [**5**]). Kontsevich pushed this result a step further by introducing the notion of motivic integration in showing that two K-equivalent varieties have the same Hodge numbers. In this paper we show further that the dimensions of subspaces generated by algebraic cycles of cohomology groups of two K-equivalent varieties are the same. We found that to construct motivic measure and motivic integration, we do not need the product structure of $K_0(Var)$, the Grothendieck group of algebraic varieties. It is sufficient to give $K_0(Var)$ a $\mathscr{L}$-module structure where $\mathscr{L}$ is the polynomial ring generated by $\mathbb{L} = \mathbb{C}$, and then we can follow Kontsevich's construction to get an abelian group completion. This makes a huge difference since many invariants are not multiplicative, but they can be defined in our new settings. The tools we need to use are the homotopy property and the blow-up formula in Lawson homology. We first show that we can define invariants over the Grothendieck group $K_0(SPV)$ of smooth projective varieties, and then show that the isomorphism given by Bittner $\varphi : K_0(Var) \to K_0(SPV)$ is an isomorphism of $\mathscr{L}$-modules. Through this isomorphism, invariants defined on smooth projective manifolds induce invariants for quasi-projective varieties. These invariants induce invariants defined on the image of some localization $S^{-1}K_0(Var)$ of $K_0(Var)$ in the Kontsevich's completion which enables us to use motivic integration.

We review and modify some constructions of motivic measure and motivic integration in Section 2, use the blow-up formula in Lawson homology and the natural transformations from Lawson homology to singular homology to define some motivic invariants in Section 3. This enables us to show that the generalized Hodge conjecture, the Grothendieck standard conjecture, the Friedlander-Lawson conjecture, the Friedlander-Mazur conjecture are K-statements. In Section 4 we define stringy functions which extend many classical notions to varieties with singularities. One of our most interest stringy functions defines the stringy version of the dimension of cohomology classes generated by algebraic cycles for singular varieties. We are then able to ask stringy Grothendieck standard conjecture and stringy Hodge conjecture. The stringy Grothendieck standard conjecture is verified for normal projective toric varieties with $\mathbb{Q}$-Gorenstein singularities. We conjecture that for mirror pairs (V, W) of dimension n constructed by Batyrev and Borisov, the relation of Hodge numbers $h^{p,q}(V) = h^{n-p,q}(W)$ can be enhanced to a relation of stringy ϕ-numbers.

In Section 5 we focus on varieties with finitely generated Lawson homology groups. We show that motivic integration can be defined over these varieties, and we do the same thing for higher Chow groups. Since one of the main tools we use in this paper is the projective bundle theorem in Lawson homology, it is natural to ask if similar result holds in morphic cohomology. Friedlander proved a projective bundle theorem ([**12**]) in morphic cohomology for smooth quasi-projective varieties but since there is no Mayer-Vietoris sequence in morphic cohomology at this moment, a proof of the result for general quasi-projective varieties is difficult to get. In section 6, we are able to prove a projective bundle theorem in morphic cohomology for trivial bundles over any normal quasi-projective varieties without assuming smoothness. This seeming trivial result already applies almost all techniques in Lawson homology and morphic cohomology.

2. *Motivic integration*

2·1. *Arc spaces*

Let SPV be the collection of all isomorphism classes of smooth projective varieties and Var be the collection of all quasi-projective varieties. Let $K_0(SPV) = \mathbb{Z}(SPV)/\sim_{bl}$ be the free abelian group generated by elements in SPV quotient by the subgroup $\sim_{bl}$ which is generated by elements of the form $Bl_Y X - X + Y - E(Y)$ where Y is a smooth subvariety of a smooth projective variety X, $Bl_Y X$ is the blow-up of X along Y and $E(Y)$ is the exceptional divisor of this blow-up. Let $K_0(Var) = \mathbb{Z}(Var)/\sim$ be the Grothendieck group of quasi-projective varieties. The subgroup $\sim$ is generated by elements of the form $X - (X \backslash Y) - Y$ where Y is a locally closed subvariety of X.

Let $\mathbb{L} = \mathbb{C}$ and let $\mathscr{L} = \mathbb{Z}\{\mathbb{L}^i | i \in \mathbb{Z}_{\geq 0}\}$ be the free abelian group generated by $\mathbb{L}^i$ for all nonnegative integer i. $\mathscr{L}$ is a ring with the obvious multiplication. For $[X] \in K_0(SPV)$, we define $\mathbb{L}^0 \cdot [X] = [X]$ and

$$\mathbb{L}^i \cdot [X] := [X \times \mathbb{P}^i] - [X \times \mathbb{P}^{i-1}]$$

for $i > 0$. Since

$$[(Bl_Y X - X + Y - E(Y)) \times \mathbb{P}^i] = [Bl_{Y\times\mathbb{P}^i}(X \times \mathbb{P}^i) - X \times \mathbb{P}^i + Y \times \mathbb{P}^i - E(Y \times \mathbb{P}^i)]$$

this multiplication is well defined on $K_0(SPV)$, and it is easy to check that $\mathbb{L}^i(\mathbb{L}^j[X]) = \mathbb{L}^{i+j}[X]$, the group $K_0(SPV)$ becomes a $\mathscr{L}$-module under this action. The group $K_0(Var)$ is naturally a $\mathscr{L}$-module under the product of varieties.

Let $S = \{\mathbb{L}^i\}_{i\geq 0} \subset \mathscr{L}$ and let $\mathcal{M} = S^{-1}K_0(SPV) = \{\frac{a}{\mathbb{L}^i} | a \in K_0(SPV), i \in \mathbb{Z}_{\geq 0}\}$,

$\mathcal{N} = S^{-1}K_0(Var) = \{\frac{a}{\mathbb{L}^i} | a \in K_0(Var), i \in \mathbb{Z}_{\geq 0}\}$ be the group obtained by taking the localization of $K_0(SPV), K_0(Var)$ respectively with respect to the multiplicative set S. Define $F^k\mathcal{M}, F^k\mathcal{N}$ to be the subgroups of $\mathcal{M}, \mathcal{N}$ generated by elements of the form $\frac{[X]}{\mathbb{L}^i}$ where $i - dimX \geq k$. Then we get a decreasing filtration

$$\cdots \supseteq F^k\mathcal{M} \supseteq F^{k+1}\mathcal{M} \supseteq \cdots$$

of abelian subgroups of $\mathcal{M}$ and a decreasing filtration

$$\cdots \supseteq F^k\mathcal{N} \supseteq F^{k+1}\mathcal{N} \supseteq \cdots$$

of abelian subgroups of $\mathcal{N}$.

Suppose that we are given a decreasing filtration $\cdots \supseteq F^k \supseteq F^{k+1} \supseteq \cdots$. A Cauchy sequence with respect to this filtration is a sequence $\{a_i\}$ where $a_i \in F^i$ for all i such that for any $n > 0$, there is a $N > 0$ such $a_i - a_j \in F^n$ for all $i, j > N$.

Definition 1. *The Kontsevich group of smooth projective varieties is defined to be*

$$\hat{\mathcal{M}} := \lim_{\leftarrow} \frac{\mathcal{M}}{F^k\mathcal{M}}$$

the completion of $\mathcal{M}$ with respect to the filtration above. Similarly, we define the Kontsevich group of varieties to be

$$\hat{\mathcal{N}} := \lim_{\leftarrow} \frac{\mathcal{N}}{F^k\mathcal{N}}$$

the completion of $\mathcal{N}$ with respect to the filtration of $\mathcal{N}$.

We use also $F^\bullet\mathcal{M}, F^\bullet\mathcal{N}$ to denote the filtrations in $\hat{\mathcal{M}}, \hat{\mathcal{N}}$ respectively induced by the filtrations above. We denote $\overline{\mathcal{M}}$ to be the image under the canonical map $\mathcal{M} \to \hat{\mathcal{M}}$, and $\overline{\mathcal{N}}$ to be the image under the canonical map $\mathcal{N} \to \hat{\mathcal{N}}$.

Definition 2. *We give $\hat{\mathcal{N}}$ a $\mathscr{L}$-module structure as following: for a Cauchy sequence $(a_1, a_2, ...) \in \hat{\mathcal{N}}$, define*

$$\mathbb{L} \cdot (a_1, a_2, ...) := (\mathbb{L}a_1, \mathbb{L}a_2, ...)$$

which is again a Cauchy sequence. It is easy to see that the canonical map $\phi : \mathcal{N} \to \hat{\mathcal{N}}$ defined by $\phi(a) = (a, a, ...)$ is a morphism of $\mathscr{L}$-modules and $\overline{\mathcal{N}}$ is a submodule of $\hat{\mathcal{N}}$. We define the $\mathscr{L}$-module structure similarly for $\hat{\mathcal{M}}$.

Let us recall a result of Bittner [**7**]. For a better presentation of the proof see [**23**].

Theorem 1. *There is a group isomorphism $\varphi : K_0(Var) \to K_0(SPV)$.*

The isomorphism φ is given inductively on the dimension of varieties. Assume it is defined for varieties of dimension less than n. If $dimX = n$, we consider two cases:

(i) If X is nonsingular, let $\overline{X}$ be a nonsingular compactification of X, then define

$$\varphi(X) := \overline{X} - \varphi(\overline{X} - X)$$

(ii) If X is singular, let $X = \coprod_i S_i$ be a stratification of X, then define

$$\varphi(X) := \sum_i \varphi(S_i)$$

Proposition 1.

(i) *The isomorphism $\varphi : K_0(Var) \to K_0(SPV)$ induces an isomorphism $\varphi : \mathcal{N} \to \mathcal{M}$ of $\mathscr{L}$-modules.*

(ii) *It induces an isomorphism of $\mathscr{L}$-modules $\hat{\varphi} : \hat{\mathcal{N}} \to \hat{\mathcal{M}}$.*
(iii) *It induces an isomorphism of $\mathscr{L}$-modules $\overline{\varphi} : \overline{\mathcal{N}} \to \overline{\mathcal{M}}$.*

Proof.

(i) It suffices to prove $\varphi(\mathbb{L} \cdot X) = \mathbb{L} \cdot \varphi(X)$ for X nonsingular. It is easy to check for X of dimension 1. We assume that it is true for varieties of dimension less than n. Then for $dim X = n$, we have $\varphi(\mathbb{L} \cdot X) = \overline{X \times \mathbb{C}} - \varphi(\overline{X \times \mathbb{C}} - X \times \mathbb{C}) = \overline{X} \times \mathbb{P}^1 - \varphi((\overline{X} - X) \times \mathbb{C} + \overline{X}) = \overline{X} \times \mathbb{P}^1 - \overline{X} - \varphi((\overline{X} - X) \times \mathbb{C}) = \mathbb{L} \cdot \overline{X} - \mathbb{L} \cdot \varphi(\overline{X} - X) = \mathbb{L} \cdot \varphi(X)$.

(ii) For $\frac{[X]}{\mathbb{L}^i} \in \mathcal{M}$, we define $\hat{\varphi}(\frac{[X]}{\mathbb{L}^i}) = \frac{\varphi(X)}{\mathbb{L}^i}$. Hence $\hat{\varphi}(F^k\mathcal{N}) \subset F^k\mathcal{M}$, and we have $\hat{\varphi}^{-1}(F^k\mathcal{M}) \subset F^k\mathcal{N}$. Therefore φ induces an isomorphism $\hat{\varphi} : \hat{\mathcal{N}} \to \hat{\mathcal{M}}$ on the completions. This is obviously an isomorphism of $\mathscr{L}$-modules since φ is an isomorphism of $\mathscr{L}$-modules.

(iii) We have a commutative diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\varphi"] \arrow[d] & \mathcal{M} \arrow[d] \\ \hat{\mathcal{N}} \arrow[r, "\hat{\varphi}"] & \hat{\mathcal{M}} \end{tikzcd}$$

where $\varphi, \hat{\varphi}$ are $\mathscr{L}$-module isomorphisms which implies that we have a $\mathscr{L}$-module isomorphism between $\overline{\mathcal{N}}$ and $\overline{\mathcal{M}}$.

From this Proposition, once we have a group homomorphism from $\overline{\mathcal{M}}$ to some group G, we can use $\overline{\varphi}$ to define a group homomorphism from $\overline{\mathcal{N}}$ to G which means that we can define an invariant for all quasi-projective varieties.

2·2. *Motivic integration*

We give a brief review of the arc spaces of quasi-projective varieties here. For the details, we refer to [**4**] and [**9**]. We work over the field of complex numbers. For a complex algebraic variety X of dimension d, the space of n-arcs on X is defined to be

$$\mathcal{L}_n(X) = Mor_{\mathbb{C}-\text{schemes}}(Spec\, \mathbb{C}\,[[t]]/(t^{n+1}), X).$$

For $m \geq n$, there are canonical morphisms $\theta^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$, we obtain the arc space $\mathcal{L}(X)$ of X. For every n we have a natural morphism

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

obtained by truncation. A subset A of $\mathcal{L}(X)$ is called cylindrical if $A = \pi_n^{-1}(C)$ for some n and some constructible subset C of $\mathcal{L}_n(X)$. We say that A is stable at level n if furthermore the restriction of $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ over $\pi_m(A)$ is a piecewise Zariski fibration over $\pi_m(A)$ with fiber $\mathbb{C}^d$ for all $m \geq n$. We call A stable if it is stable at some level n. If X is smooth, then all cylindrical sets of $\mathcal{L}(X)$ are stable.

In the following, let us recall some constructions and results in motivic integration. Even though we have almost all the constructions and results from classical motivic integration, we note that we only consider $K_0(Var), \overline{\mathcal{N}}, \hat{\mathcal{N}}$ as $\mathscr{L}$-modules, not rings.

DEFINITION 3. *If A is stable at level n, we define*

$$\widetilde{\mu}(A) = [\pi_n(A)]\mathbb{L}^{-nd}$$

in $\mathcal{N}$. Let

$$\mathcal{L}^{(e)}(X) := \mathcal{L}(X) \backslash \pi_e^{-1}(\pi_e(\mathcal{L}(X_{sing})))$$

where X_{sing} denote the singular locus of X and we view $\mathcal{L}(X_{sing})$ as a subset of $\mathcal{L}(X)$. For a cylindrical set A, it can be proved that $A \cap \mathcal{L}^{(e)}(X)$ is stable and we define

$$\mu(A) = \lim_{e \to \infty} \widetilde{\mu}(A \cap \mathcal{L}^{(e)}(X)) \in \hat{\mathcal{N}}$$

Define a norm $|| \cdot ||$ on $\hat{\mathcal{N}}$ by $||a|| := 2^{-n}$ where n is the largest n such that $a \in F^n\mathcal{N}$. Then

(i) for all $a, b \in \hat{\mathcal{N}}$, $||a + b|| \leq max(||a||, ||b||)$,

(ii) for any A, B cylindrical sets, we have $||\mu(A \cup B)|| \leq max(||\mu(A)||, ||\mu(B)||)$ and $||\mu(A)|| \leq ||\mu(B)||$ when $A \subset B$.

Definition 4. *We say that a subset A of $\mathcal{L}(X)$ is measurable if, for every positive real number ϵ, there exists a sequence of cylindrical subsets $A_i(\epsilon)$, $i \in \mathbb{N}$ such that*

$$(A \Delta A_0(\epsilon)) \subset \bigcup_{i \geq 1} A_i(\epsilon)$$

and $||\mu(A_i(\epsilon))|| \leq \epsilon$ for all $i \geq 1$. We say that A is strongly measurable if moreover we can take $A_0(\epsilon) \subset A$.

The following is the result A.6 from [**10**].

Theorem 2. *If A is a measurable subset of $\mathcal{L}(X)$, then*

$$\mu(A) := \lim_{\epsilon \to 0} \mu(A_0(\epsilon))$$

exists in $\hat{\mathcal{N}}$ and is independent of the choice of the sequences $A_i(\epsilon), i \in \mathbb{N}$.

Definition 5. *Let X be a quasi-projective variety of pure dimension d. We define the motivic volume of X to be $\mu(\mathcal{L}(X)) \in \hat{\mathcal{N}}$. It can be shown that*

$$\mu(\mathcal{L}(X)) = \lim_{n \to \infty} \frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}}$$

and it equals to $[X]$ when X is nonsingular.

Definition 6. *Let $A \subset \mathcal{L}(X)$ be measurable and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ a function with measurable fibres $\alpha^{-1}(n)$ for $n \in \mathbb{Z}$. We define the motivic integral of α to be*

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n \in \mathbb{Z}} \mu(\alpha^{-1}(n)) \mathbb{L}^{-n}$$

in $\hat{\mathcal{N}}$ whenever the right hand side converges in $\hat{\mathcal{N}}$, in which case we say that $\mathbb{L}^{-\alpha}$ is integrable on A. If α is bounded from below, this is always the case (see [**9**]*).*

Definition 7. *Let $\mathcal{I}$ be a sheaf of ideals on X. We define*

$$ord_t\mathcal{I} : \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$$

by $ord_t\mathcal{I}(\gamma) = \min_g \{ord_t g(\gamma)\}$ where the minimum is taken over $g \in \mathcal{I}$ in a neighborhood of $\pi_0(\gamma)$. For an effective Cartier divisor D, we define $ord_t D = ord_t I$ where I is the ideal sheaf associated to D.

The following result is from Theorem 2.7.1 of [**22**].

THEOREM 3. *(Change of variables formula) Let X be a complex algebraic variety of dimension d. Let $h : Y \to X$ be a proper birational morphism and Y a smooth variety. Let A be a subset of $\mathcal{L}(X)$ such that A and $h^{-1}(A)$ are strongly measurable. Assume that $\mathbb{L}^{-\alpha}$ is integrable on A. Then*

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha\circ h - ord_t h^*(\Omega^d_X)} d\mu$$

where $h^*(\Omega^d_X)$ is the pullback of the sheaf of regular differential d-forms of X and $ord_t h^*(\Omega^d_X) := ord_t I(h^*(\Omega^d_X))$ where $I(h^*(\Omega^d_X))$ is the ideal sheaf induced by $h^*(\Omega^d_X)$ (see [**9**, 3. 3]*).*

For a divisor $D = \sum_{i=1}^r a_i D_i$ on X and any subset $J \subset \{1, ..., r\}$, denote

$$D_J = \begin{cases} \underset{j\in J}{\cap} D_j, & \text{if } J \neq \emptyset \\ X, & \text{if } J = \emptyset. \end{cases}$$

and $D^0_J := D_J - \cup_{i\notin J} D_i$.

Even though we do not have $[X \times Y] = [X][Y]$ in $\hat{\mathcal{N}}$ for general varieties X, Y, we do have $[Y \times \mathbb{C}^*] = [Y \times (\mathbb{C} - 0)] = [Y \times \mathbb{C}] - [Y \times 0] = \mathbb{L}[Y] - [Y] = [Y][\mathbb{C}^*]$. By a similar calculation as in Theorem 6.28 of [**4**], we have the following result.

THEOREM 4. *Let X be a nonsingular algebraic varieties of dimension d and $D = \sum_{i=1}^r a_i D_i$ an effective divisor on X with only simple normal crossings. Then*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t D} d\mu = \sum_{J\subset\{1,\dots,r\}} [D^0_J] \cdot (\prod_{j\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1})$$

where J is any subset (including empty set) of $\{1, ..., r\}$.

COROLLARY 1. *Let X be a variety of pure dimension d, and let $h : Y \to X$ be a resolution of singularities of X such that the relative canonical divisor $D = \sum_{i=1}^r a_i D_i = K_Y - h^* K_X$ of h has simple normal crossings. Furthermore, assume that the ideal sheaf induced by $h^*(\Omega^d_X)$ is invertible. Then*

$$\mu(\mathcal{L}(X)) = \sum_{J\subset\{1,\dots,r\}} [D^0_J] \cdot (\prod_{j\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1}).$$

Hence $\mu(\mathcal{L}(X))$ belongs to $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$

We say that two smooth projective varieties X and Y are K-equivalent if there is a smooth projective variety Z and birational morphisms $\rho_1 : Z \to X$ and $\rho_2 : Z \to Y$ such that $\rho_1^* K_X = \rho_2^* K_Y$ in Z where K_X, K_Y are the canonical divisors on X and Y respectively. As a simple consequence of the "change of variables formula", we have the following result.

THEOREM 5. *If two smooth projective varieties X, Y are K-equivalent, then $[X] = [Y]$ in $\overline{\mathcal{N}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$.*

3. *Lawson homology groups*

For an overview of Lawson homology and morphic cohomology, we refer to [**16**, **14**]. Recall that in Lawson homology we have the homotopy property $L_p H_n(X \times \mathbb{C}^t) = L_{p-t} H_{n-2t}(X)$ for X a smooth projective variety.

DEFINITION 8. *For a quasi-projective variety X and an integer $p < 0$, we define the negative cycle group $Z_p(X) := Z_0(X \times \mathbb{C}^{-p})$, and $H_p(X) := H_0^{BM}(X \times \mathbb{C}^{-p})$. Then we have $L_pH_k(X) = L_0H_{k-2p}(X \times \mathbb{C}^{-p}) = H^{BM}_{k-2p}(X \times \mathbb{C}^{-p})$. Throughout this paper we will identify $L_rH_k(X)$ with $L_{r+t}H_{k+2t}(X \times \mathbb{C}^t)$.*

Recall that we have natural transformations $\Phi : L_pH_k(X) \to H_k(X)$ from Lawson homology to singular homology (see [**11**], [**14**], [**20**]). The intersection theory of cycle spaces was developed by Friedlander and Gabber in [**13**] in which they obtained a projective bundle theorem for Lawson homology. We extend their result to negative cycle groups.

PROPOSITION 2. *Let E be an algebraic vector bundle of rank $r + 1$ over a quasi-projective variety Y of dimension n and $\pi : P(E) \to Y$ the projective bundle associated to E. We denote the tautological line bundle on $P(E)$ by $\mathcal{O}_{P(E)}(1)$ and first Chern class by c_1 (see* [**13**]*).*

(i) *(Friedlander-Gabber) If $p \geq r$, then the map*

$$\Psi \equiv \sum_{j=0}^{r} c_1(O_{P(E)}(1))^{r-j} \circ \pi^* : \bigoplus_{j=0}^{r} Z_{p-j}(Y) \to Z_p(P(E))$$

is a homotopy equivalent.

(ii) *If $0 \leq p < r$, then the map*

$$\Psi : \bigoplus_{j=0}^{r} Z_{p-j}(Y) \longrightarrow Z_p(P(E))$$

is a homotopy equivalent where $\Psi \equiv \sum_{j=0}^{p}(\pi_1^*)^{-1} \circ c_1(O_{P(E)\times\mathbb{C}^{r-p}}(1))^{r-j} \circ \pi^* \circ \pi_2^* + \sum_{j=p+1}^{r}(\pi_1^*)^{-1} \circ c_1(O_{P(E)\times\mathbb{C}^{r-p}}(1))^{r-j} \circ \pi^* \circ \pi_3^*$, and $\pi_1 : P(E) \times \mathbb{C}^{r-p} \to P(E), \pi_2 : Y \times \mathbb{C}^{r-p} \to Y, \pi_3 : Y \times \mathbb{C}^{r-p} \to Y \times \mathbb{C}^{j-p}$ are the projections.

(iii) *For any p, we have a commutative diagram:*

$$\begin{array}{ccc} \Psi : \bigoplus_{j=0}^{r} L_{p-j}H_{k-2j}(Y) & \longrightarrow & L_pH_k(P(E)) \\ \downarrow \Phi & & \downarrow \Phi \\ \Psi' : \bigoplus_{j=0}^{r} H^{BM}_{k-2j}(Y) & \longrightarrow & H_k^{BM}(P(E)) \end{array}$$

where Ψ' is the counterpart of Ψ in singular homology.

Proof. We prove (ii). We have two homotopy equivalences:

$$\bigoplus_{j=0}^{p} Z_{p-j}(Y) \xrightarrow{\pi_2^*} \bigoplus_{j=0}^{p} Z_{r-j}(Y \times \mathbb{C}^{r-p}),$$

and

$$\bigoplus_{j=p+1}^{r} Z_0(Y \times \mathbb{C}^{j-p}) \xrightarrow{\pi_3^*} \bigoplus_{j=p+1}^{r} Z_{r-j}(Y \times \mathbb{C}^{r-p})$$

Combining them together, we get

$$\bigoplus_{j=0}^{r} Z_{p-j}(Y) = \bigoplus_{j=0}^{p} Z_{p-j}(Y) \oplus \bigoplus_{j=p+1}^{r} Z_0(Y \times \mathbb{C}^{j-p}) \cong \bigoplus_{j=0}^{r} Z_{r-j}(Y \times \mathbb{C}^{r-p})$$

Consider $\mathbb{C}^{r-p}$ as a zero rank vector bundle over itself, then $E \times \mathbb{C}^{r-p}$ is an algebraic vector bundle over $Y \times \mathbb{C}^{r-p}$ of rank $r+1$. We have $P(E \times \mathbb{C}^{r-p}) = P(E) \times \mathbb{C}^{r-p}$. By (i), we have a homotopy equivalence

$$\sum_{j=0}^{r} c_1(O_{P(E)\times\mathbb{C}^{r-p}}(1))^{r-j} \circ \pi^* : \bigoplus_{j=0}^{r} Z_{r-j}(Y \times \mathbb{C}^{r-p}) \longrightarrow Z_r(P(E) \times \mathbb{C}^{r-p})$$

Combining with the homotopy equivalence $(\pi_1^*)^{-1} : Z_r(P(E) \times \mathbb{C}^{r-p}) \to Z_p(P(E))$, we are done. (iii) follows from the fact that all the maps in (i) and (ii) are induced from algebraic maps and Φ is a natural transformation.

The following is the blow-up formula in Lawson homology from [**19**]. We state the result for integral coefficients but we use the formula only in rational coefficients.

PROPOSITION 3. *Let X be a smooth projective variety and $i' : Y \hookrightarrow X$ a smooth subvariety of codimension $r+1$. Let $\sigma : Bl_Y X \to X$ be the blow-up of X along Y, $\pi : E = \sigma^{-1}(Y) \to Y$ the projection, and $i : E \to Bl_Y X$ the inclusion map. For $p \geq 0$ and $k \geq 2p$,*

(i) *the map*

$$I_{p,k} : \bigoplus_{j=1}^{r} L_{p-j}H_{k-2j}(Y) \oplus L_pH_k(X) \to L_pH_k(Bl_Y X)$$

defined by

$$I_{p,k}(u_1, ..., u_r, u) = \sum_{j=1}^{r} i_* h^{r-j} \pi^* u_j + \sigma^* u$$

is an isomorphism where $h \in L_{m-1}H_{2(m-1)}(E)$ is the class defined by a hyperplane section of E.

(ii) *There is a split short exact sequence*

$$0 \to L_pH_k(Y) \xrightarrow{\psi_1} L_pH_k(E) \oplus L_pH_k(X) \xrightarrow{\psi_2} L_pH_k(Bl_Y X) \to 0$$

where $\psi_1(x) = (h^r \pi^* x, -i'_* x)$, and $\psi_2(x) = (\widetilde{x}, y) = i_* \widetilde{x} + \sigma^* y$.

DEFINITION 9. *Suppose that X is a smooth projective variety. Let $T_pH_k(X) := \Phi(L_pH_k(X; \mathbb{Q})) \subset H_k(X; \mathbb{Q})$ be the image of the natural transformation Φ from Lawson homology to singular homology and $T_{p,k} = dim\Phi(L_pH_k(X; \mathbb{Q}))$ be its dimension for $k \geq 2p \geq 0$.*

Combine with the blow-up formula in singular homology (see [**18**], Chapter 4.6), and as an immediate consequence of the Proposition above, we get the following crucial equality.

PROPOSITION 4. *Suppose that X is a smooth projective variety. Then we have*

$$T_{p,k}(Bl_Y X) - T_{p,k}(E(Y)) = T_{p,k}(X) - T_{p,k}(Y)$$

where $Bl_Y X$ is the blow-up of X along a smooth subvariety Y and $E(Y)$ is the exceptional divisor.

Since $K_0(SPV) = \mathbb{Z}(SPV)/\sim_{bl}$ where $\sim_{bl}$ is the blow-up relation, we see that $T_{p,k}$ induces a group homomorphism from $K_0(SPV)$ to $\mathbb{Z}$.

PROPOSITION 5. *For a smooth projective variety X, define*

$$T_{p,k}(X \cdot \mathbb{L}^i) := T_{p,k}(X \times \mathbb{P}^i) - T_{p,k}(X \times \mathbb{P}^{i-1})$$

for $k \geq 2p \geq 0$.

(i) *The map $T_{p,k}$ induces a group homomorphism from $\mathcal{M}$ to $\mathbb{Z}$.*
(ii) *The map $T_{p,k}$ induces a group homomorphism from $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$ to $\mathbb{Z}$.*

Proof.
(i) From the projective bundle theorem for trivial bundles and applying the natural transformations from Lawson homology to singular homology, we have the following commutative diagram:

$$\begin{array}{ccc} L_pH_k(X\times\mathbb{P}^i;\mathbb{Q}) & \cong & L_pH_k(X\times\mathbb{P}^{i-1};\mathbb{Q})\oplus L_{p-i}H_{k-2i}(X;\mathbb{Q}) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ H_k(X\times\mathbb{P}^i;\mathbb{Q}) & \cong & H_k(X\times\mathbb{P}^{i-1};\mathbb{Q})\oplus H_{k-2i}(X;\mathbb{Q}) \end{array}$$

Then $T_{p,k}(X\cdot\mathbb{L}^i)=T_{p,k}(X\times\mathbb{P}^i)-T_{p,k}(X\times\mathbb{P}^{i-1})=T_{p-i,k-2i}(X)$.
For $\frac{A}{\mathbb{L}^i}\in\mathcal{M}$ where $A\in K_0(SPV)$, we define

$$T_{p,k}(\frac{A}{\mathbb{L}^i}):=T_{p+i,k+2i}(A)$$

Since $T_{p,k}(\frac{A\cdot\mathbb{L}^j}{\mathbb{L}^{i+j}})=T_{p+i+j,k+2(i+j)}(A\cdot\mathbb{L}^j)=T_{p+i,k+2i}(A)=T_{p,k}(\frac{A}{\mathbb{L}^i})$, $T_{p,k}$ is well defined over $\mathcal{M}$. Extending $T_{p,k}$ by linearity, we get a group homomorphism from $\mathcal{M}$ to $\mathbb{Z}$.

(ii) The kernel of the canonical map $\phi:\mathcal{M}\to\hat{\mathcal{M}}$ is $\cap_n\mathcal{F}^n\mathcal{M}$. If $A\in\cap_n\mathcal{F}^n\mathcal{M}$, then $A=\sum_i b_i\frac{B_i}{\mathbb{L}^{n_i}}$ for some $b_i\in\mathbb{Z}, B_i\in K_0(SPV)$ such that $n_i-dimB_i\geq 1$. Therefore $T_{p,k}(A)=\sum_i b_iT_{p,k}(\frac{B_i}{\mathbb{L}^{n_i}})=\sum_i b_iT_{p+n_i,k+2n_i}(B_i)=0$ since $n_i>\dim B_i$. Hence $T_{p,k}$ induces a group homomorphism from $\overline{\mathcal{M}}=\phi(\mathcal{M})\subset\hat{\mathcal{M}}$ to $\mathbb{Z}$.
Write $\frac{1}{\mathbb{L}^i-1}=\sum_{j=0}^{\infty}a_j\mathbb{L}^j$. For $[X]\in\overline{\mathcal{M}}$, define

$$T_{p,k}(X(\mathbb{L}^i-1)^{-1}):=\sum_{j=0}^{\infty}a_jT_{p,k}(X\cdot\mathbb{L}^j)$$

which is a finite sum. Hence $T_{p,k}$ extends to a group homomorphism from $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$ to $\mathbb{Z}$.

Proposition 6. *For a smooth projective variety X of dimension m, let $h^{p,q}(X)$ be the (p,q)-Hodge number of X and $h_{m-p,m-q}(X)$ be the dimension of the Poincaré dual of $H^{p,q}(X)$ in the homology group $H_{2m-(p+q)}(X;\mathbb{C})$. Then $h_{p,q}$ and $h^{p,q}$ induce group homomorphisms from $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$ to $\mathbb{Z}$.*

Proof. We show this for $h_{p,q}$. The maps in the exact sequence

$$0\to H_n(X;\mathbb{C})\to H_n(Bl_YX;\mathbb{C})\oplus H_n(Y;\mathbb{C})\to H_n(E(Y);\mathbb{C})\to 0$$

are easy to see to be morphisms of Hodge structures, hence we have

$$h_{p,q}(Bl_YX)-h_{p,q}(E(Y))=h_{p,q}(X)-h_{p,q}(Y)$$

which implies that $h_{p,q}$ induces a group homomorphism from $K_0(SPV)$ to $\mathbb{Z}$.
Define

$$h_{p,q}(X\cdot\mathbb{L}^i):=h_{p,q}(X\times\mathbb{P}^i)-h_{p,q}(X\times\mathbb{P}^{i-1}).$$

From the isomorphism $H_{p,q}(X\times\mathbb{P}^i)\cong H_{p,q}(X\times\mathbb{P}^{i-1})\oplus H_{p-i,q-i}(X)$, we get

$$h_{p,q}(X\cdot\mathbb{L}^i)=h_{p-i,q-i}(X)$$

and then we extend $h_{p,q}$ as in the Proposition above.

We recall that the niveau filtration $\{N_pH_*(X;\mathbb{Q})\}_{p\geq 0}$ of $H_*(X;\mathbb{Q})$ is defined by

$$N_pH_k(X;\mathbb{Q}) = \text{ span } \{ \text{ images } i_* : H_k(Y;\mathbb{Q}) \to H_k(X;\mathbb{Q}) | i : Y \hookrightarrow X, dimY \leq p\}$$

The geometric filtration $\{G_pH_*(X;\mathbb{Q})\}_{p\geq 0}$ of $H_*(X;\mathbb{Q})$ is defined by

$$G_pH_k(X;\mathbb{Q}) = N_{k-p}H_k(X;\mathbb{Q})$$

We define the homological Hodge filtration to be

$$F_pH_k(X;\mathbb{C}) := \bigoplus_{t\leq p} H_{t,k-t}(X)$$

and define the homological rational Hodge filtration to be

$$F_p^hH_k(X;\mathbb{Q}) = \text{ largest sub-Hodge structure of } F_pH_k(X;\mathbb{C}) \cap H_k(X;\mathbb{Q})$$

The homological generalized Hodge conjecture says that for a smooth projective variety X,

$$F_p^hH_k(X;\mathbb{Q}) = N_pH_k(X;\mathbb{Q})$$

The Friedlander-Mazur conjecture (see [**17**]) says that

$$T_pH_k(X;\mathbb{Q}) = G_pH_k(X;\mathbb{Q})$$

and the Friedlander-Lawson conjecture says that

$$L_pH_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})$$

is surjective if $k \geq m + p$ where m is the dimension of X. This conjecture was proved by the author in [**25**] by assuming the Grothendieck standard conjecture.

DEFINITION 10. *We say that a statement is a K-statement if it is true for a smooth projective variety X, then it is true for all varieties which are K-equivalent to X.*

We will show that all these conjectures are K-statements.

PROPOSITION 7. *Let $G_{j,n}(X) := dimG_jH_n(X;\mathbb{Q}), F_{j,n}(X) := dimF_j^hH_n(X;\mathbb{Q})$. Then $G_{j,n}$ and $F_{j,n}$ extend to $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$.*

Proof. By Lemma 2.3 of [**3**], we have a short exact sequence of pure Hodge structures:

$$0 \to H_n(X;\mathbb{C}) \to H_n(Bl_YX;\mathbb{C}) \oplus H_n(Y;\mathbb{C}) \to H_n(E(Y);\mathbb{C}) \to 0$$

which give us the following formula

$$F_{j,n}(Bl_YX) + F_{j,n}(Y) = F_{j,n}(X) + F_{j,n}(E(Y)).$$

We define

$$F_{j,n}(X \cdot \mathbb{L}^k) := F_{j,n}(X \times \mathbb{P}^k) - F_{j,n}(X \times \mathbb{P}^{k-1})$$

Since $X \times \mathbb{C}^k = X \times \mathbb{P}^k - X \times \mathbb{P}^{k-1}$, from the mixed Hodge theory, there is a long exact sequence of mixed Hodge structures:

$$\cdots \to H_n(X\times\mathbb{P}^{k-1};\mathbb{C}) \to H_n(X\times\mathbb{P}^k;\mathbb{C}) \to H_n^{BM}(X\times\mathbb{C}^k;\mathbb{C}) \to H_{n-1}(X\times\mathbb{P}^{k-1};\mathbb{C}) \to \cdots$$

but the map induced by inclusion $H_i(X\times\mathbb{P}^{k-1};\mathbb{C}) \to H_i(X\times\mathbb{P}^k;\mathbb{C})$ is always an injection, and therefore we get an exact sequence:

$$0 \to H_n(X \times \mathbb{P}^{k-1};\mathbb{C}) \to H_n(X \times \mathbb{P}^k;\mathbb{C}) \to H_n^{BM}(X \times \mathbb{C}^k;\mathbb{C}) \to 0$$

The isomorphism $H_{n-2k}(X;\mathbb{C}) \cong H_n^{BM}(X \times \mathbb{C}^k;\mathbb{C})$ is an isomorphism of Hodge structures of type (k,k), hence

$$F_{j,n}(X \cdot \mathbb{L}^k) = \dim F_j^h H_n^{BM}(X \times \mathbb{C}^k;\mathbb{Q}) = \dim F_{j-k}^h H_{n-2k}(X;\mathbb{C}) = F_{j-k,n-2k}(X)$$

Then extend as in the Proposition 5, we get a group homomorphism from $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$ to $\mathbb{Z}$.

By a homological version of Lemma 2.4 of [**3**], we get

$$G_{j,n}(Bl_Y X) + G_{j,n}(Y) = G_{j,n}(X) + G_{j,n}(E(Y)).$$

Define

$$G_{j,n}(X \cdot \mathbb{L}^k) := G_{j,n}(X \times \mathbb{P}^k) - G_{j,n}(X \times \mathbb{P}^{k-1}).$$

Since a morphism of Hodge structures preserve niveau filtration, we have $G_{j,n}(X\cdot\mathbb{L}^k) = G_{j-k,n-2k}(X)$. Then similar to the construction above, $G_{j,n}$ extends to a group homomorphism from $\overline{\mathcal{M}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$ to $\mathbb{Z}$.

By composing with the isomorphism in Proposition 1(iii), we have the following crucial result.

THEOREM 6. *The group homomorphisms $T_{p,k}, h^{p,k}, h_{p,k}, G_{p,k}, F_{p,k}$ induce group homomorphisms from $\overline{\mathcal{N}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 0}]$ to $\mathbb{Z}$. We will abusively use the same notations for the induced homomorphisms.*

REMARK 1. *For a quasi-projective variety X, we may define $T'_{p,k}(X)$ to be the dimension of $\Phi(L_pH_k(X;\mathbb{Q})) \subset H_k^{BM}(X;\mathbb{Q})$. Even though this definition makes sense, in general it does not equal to $T_{p,k}(X)$. For example $T_{0,0}(\mathbb{C}^*) = T_{0,0}(\mathbb{P}^1) - T_{0,0}(\{0,\infty\}) = 1-2 = -1$ but $T'_{0,0}(\mathbb{C}^*)$ is a nonnegative integer.*

We recall that the Lawson homology group $L_pH_{2p}(X)$ = algebraic p-cycles quotient by algebraic equivalence and the natural transformation $\Phi : L_pH_{2p}(X) \to H_{2p}(X)$ is the cycle map. Hence $\Phi(L_pH_{2p}(X))$ is the subspace of $H_{2p}(X)$ generated by algebraic cycles. We recall that the Grothendieck standard conjecture A (GSCA) predicts that for a smooth projective variety X of dimension m, $T_{p,2p}(X) = T_{m-p,2(m-p)}(X)$ where $p \leq [\frac{m}{2}]$.

PROPOSITION 8. *If $Bl_Y X$ is the blow-up of a smooth projective variety X at a smooth center Y of codimension $r+1$ and if the GSCA is true for Y, then the GSCA on $Bl_Y X$ is equivalent to the GSCA on X.*

Proof. Let the dimension of X be m and $p \leq [\frac{m}{2}]$. Then the dimension of Y is $m-r-1$. Let $A_p(Y) = \sum_{j=1}^r T_{p-j,2(p-j)}(Y)$, $B_p(X) = T_{p,2p}(X)$, $B_p(Bl_Y X) = T_{p,2p}(Bl_Y X)$. We have $A_p(Y) + B_p(X) = B_p(Bl_Y X)$.

From the calculation $A_{m-p}(Y) = \sum_{j=1}^r T_{m-p-j,2(m-p-j)}(Y) = \sum_{j=1}^r T_{p+j-r-1,2(p+j-r-1)}(Y) = \sum_{j=1}^r T_{p-j,2(p-j)}(Y) = A_p(Y)$, we get $B_p(X) - B_{m-p}(X) = B_p(Bl_Y X) - B_{m-p}(Bl_Y X)$ which means that the GSCA holds on X if and only if it holds on $Bl_Y X$.

COROLLARY 2. *If the GSCA holds for smooth projective varieties of dimension less than $m-1$, then the GSCA is a birational statement for smooth projective varieties of dimension m.*

Proof. By the Weak Factorization Theorem of birational maps (see [**2**]), we are able to decompose a proper birational map as a sequence of blowing-ups and blowing-downs, then we apply the result above.

Since we know that the GSCA is true for smooth varieties of dimension less than or equal to 4, we have the following result.

COROLLARY 3. *The GSCA is invariant under birational equivalence of smooth varieties of dimension less than 7.*

For a projective manifold X, let $N_{j,n}(X) := \dim N_jH_n(X;\mathbb{Q})$.

PROPOSITION 9. *If Bl_YX is the blow-up of a smooth projective variety X at a smooth center Y of codimension $r+1$ and the generalized Hodge conjecture is true for Y, then the generalized Hodge conjecture on Bl_YX is equivalent to the generalized Hodge conjecture on X.*

Proof. We have $N_{j,n}(Bl_YX) = N_{j,n}(X) + \sum_{i=1}^{r} N_{j-i,n-2i}(Y)$ and $F_{j,n}(Bl_YX) = F_{j,n}(X) + \sum_{i=1}^{r} F_{j-i,n-2i}(Y)$. By the assumption that the generalized Hodge conjecture is true for Y, we have $N_{j,n}(Bl_YX) - F_{j,n}(Bl_YX) = N_{j,n}(X) - F_{j,n}(X)$. This completes the proof.

Again by using the Weak Factorization Theorem of birational maps, we get the following result. We do not know who is the first to have this result, but a proof without using the Weak Factorization Theorem can be found in [**1**].

COROLLARY 4. *If the Hodge conjecture is true for dimension less than $m-1$, then the Hodge conjecture is a birational statement for smooth varieties of dimension m. In particular it is a birational statement for dimension less than 6.*

Since two K-equivalent varieties have the same image in $\overline{\mathcal{N}}$, any group homomorphism defined previously gives the same value at them. Then the following result is an immediate consequence.

THEOREM 7. *The Friedlander-Mazur conjecture, the Friedlander-Lawson conjecture, the Grothendieck standard conjecture and the generalized Hodge conjecture are K-statements.*

For the case of generalized Hodge conjecture, this result was proved by Arapura and Kang in [**3**]. By a result of Wang (see [**27**], Corollary 1.10), two birational smooth minimal models are K-equivalent, hence in particular we have the following result.

COROLLARY 5. *If any conjecture in Theorem 7 is true for a smooth minimal model, then it is true for any smooth minimal model which is birational to it.*

4. *Stringy functions*

DEFINITION 11. *A motivic invariant is a group homomorphism from $\overline{\mathcal{N}}[(\frac{1}{\mathbb{L}^i-1})_{i\geq 1}]$ to $\mathbb{Z}$.*

We have seen several motivic invariants: $T_{j,n}$, $G_{j,n}$, $F_{j,n}$, $h^{p,q}$ and $h_{p,q}$. One of the most important properties of these invariants is that they satisfy $\phi_{j,n}(X\times\mathbb{L}^k) = \phi_{j-ak,n-bk}(X)$ for some numbers a, b. This enables us to associate a stringy ϕ-function to ϕ. Before we consider the general case, let us exemplify this by Batyrev's stringy E-function.

EXAMPLE 1. *Let us recall some definitions from* [**4**]. *For a variety X of dimension m, let*

$$e^{p,q}(X) := \sum_{0\le k\le 2m} (-1)^k h^{p,q}(H_c^k(X;\mathbb{C}))$$

where $h^{p,q}(H_c^k(X;\mathbb{C}))$ is the (p,q)-Deligne-Hodge number of the cohomology groups with compact support of X. For a projective manifold, the number $e^{p,q}$ is same as the Hodge number $(-1)^{p+q}h^{p,q}$. The E-polynomial $E(X;u,v)\in\mathbb{Z}[u,v]$ is defined to be

$$E(X;u,v) := \sum_{p,q} e^{p,q}(X)u^pv^q.$$

This is a finite sum and $E(X\times\mathbb{L}^k;u,v) = (uv)^kE(X;u,v)$. Therefore by defining $E(\mathbb{L}^{-1};u,v) = (uv)^{-1}$, we are able to extend E to a group homomorphism $E:\overline{\mathcal{N}}[(\frac{1}{\mathbb{L}^i-1})_{i\ge1}]\to\mathbb{Z}[[u,v,(uv)^{-1}]]$.

If X is a normal irreducible algebraic variety with at worst log-terminal singularities, and $\rho: Y\to X$ is a resolution of singularities such that the relative canonical divisor $D=\sum_{i=1}^r a_iD_i$ has simple normal crossings. Then the stringy E-function of X is defined to be

$$E_{st}(X;u,v) := \sum_{J\subset I} E(D_J^0;u,v)\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1}$$

where $I=\{1,...,r\}$. If X is a projective variety with at worst Gorenstein canonical singularities and $E_{st}(X;u,v)=\sum_{p,q}a_{p,q}u^pv^q$ is a polynomial, we define the stringy Hodge numbers of X to be

$$h_{st}^{p,q}(X) := (-1)^{p+q}a_{p,q}.$$

Now we come to the general case.

DEFINITION 12. *We say that a family of motivic invariants $\phi=\{\phi_{j,n}|j,n\in\mathbb{Z}\}$ is of type $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ if $\phi_{j,n}(X\times\mathbb{L}^k)=\phi_{j-ak,n-bk}(X)$ for any j,n and any varieties X. And we say that ϕ is bounded if $\phi_{j,n}(X)$ vanishes for $|j|,|n|$ large enough, depending on X.*

For example $T=\{T_{j,n}|j,n\in\mathbb{Z}\}$ is of type $(1,2)$ and $h=\{h^{p,q}|p,q\in\mathbb{Z}\}$ is of type $(1,1)$ where $T_{j,n}, h^{p,q}$ are defined to be zero if any j,n,p,q is negative.

DEFINITION 13. *Suppose that $\phi=\{\phi_{j,n}|j,n\in\mathbb{Z}\}$ is a family of bounded motivic invariants of type (a,b), then define*

$$\phi(X;u,v) := \sum_{j,n}\phi_{j,n}(X)u^jv^n$$

and

$$\phi(\mathbb{L}^{-1};u,v) := (u^av^b)^{-1},$$

we get a group homomorphism

$$\phi : \overline{\mathcal{N}}[(\frac{1}{\mathbb{L}^i - 1})_{i\geq 1}] \to \mathbb{Z}[[u, v, (u^a v^b)^{-1}]].$$

If X is a normal irreducible algebraic variety with at worst log-terminal singularities, and $\rho : Y \to X$ is a resolution of singularities such that the relative canonical divisor $D = \sum_{i=1}^r a_i D_i$ has simple normal crossings. Then the stringy ϕ-function associated to ϕ is defined to be

$$\phi^{st}(X; u, v) := \sum_{J\subset I} \phi(D_J^0; u, v) \prod_{j\in J} \frac{u^a v^b - 1}{(u^a v^b)^{a_j+1} - 1}$$

where $I = \{1, ..., r\}$. If X is projective with at worst canonical singularities and $\phi^{st}(X; u, v) = \sum_{p,q} a_{p,q} u^p v^q$ is a polynomial, we define the (p, q)-stringy ϕ-numbers of X to be

$$\phi^{st}_{p,q}(X) := (-1)^{p+q} a_{p,q}$$

PROPOSITION 10. *The stringy ϕ-numbers of X defined above are independent of resolution of singularities.*

Proof. Let $\rho_1 : X_1 \to X, \rho_2 : X_2 \to X$ be two resolution of singularities. Take another resolution of singularities $\alpha : Y \to X$ which dominates ρ_1, ρ_2, i.e., we have the following commutative diagram:

$$\begin{tikzcd} & Y \arrow[dl, "\alpha_1"'] \arrow[dd, "\alpha"] \arrow[dr, "\alpha_2"] & \\ X_1 \arrow[dr, "\rho_1"'] & & X_2 \arrow[dl, "\rho_2"] \\ & X & \end{tikzcd}$$

Let $K_{X_1} = \rho_1^* K_X + D_1, K_{X_2} = \rho_2^* K_X + D_2$ and $D = K_Y - \alpha^* K_X$. Then $\alpha_1^* D_1 + K_{Y|X_1} = \alpha_1^* D_1 + K_Y - \alpha_1^* K_{X_1} = K_Y - \alpha^* K_X = \alpha_2^* D_2 + K_{Y|X_2}$. Therefore by the change of variables formula,

$$\int_{\mathcal{L}(X_1)} \mathbb{L}^{-ord_t D_1} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t(\alpha_1^* D_1 + K_{Y|X_1})} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t(\alpha_2^* D_2 + K_{Y|X_2})} = \int_{\mathcal{L}(X_2)} \mathbb{L}^{-ord_t D_2}$$

Taking ϕ^{st} on both sides, this shows that ϕ^{st} is independent of resolution of singularities.

With all these definitions, we are able to ask the stringy version of some conjectures.

CONJECTURE 1. *Suppose that X is a m-dimensional normal irreducible projective variety with at worst canonical singularities. Let $T^{st}_p, T^{st}_{p,q}, G^{st}_{p,q}, F^{st}_{p,q}$ be the (p, q)-stringy numbers associated to the families $T = \{T_{p,2p} | p \in \mathbb{Z}\}, T' = \{T_{p,q} | p, q \in \mathbb{Z}\}, G = \{G_{p,q} | p, q \in \mathbb{Z}\}, F = \{F_{p,q} | p, q \in \mathbb{Z}\}$ respectively, and assume that all these numbers of X are defined.*

(i) *(Stringy GSCA) Is $T^{st}_{p,2p}(X) = T^{st}_{m-p,2(m-p)}(X)$?*
(ii) *(Stringy morphic conjecture) Is $T^{st}_{p,q}(X) = T^{st}_{m-p,2m-q}(X)$?*
(iii) *(Stringy generalized Hodge conjecture) Is $G^{st}_{p,q}(X) = F^{st}_{p,q}(X)$?*
(iv) *(Stringy Hodge conjecture) Is $T^{st}_{p,2p}(X) = F^{st}_{p,2p}(X)$?*

By [**4**, Theorem 3.7], for a projective $\mathbb{Q}$-Gorenstein variety X of dimension d with at worst log-terminal singularities, Batyrev's stringy E-function satisfies the equality:

$$E_{st}(X;u,v) = (uv)^d E_{st}(X;u^{-1},v^{-1})$$

this follows basically from the strong Lefschetz theorem. Similar calculation shows that $G^{st}(X;u,v) = (uv^2)^d G^{st}(X;u^{-1},v^{-1})$ and

$$F^{st}(X;u,v) = (uv^2)^d F^{st}(X;u^{-1},v^{-1})$$

which follows from the fact that the Lefschetz isomorphism is an isomorphism of Hodge structures. These facts are some special cases of the following conjecture.

CONJECTURE 2. *(Generalized stringy GSCA) Let X be as above. If ϕ is a family of bounded motivic invariants of type (a,b), then $\phi^{st}(X;u,v) = (u^a v^b)^d \phi^{st}(X;u^{-1},v^{-1})$.*

If X is smooth projective and $\phi = T = \{T_{p,2p} | p \in \mathbb{Z}\}$, this is just the GSCA.

We verify this conjecture for normal projective $\mathbb{Q}$-Gorenstein toric varieties.

THEOREM 8. *Suppose that X is a normal projective $\mathbb{Q}$-Gorenstein toric varieties, then the generalized stringy GSCA holds.*

Proof. Let d be the dimension of X and $\phi = \{\phi_{i,j}\}$ be a family of bounded motivic invariants of type (a,b). By [**4**, Theorem 3.7], the stringy E-function of X satisfies the following relation: $E_{st}(X;u,v) = (uv)^d E_{st}(X;u^{-1},v^{-1})$. And by [**4**, Theorem 4.3], $E_{st}(X;u,v) = \sum_{\sigma\in\sum}\sum_{n\in\sigma^0\cap\mathbb{N}}(uv)^{-\varphi(n)}$ where X is defined by the fan $\sum$ on the lattice $\mathbb{N}$, and φ is a supporting function of X. Comparing the equality of the E-function, we get $(-1)^d\sum_{\sigma\in\sum}\sum_{n\in\sigma^0\cap N}(uv)^{\varphi(n)} = \sum_{\sigma\in\sum}\sum_{n\in\sigma^0\cap\mathbb{N}}(uv)^{-\varphi(n)}$. Now follow exactly the same calculation as in [**4**, Theorem 4.3], the stringy function satisfies the equality: $\phi^{st}(X;u,v) = (u^a v^b - 1)^d \sum_{\sigma\in\sum}\sum_{n\in\sigma^0\cap\mathbb{N}}(uv)^{-\varphi(n)} = (-1)^d(u^a v^b - 1)^d\sum_{\sigma\in\sum}\sum_{n\in\sigma^0\cap\mathbb{N}}(uv)^{\varphi(n)} = (u^a v^b)^d \phi^{st}(X;u^{-1},v^{-1})$.

PROPOSITION 11. *For a toric variety $X_{N,\Sigma}$ of dimension m,*

(i)

$$h_{p,q}(X_{N,\Sigma}) = \begin{cases} \sum_{k=0}^m d_{m-k}(-1)^{k-p}\binom{k}{k-p}, & \text{if } p=q \text{ and } 0\le p\le m,\\ 0, & \text{otherwise.}\end{cases}$$

where d_k is the number of cones of dimension k in Σ.

(ii) *the number $e^{p,q}(X_{N,\Sigma})$ is equal to $h_{p,q}(X_{N,\Sigma})$.*

(iii) *$T_{p,2p}(X_{N,\Sigma}) = T_{p,k}(X_{N,\sigma}) = N_{p,k}(X_{N,\Sigma}) = F_{p,k}(X_{N,\Sigma}) = h_{p,p}(X_{N,\Sigma})$ for $k\ge 2p$. In particular, the Friedlander-Lawson conjecture, the Friedlander-Mazur conjecture, the generalized Hodge conjecture, the Grothendieck standard conjecture are true for smooth toric varieties.*

Proof.

(i) The action of the torus $\mathbb{T}\sim(\mathbb{C}^*)^m$ on $X_{N,\Sigma}$ induces a stratification of $X_{N,\Sigma}$ into orbits of the torus action $O_\tau \cong (\mathbb{C}^*)^{m-dim\tau}$, one for each cone $\tau\in\Sigma$. Then we have

$$[X_{N,\Sigma}] = \sum_{k=0}^m d_{m-k}[\mathbb{L}-1]^k$$

in $K_0(SPV)$. Since $[\mathbb{L}-1]^k=\sum_{i=0}^{k}\binom{k}{i}(-1)^k\mathbb{L}^{k-i}$, we have

$$h_{p,q}([\mathbb{L}-1]^k)=\begin{cases}\binom{k}{k-p}(-1)^{k-p}, & \text{if } p=q \text{ and } 0\le p\le k\\ 0, & \text{otherwise}\end{cases}$$

and substitute into the formula

$$h_{p,q}([X_{N,\Sigma}])=\sum_{k=0}^{m}d_{m-k}h_{p,q}([\mathbb{L}-1]^k),$$

then we get the result.

(ii) It was calculated by Batyrev (see [**4**]) that $E(X_{N,\Sigma};u,v)=\sum_{k=0}^{m}d_{m-k}(uv-1)^k$. Then we make a simple comparison to the coefficients of $E(X_{N,\sigma})$ with the corresponding $h_{p,q}(X_{N,\Sigma})$.

(iii) We note that $T_{p,2p}(\mathbb{L}^k)=h_{p,p}(\mathbb{L}^k)$ for any p,k, this implies the equality of $T_{p,2p}(X_{N,\Sigma})=h_{p,p}(X_{N,\Sigma})$. The number $h_{p,q}(X_{N,\Sigma})=0$ if $p\neq q$, this implies that $T_{p,2p}(X_{N,\Sigma})=T_{p,k}(X_{N,\sigma})=N_{p,k}(X_{N,\Sigma})=F_{p,k}(X_{N,\Sigma})=h_{p,p}(X_{N,\Sigma})$ for $k\ge 2p$. For smooth toric varieties, $T_{p,2p}(X_{N,\Sigma})=h_{p,p}(X_{N,\Sigma})$ means that the homology group $H_{2p}(X_{N,\Sigma};\mathbb{Q})$ is generated by algebraic cycles hence all the conjectures are trivially true.

In their paper [**6**], Batyrev and Borisov proved the mirror duality conjecture for stringy Hodge numbers of Calabi-Yau complete intersections in Gorenstein Fano toric varieties, i.e., for a mirror pair (V,W) of such varieties of dimension n, their stringy E-functions satisfies the relation

$$E_{st}(V;u,v)=(-u)^nE_{st}(W;u^{-1},v)$$

which in particular gives the rotation of the Hodge diamond: $h^{p,q}(V)=h^{n-p,q}(W)$. We wonder if similar relation is true for the stringy T-function associated to the family $T=\{T_{j,n}|j,n\in\mathbb{Z}\}$. We form our conjecture below.

CONJECTURE 3. *If (V,W) is a mirror pair from Batyrev-Borisov's construction, then*

$$T^{st}(V;u,v)=(-u)^nT^{st}(W;u^{-1},v).$$

5. *Lawson-Deligne-Hodge polynomials*

5·1. *Varieties with finitely generated Lawson homology groups*

Let $VarFL$ be the collection of all quasi-projective varieties X such that the dimension of $L_rH_n(X;\mathbb{Q})$ is finite for all nonnegative integers n,r. Let $X,Y\in VarFL$ and Y be a locally closed subvariety of X. From the localization sequence of Lawson homology,

$$\cdots\to L_rH_{n+1}(X;\mathbb{Q})\to L_rH_{n+1}(X-Y;\mathbb{Q})\to L_rH_n(Y;\mathbb{Q})\to L_rH_n(X;\mathbb{Q})\to\cdots$$

we see that $X-Y$ is also in $VarFL$. Hence we may form the Grothendieck group $K_0(VarFL)$ of $VarFL$. The ring $\mathscr{L}=\mathbb{Z}\{\mathbb{L}^i|i\in\mathbb{Z}_{\ge 0}\}$ acts on $K_0(VarFL)$ and we consider $K_0(VarFL)$ as a $\mathscr{L}$-module under this action. Let $S=\{\mathbb{L}^i\}_{i\ge 0}\subset\mathscr{L}$ and $FL\mathcal{N}=S^{-1}K_0(VarFL)$ be the localization of $K_0(VarFL)$ with respect to the multiplicative set S. Let $F^kFL\mathcal{N}$ be the subgroup of $FL\mathcal{N}$ generated by elements of the form $\frac{[X]}{\mathbb{L}^i}$ where $i-dimX\ge k$. Then we have a decreasing filtration

$$\cdots\supset F^kFL\mathcal{N}\supset F^{k+1}FL\mathcal{N}\supset\cdots$$

of $FL\mathcal{N}$.

Definition 14. *We definite the FL-Kontsevich group to be the completion*

$$FL\hat{\mathcal{N}} := \lim_{\leftarrow} \frac{\mathcal{N}}{F^k FL\mathcal{N}}$$

with respect to the filtration defined above.

Let X be a smooth projective variety of dimension n and D an effective divisor on X with simple normal crossings. We use the notation D_J^0 as defined in 4.

Definition 15. *We say that a subset $A \subset \mathcal{L}(X)$ is FL-cylindrical if $A = \pi_k^{-1}(C)$ for some $C \subset \mathcal{L}_k(X), C \in VarFL$. For such set A, define*

$$\widetilde{\mu}(A) := [C]\mathbb{L}^{-kn}.$$

Let $\mathcal{C}$ be the collection of all countable disjoint unions of FL-cylindrical sets $\coprod_{i\in\mathbb{N}} A_i$ for which $\widetilde{\mu}(A_i) \to 0$ in $FL\hat{N}$. An element in $\mathcal{C}$ is called a FL-measurable set. We definite the FL-motivic measure to be $\mu : \mathcal{C} \to FL\hat{N}$ by

$$\mu(\coprod_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \widetilde{\mu}(A_i)$$

in $FL\hat{N}$. A function $\alpha : \mathcal{L}(X) \to \mathbb{Z} \cup \{\infty\}$ is FL-integrable if $\alpha^{-1}(n)$ is FL-measurable for each $n \in \mathbb{Z} \cup \{\infty\}$.

Proposition 12. *Let X be a smooth projective variety of dimension n and $D = \sum_{i=1}^r a_i D_i$ be an effective divisor with simple normal crossings on X. If all D_J^0 are in $VarFL$, then $\mathbb{L}^{-ord_t D}$ is FL-integrable, and $\mu((\mathbb{L}^{-ord_t D})^{-1}(\infty)) = 0$. We define the motivic integral of the pair (X, D) to be*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t D} d\mu := \sum_{s\in\mathbb{Z}_{\geq 0}} \mu((\mathcal{L}^{-ord_t D})^{-1}(s))$$

in $FL\hat{\mathcal{N}}$.

This calculation is same as in the proof of [**8**, Lemma 2.13] and [**8**, Theorem 2.15].

The following result gives a simpler way to see if $D_J^0 \in VarFL$.

Proposition 13. *$D_J^0 \in VarFL$ for all $J \subset \{1, ..., r\}$ if and only if $D_J \in VarFL$ for all $J \subset \{1, ..., r\}$.*

Proof. Let $R = \{1, ..., r\}$. We prove by induction on the length of subsets of R. $D_R = D_R^0 \in VarFL$. We assume that for $J \subset R$ with $|J| > k$, $D_J \in VarFL$. If $|J| = k$, we have

$$D_J^0 = D_J - \cup_{i\notin J} D_i = D_J - \cup_{i\notin J} D_i \cap D_J = D_J - \bigcup_{\substack{I\supset J \\ |I|=k+1}} D_I$$

Note that for two algebraic varieties $A, B \in VarFL$, if $A \cap B \in VarFL$, then $A \cup B \in VarFL$. Hence it suffices to prove that $\bigcup_{\substack{I\supset J \\ |I|=k+1}} D_I \in VarFL$. This follows once we claim that for any subsets $I_1, ..., I_t$ of R where each $|I_i| > k$, the union $\cup_{i=1}^t D_{I_i} \in VarFL$. We use induction again to prove this statement. When $t = 1$, $D_{I_1} \in VarFL$ is by the hypothesis of the first induction. We assume that this statement is true for $t = n$. Then for $t = n+1$, we have

$$(\bigcup_{i=1}^{n} D_{I_i}) \cap D_{I_{n+1}} = \bigcup_{i=1}^{n} (D_{I_i} \cap D_{I_{n+1}}) = \bigcup_{i=1}^{n} D_{I_i \cup I_{n+1}} \in VarFL$$

By induction hypothesis, $\cup_{i=1}^{n} D_{I_i}, D_{I_{n+1}}$ are in $VarFL$, hence $\cup_{i=1}^{n+1} D_{I_i} \in VarFL$. For another direction, a similar argument works.

We recall a definition from [**9**].

DEFINITION 16. *Let X, Y and F be algebraic varieties, and $A \subset X, B \subset Y$ be constructible subsets of X and Y respectively. We say that a map $p : A \to B$ is a piecewise trivial fibration with fiber F, if there exists a finite partition of B in subsets S which are locally closed in Y such that $p^{-1}(S)$ is locally closed in X and isomorphic, as a variety, to $S \times F$, with p corresponding under the isomorphism to the projection $S \times F \to S$.*

By the homotopy property of Lawson homology, we have $L_tH_n(X \times \mathbb{C}^k) = L_{t-k}H_{n-2k}(X)$ which implies the following Lemma.

LEMMA 1. *For a trivial bundle $X \times \mathbb{C}^k$ over X, $X \times \mathbb{C}^k \in VarFL$ if and only $X \in VarFL$.*

PROPOSITION 14. *Let X, Y and F be algebraic varieties, and $A \subset X, B \subset Y$ be constructible subsets of X and Y respectively. If $p : A \to B$ is a piecewise trivial fibration with fibre $\mathbb{C}^k$, then $A \in VarFL$ if and only if $B \in VarFL$.*

Proof. Consider $B = B_1 \coprod B_2$. Then $A = p^{-1}(B_1) \coprod p^{-1}(B_2)$ where $p^{-1}(B_1) \cong B_1 \times \mathbb{C}^k, p^{-1}(B_2) \cong B_2 \times \mathbb{C}^k$. From the localization sequences

$$\begin{tikzcd}
\cdots \arrow[r] & L_pH_n(p^{-1}(B_1)) \arrow[r] \arrow[d, "\cong"'] & L_pH_n(A) \arrow[r] \arrow[d] & L_pH_n(p^{-1}(B_2)) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & L_{p-k}H_{n-2k}(B_1) \arrow[r] & L_{p-k}H_{n-2k}(B) \arrow[r] & L_{p-k}H_{n-2k}(B_2) \arrow[r] & \cdots
\end{tikzcd}$$

we see that $L_pH_n(A) \cong L_{p-k}H_{n-2k}(B)$. Hence $A \in VarFL$ if and only if $B \in VarFL$. The general case follows by an induction on the number of components of the partition of B.

THEOREM 9. *(The change of variables formula) Suppose that X, Y are smooth projective varieties of dimension d and $h : Y \to X$ is a birational morphism with effective relative canonical divisor $D = \sum_{j=1}^{r} a_j D_j$ which has simple normal crossings. Assume that X, Y, D_J^0 are in $VarFL$ for any $J \subset \{1, ..., r\}$, then*

$$[X] = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t D} d\mu$$

Proof. Let $C_k = (\mathbb{L}^{-ord_t D})^{-1}(k)$ for $k \in \mathbb{Z}$.
We claim that $\mu(h_\infty(C_k)) \in FL\hat{\mathcal{N}}$ and

$$\mu(C_k) = \mu(h_\infty(C_k)) \cdot \mathbb{L}^k.$$

We have a commutative diagram:

$$\begin{tikzcd} C_k \subset \mathcal{L}(Y) \arrow[r, "h_\infty"] \arrow[d, "\pi_t"'] & h_\infty(C_k) \subset \mathcal{L}(X) \arrow[d, "\pi_t"] \\ B_t' \subset \mathcal{L}_t(Y) \arrow[r, "h_t"] & B_t \subset \mathcal{L}_t(X) \end{tikzcd}$$

where $B_t' = \pi_t(C_k), B_t = \pi_t(h_\infty(C_k))$ are constructible sets. By the calculation in Proposition 12, we see that

$$B_t' \cong \coprod_{J \subset \{1,...,r\}} \coprod_{(m_1,...,m_r) \in M_{J,k}} D_J^0 \cdot \mathbb{L}^{tn - \sum_{j\in J} m_j} \cdot (\mathbb{L}-1)^{|J|}$$

and by a local calculation in [**9**], Lemma 3.4(b), the restriction of h_t to B_t' is a piecewise trivial fibration with fiber $\mathbb{C}^k$ over B_t, and from this calculation we see that

$$B_t \cong \coprod_{J \subset \{1,...,r\}} \coprod_{(m_1,...,m_r) \in M_{J,k}} D_J^0 \cdot \mathbb{L}^{tn - \sum_{j\in J} m_j - k} \cdot (\mathbb{L}-1)^{|J|}$$

hence $[B_t] \in FL\mathcal{N}$ and $[B_t'] = [B_t]\mathbb{L}^k$. Therefore $h_\infty(C_k) = \pi_t^{-1}(B_t)$ is FL-measurable and $\mu(C_k) = \mu(h_\infty(C_k)) \cdot \mathbb{L}^k$.

Since

$$\mathcal{L}(X) = \coprod_{k \in \mathbb{Z}_{\geq 0} \cup \infty} h_\infty(C_k),$$

we have $[X] = \sum_{k\in\mathbb{Z}_{\geq 0}} \mu(h_\infty(C_k)) = \sum_{k\in\mathbb{Z}_{\geq 0}} \mu(C_k)\mathbb{L}^{-k} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t D} d\mu$.

Definition 17. *Two smooth projective varieties $X_1, X_2 \in VarFL$ are said to be FLK-equivalent if there exists a smooth projective variety $Y \in VarFL$ and two birational morphisms $\rho_1 : Y \to X_1, \rho_2 : Y \to X_2$ such that $\rho_1^* K_{X_1} = \rho_2^* K_{X_2}$ and the effective divisor $D = K_Y - \rho_1^* K_{X_1} = \sum_{i=1}^r a_i D_i$ has simple normal crossings and $D_J^0 \in VarFL$ for any $J \subset \{1, ..., r\}$.*

Directly from the change of variables formula, we get the following result.

Corollary 6. *If two smooth projective varieties $X_1, X_2 \in VarFL$ are FLK-equivalent, then $[X_1] = [X_2]$ in $FL\hat{\mathcal{N}}$.*

We recall that each Lawson homology group $L_r H_n(X; \mathbb{Q})$ has an inductive limit of mixed Hodge structure (see [**17**]). And by a result of Walker (see [**26**]), the localization sequence of Lawson homology groups is a sequence of inductive limit of mixed Hodge structures. Since we are considering finite Lawson homology groups, an inductive limit of mixed Hodge structure is just a mixed Hodge structure.

Fix $n \in \mathbb{Z}_{\geq 0}$. For $X \in VarFL$, let $h_{p,q}(L_n H_i(X; \mathbb{C}))$ be the dimension of the (p,q)-type Hodge component in $L_n H_i(X; \mathbb{C})$. We define the Lawson-Deligne-Hodge polynomial of $X \in VarFL$ to be

$$F_n E(X) := \sum_{p,q} F_n E^{p,q}(X) u^p v^q$$

where

$$F_n E^{p,q}(X) := \sum_{i \geq 2n} (-1)^i h_{p,q}(L_n H_i(X; \mathbb{C}))$$

And we define the F_rL-Euler characteristic of X to be

$$F_rL(X) = \sum_k (-1)^k dim L_r H_k(X; \mathbb{Q})$$

From the localization sequence of Lawson homology, it is not difficult to see that F_rL, F_nE induces a group homomorphism from $FL\overline{\mathcal{N}}$, the image of $FL\mathcal{N}$ in $FL\hat{\mathcal{N}}$, to $\mathbb{Z}$ and $\mathbb{Z}[u, v]$ respectively.

COROLLARY 7. *Two FLK-equivalent smooth projective varieties have the same Lawson-Deligne-Hodge polynomial and the F_rL-Euler characteristic for any $r \geq 0$.*

5·2. *Higher Chow groups*

In the previous section, we use only the properties of localization sequences of Lawson homology, the homotopy property and the projective bundle theorem. Since there are analogous theorems for higher Chow groups, we can play the same game for higher Chow groups. Let $VarCH$ be the collection of all quasi-projective X whose higher Chow groups $CH^r(X, n)$ are all finitely generated for any r, n. Then from the localization sequence of higher Chow groups:

$$\cdots \to CH^{q-d}(Z, p) \to CH^q(X, p) \to CH^q(U, p) \to CH^{q-d}(Z, p-1) \to \cdots$$

where $Z \subset X$ is a closed subvariety of codimension d and U is its complement. Hence if X, Z are in $VarCH$, then U is in $VarCH$. Then we can form the Grothendieck group $K_0(VarCH)$ of $VarCH$. Similar to what we have done for Lawson homology, we have some analogous results. We form $CH\mathcal{N}$, $CH\hat{\mathcal{N}}$ and $CH\overline{\mathcal{N}}$ as their analogs in Lawson homology.

DEFINITION 18. *Two smooth projective varieties $X_1, X_2 \in VarCH$ are said to be CHK-equivalent if there exists a smooth projective variety $Y \in VarCH$ and two birational morphisms $\rho_1 : Y \to X_1, \rho_2 : Y \to X_2$ such that $\rho_1^* K_{X_1} = \rho_2^* K_{X_2}$ and the effective canonical relative divisor $D = K_Y - \rho_1^* K_{X_1} = \sum_{i=1}^r a_i D_i$ has simple normal crossings and $D_J^0 \in VarCH$ for any $J \subset \{1, ..., r\}$.*

THEOREM 10. *Suppose that $X_1, X_2 \in VarCH$ are two smooth projective varieties which are CHK-equivalent. Then $[X_1] = [X_2]$ in $CH\overline{\mathcal{N}}$.*

6. *Projective bundle theorem*

In [**12**], Friedlander proved a projective bundle theorem in morphic cohomology for smooth normal quasi-projective varieties. In this section we prove a projective bundle theorem of trivial bundles for all normal quasi-projective varieties without assuming smoothness. Since we do not have a Mayer-Vietoris sequence in morphic cohomology, the proof is much more complicated than its counterpart in Lawson homology. We are not sure if our approach may work for general bundles.

DEFINITION 19. *Suppose that X is a normal quasi-projective variety and W, Y are projective varieties. Let $Z_k(W)(Y)$ be the subgroup of $Z_{k+m}(W \times Y)$ consisting of algebraic cycles equidimensional over Y where m is the dimension of Y and k is the dimension of a fibre. The Chow variety $\mathscr{C}_{r,d}(W \times Y)$ of r-dimensional algebraic cycles of degree d of $W \times Y$ is a projective variety and $\mathscr{M}or(X, \mathscr{C}_{r,d}(W \times Y))$, the collection of all algebraic morphisms from X to $\mathscr{C}_{r,d}(W \times Y)$, is enrolled with the topology of convergence* *with bounded degree (see [**16**]). We define $\mathscr{M}or(X, \mathscr{C}_{r,d}(W)(Y))$ to be the subspace of $\mathscr{M}or(X, \mathscr{C}_{r,d}(W \times Y))$ consisting of morphisms f such that $f(x) \in \mathscr{C}_{r,d}(W)(Y)$ for all $x \in X$ where $\mathscr{C}_{r,d}(W)(Y)$ is the collection of all algebraic r-cycles of degree d of $Y \times W$ equidimensional over Y. Let*

$$\mathscr{M}or(X, \mathscr{C}_r(W)(Y)) := \coprod_{d \geq 0} \mathscr{M}or(X, \mathscr{C}_{r,d}(W)(Y))$$

which is a topological monoid and let

$$\mathscr{M}or(X, Z_r(W)(Y)) := [\mathscr{M}or(X, \mathscr{C}_r(W)(Y))]^+$$

be its naive group completion. The inclusion map $i : \mathscr{M}or(X, \mathscr{C}_r(W)(Y)) \hookrightarrow \mathscr{M}or(X, \mathscr{C}_{r+m}(W \times Y))$ induces a continuous homomorphism

$$\mathscr{D}' : \mathscr{M}or(X, Z_k(W)(Y)) \to \mathscr{M}or(X, Z_{k+m}(W \times Y))$$

Friedlander and Lawson proved a moving lemma in [**15**] which has numerous good properties including moving algebraic cycles algebraically. Their moving lemma has been used by them to prove a duality theorem between Lawson homology and morphic cohomology, and by Friedlander and Voevodsky to prove a duality theorem for motivic cohomology and homology. Here we need the algebraicity of the maps in their moving lemma when we consider algebraic cycles with bounded degree.

Lemma 2. *Let Y be a projective variety. From the Friedlander-Lawson moving lemma (see [**15**]), we have a map $\widetilde{\Psi}_t = (\Psi_{1,t}, \Psi_{2,t}) : \mathscr{C}_r(Y) \to \mathscr{C}_r(Y) \times \mathscr{C}_r(Y)$ for $t \in I$ where I is the unit interval $[0,1]$ and $\mathscr{C}_r(Y) = \coprod_{d \geq 0} \mathscr{C}_{r,d}(Y)$ is the Chow monoid of r-cycles of Y. The restriction of the maps*

$$\Psi_{1,t}, \Psi_{2,t} : \mathscr{C}_{r,d}(Y) \to \mathscr{C}_{r,d'}(Y)$$

are algebraic morphisms.

Theorem 11. *Suppose that X is a normal quasi-projective variety and W, Y are smooth projective varieties. Then the map*

$$\mathscr{D}' : \mathscr{M}or(X, Z_r(W)(Y)) \to \mathscr{M}or(X, Z_{r+m}(W \times Y))$$

is a homotopy equivalence.

Proof. Let $\widetilde{\Psi}_t = (\Psi_{1,t}, \Psi_{2,t})$ be the map from Friedlander-Lawson moving lemma where $t \in I$. By abuse of notation, we define a map $\Psi_t : \mathscr{M}or(X, Z_r(W)(Y)) \to \mathscr{M}or(X, Z_r(W)(Y))$ by

$$\Psi_t(f) := \Psi_{1,t}(f) - \Psi_{2,t}(f)$$

and $\Psi_{i,t}(f)(x) := \Psi_{i,t}(f(x))$ for $x \in X, i = 1, 2$.

Let

$$K_d := \coprod_{d_1 + d_2 \leq d} \mathscr{M}or(X, \mathscr{C}_{r,d_1}(W \times Y)) \times \mathscr{M}or(X, \mathscr{C}_{r,d_2}(W \times Y))/\sim,$$

$$K'_d := \coprod_{d_1 + d_2 \leq d} \mathscr{M}or(X, \mathscr{C}_{r,d_1}(W)(Y)) \times \mathscr{M}or(X, \mathscr{C}_{r,d_2}(W)(Y))/\sim$$

where $(f_1, g_1) \sim (f_2, g_2)$ if and only if $f_1 + g_2 = f_2 + g_1$.

The topology of $\mathscr{M}or(X, Z_{r+m}(W \times Y))$ is same as the weak topology defined by the filtration

$$K_0 \subset K_1 \subset K_2 \subset \cdots = \mathscr{M}or(X, Z_{r+m}(W \times Y))$$

and the topology of $\mathscr{M}or(X, Z_r(W)(Y))$ is same as the weak topology defined by the filtration

$$K'_0 \subset K'_1 \subset K'_2 \subset \cdots = \mathscr{M}or(X, Z_{r+m}(W)(Y))$$

From [**24**, Lemma 2.3], these two filtrations are locally compact. Let $\phi_{e,t}, \phi'_{e,t}$ be the restriction of $\Psi_{1,t}, \Psi_{2,t}$ to K_e and K'_e respectively. Let $\lambda_e : K_e \times \{1\} \to \mathscr{M}or(X, Z_r(W)(X))$ be $\phi_{e,1}$. Then we have the following diagrams:

$$\begin{tikzcd}
K'_e \times I \arrow[r, "\phi'_e"] \arrow[d, "\mathscr{D}' \times Id"'] & \mathscr{M}or(X, Z_r(W)(Y)) \arrow[d, "\mathscr{D}'"] \\
K_e \times I \arrow[r, "\phi_e"] & \mathscr{M}or(X, Z_{r+m}(W \times Y))
\end{tikzcd}$$

$$\begin{tikzcd}
K'_e \times \{1\} \arrow[r, hook] \arrow[d, "\mathscr{D}'"'] & K'_e \times I \arrow[r, "\phi'_e"] & \mathscr{M}or(X, Z_r(W)(Y)) \arrow[d, "\mathscr{D}'"] \\
K_e \times \{1\} \arrow[r, hook] & K_e \times I \arrow[r, "\phi_e"] \arrow[ur, "\lambda_e"] & \mathscr{M}or(X, Z_{r+m}(W \times Y))
\end{tikzcd}$$

Then by [**16**, Lemma 5.2], $\mathscr{D}'$ is a weak homotopy equivalence. Since $\mathscr{M}or(X, Z_r(W)(Y))$ and $\mathscr{M}or(X, Z_{r+m}(W \times Y))$ have the homotopy type of a CW-complex, by Whitehead theorem, $\mathscr{D}'$ is a homotopy equivalence.

PROPOSITION 15. *Suppose that X is a normal quasi-projective variety and W, Y are projective varieties. Then $\mathscr{M}or(X \times Y, Z_r(W))$ is isomorphic as a topological group to $\mathscr{M}or(X, Z_r(W)(Y))$.*

Proof. There is a natural bijection

$$\psi : \mathscr{M}or(X \times Y, \mathscr{C}_r(W)) \to \mathscr{M}or(X, \mathscr{C}_r(W)(Y))$$

defined by $\psi(f)(x)(y) := f(x, y)$. These two spaces are obviously homeomorphic under the topology of convergence with bounded degree and ψ is monoid isomorphism. Hence we complete the proof.

Consider the topology of convergence with bounded degree, we have the following fact.

LEMMA 3. *For a normal quasi-projective variety X and a projective variety Y, let $\mathscr{M}or(X, Z_{r_1}(Y) \times \cdots \times Z_{r_k}(Y))$ be the topological naive group completion of $\mathscr{M}or(X, \mathscr{C}_{r_1}(Y) \times \mathscr{C}_{r_2}(Y) \times \cdots \times \mathscr{C}_{r_k}(Y))$. Then there is an isomorphism of topological groups:*

$$\mathscr{M}or(X, Z_{r_1}(Y) \times Z_{r_2}(Y) \times \cdots \times Z_{r_k}(Y)) \cong \mathscr{M}or(X, Z_{r_1}(Y)) \times \mathscr{M}or(X, Z_{r_2}(Y)) \times \cdots \times \mathscr{M}or(X, Z_{r_k}(Y))$$

DEFINITION 20. *For normal quasi-projective varieties X, U, if $U = Y - Z$ where Y, Z are projective varieties and Z is a subvariety of Y, then we define*

$$\mathscr{M}or(X, Z_i(U)) := \frac{\mathscr{M}or(X, Z_i(Y))}{\mathscr{M}or(X, Z_i(Z))}$$

PROPOSITION 16. *(localization sequence) For X, U, Y, Z as above. There is a localization sequence:*

$$\cdots \to \pi_k\mathscr{M}or(X, Z_i(Z)) \to \pi_k\mathscr{M}or(X, Z_i(Y)) \to \pi_k\mathscr{M}or(X, Z_i(U)) \to \pi_{k-1}\mathscr{M}or(X, Z_i(Z)) \to \cdots$$

DEFINITION 21. *For a normal quasi-projective variety X and a projective variety Y, we define*

$$\mathscr{M}or(X, Z^t(Y)) := \frac{\mathscr{M}or(X, Z_0(\mathbb{P}^t)(Y))}{\mathscr{M}or(X, Z_0(\mathbb{P}^{t-1})(Y))}$$

with the quotient topology.

THEOREM 12. *For X a normal quasi-projective variety and Y a smooth projective variety,*
(i) *$\mathscr{M}or(X, Z_r(Y \times \mathbb{A}^t))$ is homotopy equivalent to $\mathscr{M}or(X, Z_{r-t}(Y))$ if $r \geq t$.*
(ii) *(Duality) $\mathscr{M}or(X, Z^t(Y))$ is homotopy equivalent to $\mathscr{M}or(X, Z_{n-t}(Y))$.*

Proof.
(i) By [**12**, Proposition 3.7], the suspension $\Sigma\!\!\!/_* : \mathscr{M}or(X, Z_r(Y)) \to \mathscr{M}or(X, Z_{r+1}(\Sigma\!\!\!/ Y))$ is a homotopy equivalence. Let L be the restriction of the hyperplane line bundle $\mathscr{O}(1)$ of $\mathbb{P}^N$ to Y and observe that $L = \Sigma\!\!\!/ Y - \{\infty\}$. Hence $\mathscr{M}or(X, Z_r(Y))$ is homotopy equivalent to $\mathscr{M}or(X, Z_{r+1}(L))$. Now the result follows from an induction on dimension of Y and the localization sequence.
(ii) Consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{M}or(X, Z_0(\mathbb{P}^{t-1})(Y)) \arrow[r] \arrow[d, "\mathscr{D}"] & \mathscr{M}or(X, Z_0(\mathbb{P}^{t})(Y)) \arrow[r] \arrow[d, "\mathscr{D}"] & \mathscr{M}or(X, Z^t(Y)) \arrow[d] \\
\mathscr{M}or(X, Z_n(Y \times \mathbb{P}^{n-1})) \arrow[r] & \mathscr{M}or(X, Z_n(Y \times \mathbb{P}^{t})) \arrow[r] & \mathscr{M}or(X, Z_n(Y \times \mathbb{A}^{t}))
\end{tikzcd}$$

where $Z^t(Y) = \frac{Z_0(\mathbb{P}^t)(Y)}{Z_0(\mathbb{P}^{t-1})(Y)}$. From the induced long exact sequence of homotopy groups and the result above, we see that $\mathscr{M}or(X, Z^t(Y))$ is homotopy equivalent to $\mathscr{M}or(X, Z_n(Y \times \mathbb{A}^t)) \cong \mathscr{M}or(X, Z_{n-t}(Y))$.

THEOREM 13. *For a projective variety Y, there is a splitting*

$$\xi : \mathscr{M}or(X, Z_0(\mathbb{P}^t)(Y)) \to \mathscr{M}or(X, Z^t(Y)) \times \mathscr{M}or(X, Z^{t-1}(Y)) \times \cdots \times \mathscr{M}or(X, Z^0(Y))$$

which is a homotopy equivalence.

Proof. By the construction in the proof of the splitting theorem of Lawson and Friedlander ([**14**, Theorem 2.10]), there is a projection map

$$p : Z_0(\mathbb{P}^t)(Y) \to Z_0(\mathbb{P}^t)(Y) \times Z_0(\mathbb{P}^{t-1})(Y) \times \cdots \times Z_0(\mathbb{P}^0)(Y)$$

Write $p = (p_t, p_{t-1}, ..., p_0)$ and for $f \in \mathscr{M}or(X, Z_0(\mathbb{P}^t)(Y))$, define

$$p_i(f)(x) := p_i(f(x))$$

Then we get a map

$$\xi^t : \mathscr{M}or(X, Z_0(\mathbb{P}^t)(Y)) \to \mathscr{M}or(X, Z^t(Y)) \times \mathscr{M}or(X, Z^{t-1}(Y)) \times \cdots \times \mathscr{M}or(X, Z^0(Y))$$

defined by

$$\xi^t(f) = (p_t(f) + \mathscr{M}or(X, Z_0(\mathbb{P}^{t-1})(Y)), p_{t-1}(f) + \mathscr{M}or(X, Z_0(\mathbb{P}^{t-2})(Y)), ..., p_0(f) + \mathscr{M}or(X, Z_0(\mathbb{P}^0)(Y))$$

We are going to show this map is a homotopy equivalence. We prove by induction on t. When $t=0$, this follows from definition. Assume that we have the splitting for $t-1$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{M}or(X, Z_0(\mathbb{P}^{t-1})(Y)) \arrow[r, "\xi^{t-1}"] \arrow[d] & \mathscr{M}or(X, Z^{t-1}(Y)) \times \cdots \times \mathscr{M}or(X, Z^0(Y)) \arrow[d] \\
\mathscr{M}or(X, Z_0(\mathbb{P}^{t})(Y)) \arrow[r, "\xi^{t}"] \arrow[d] & \mathscr{M}or(X, Z^{t}(Y)) \times \mathscr{M}or(X, Z^{t-1}(Y)) \times \cdots \times \mathscr{M}or(X, Z^0(Y)) \arrow[d] \\
\mathscr{M}or(X, Z^{t}(Y)) \arrow[r, "="] & \mathscr{M}or(X, Z^{t}(Y))
\end{tikzcd}$$

Follow from the long exact sequences of homotopy groups induced from the vertical rows, we see that ξ^t is a weak homotopy equivalence. Since all spaces have the homotopy type of a CW-complex, by the Whitehead theorem, ξ^t is a homotopy equivalence.

Combining the splitting and the duality theorem, we get the following splitting.

COROLLARY 8. *For a normal quasi-projective variety X and Y a smooth projective of dimension n, if $t \leq n$, there is a splitting*

$$\xi' : \mathscr{M}or(X, Z_0(\mathbb{P}^t)(Y)) \cong \mathscr{M}or(X, Z_{n-t}(Y)) \times \mathscr{M}or(X, Z_{n-t+1}(Y)) \times \cdots \times \mathscr{M}or(X, Z_n(Y))$$

which is a homotopy equivalence.

THEOREM 14. *For a normal quasi-projective variety X, there is a homotopy equivalence $\overline{\eta} : Z^t(X \times \mathbb{P}^e) \cong \bigoplus_{i=0}^{t} Z^{t-i}(X)$ for $e \geq t$.*

Proof. By Lawson suspension theorem (see [**12**, Proposition 3.7]), there is a homotopy equivalent $\Sigma\!\!\!/_* : \mathscr{M}or(X, Z_{e-t+i}(\mathbb{P}^e)) \cong \mathscr{M}or(X, Z_0(\mathbb{P}^{t-i}))$, We have a homotopy equivalence

$$\eta : \mathscr{M}or(X \times \mathbb{P}^e, Z_0(\mathbb{P}^t)) \xrightarrow{\psi} \mathscr{M}or(X, Z_0(\mathbb{P}^t)(\mathbb{P}^e)) \xrightarrow{\xi} \bigoplus_{i=0}^{t} \mathscr{M}or(X, Z^{t-i}(\mathbb{P}^e)) \xrightarrow{\mathscr{D}} \bigoplus_{i=0}^{t} \mathscr{M}or(X, Z_0(\mathbb{P}^{t-i}))$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{M}or(X \times \mathbb{P}^e, Z_0(\mathbb{P}^{t-1})) \arrow[r] \arrow[d, "\eta"] & \mathscr{M}or(X \times \mathbb{P}^e, Z_0(\mathbb{P}^{t})) \arrow[r] \arrow[d, "\eta"] & Z^t(X \times \mathbb{P}^e) \arrow[d, "\overline{\eta}"] \\
\bigoplus_{i=0}^{t-1} \mathscr{M}or(X, Z_0(\mathbb{P}^{t-1-i})) \arrow[r] & \bigoplus_{i=0}^{t} \mathscr{M}or(X, Z_0(\mathbb{P}^{t-i})) \arrow[r] & \bigoplus_{i=0}^{t} Z^{t-i}(X)
\end{tikzcd}$$

From the long exact sequences of homotopy groups induced by the horizontal rows, we see that $\overline{\eta}$ is a weak homotopy equivalence. But all these spaces have the homotopy type of CW-complexes, hence $\overline{\eta}$ is a homotopy equivalence.

The morphic cohomology groups are known only for very few cases of smooth varieties, and almost nothing about singular varieties. As an application of above result, we calculate the morphic cohomology groups of two singular surfaces.

EXAMPLE 2. *One of the main tools we use is* [**14**, Theorem 9.1] *which says that there is a fibration $Z^1(X) \to Pic(X)$ with homotopy fibre $K(\mathbb{Z}, 2)$ for projective variety X.*

(i) *Let $C_1 : y^2Z = x^2(x+z)$ be the curve in $\mathbb{P}^2$ which has a node at $[0:0:1]$. The Picard group of C_1 is $Pic(C_1) \cong \mathbb{C}^* \times \mathbb{Z}$. Therefore we know that $L^1H^k(C_1 \times \mathbb{P}^1)$ is 0 for $k > 2$. We list other cases in the following table.*

k	$\pi_k Pic(C_1)$	$\pi_k Z^1(C_1)$	$\pi_k Z^0(C_1)$	$L^1H^k(C_1 \times \mathbb{P}^1)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}$	$\mathbb{Z}$	*0*	$\mathbb{Z}$
2	*0*	$\mathbb{Z}$	*0*	$\mathbb{Z} \oplus \mathbb{Z}$

(ii) *Let $C_2 : y^2z = x^3$ be the curve in $\mathbb{P}^2$ which has a cusp at $(0:0:1)$. The Picard group of C_2 is $Pic(C_2) \cong \mathbb{C} \times \mathbb{Z}$. Hence $L^1H^k(C_2 \times \mathbb{P}^1)$ is 0 for $k > 2$. We list other cases in the following table.*

k	$\pi_k Pic(C_2)$	$\pi_k Z^1(C_2)$	$\pi_k Z^0(C_2)$	$L^1H^k(C_2 \times \mathbb{P}^1)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	*0*	*0*	*0*	*0*
2	*0*	$\mathbb{Z}$	*0*	$\mathbb{Z} \oplus \mathbb{Z}$

Acknowledgements. The author thanks Chin-Lung Wang for a question he asked, Willem Veys and Francois Loeser for answering some of his trivial questions and the referee for his very nice comments and suggestions. He thanks the National Center for Theoretical Sciences in Hsinchu, Taiwan for financial support.

REFERENCES

[**1**] D. Arapura, *Hodge cycles on some moduli spaces*, arXiv, math.AG/0102070.

[**2**] D.Abramovich, K. Karu, K. Matsuki, J. Wlodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc., 15 (2002), 531-572.

[**3**] D. Arapura, S.-J Kang, *Coniveau and the Grothendieck group of varieties*, ArXiv:math/0506210.

[**4**] V. V. Batyrev, *Stringy Hodge numbers of varities with Gorenstein canonical singularities*, Proc. Taniguchi Symposium 1997, In 'Integrable Systems and Algebraic Geometry, Kobe/Kyoto 1997', World Sci. Publ. (1999), 1-32.

[**5**] V. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New Trends in Algebraic geometry, Euroconference on Algebraic Geometry(Warwick 1996), London Math. Soc. Lecture Note Ser. 264, K. Hulek et at Ed., CUP, 1999, 1-11.

[**6**] V. V. Batyrev and L. A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math., Vol. 126 (1996), no. 1, 183-203.

[**7**] F. Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compositio Math. 140 (2004), 1011-1032.

[**8**] A. Craw, *An introduction to motivic integration*, Strings and Geometry, 203-225, Clay Math. Proc. 3, Amer. Math. Soc., Providence, RI, 2004.

[**9**] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999), 201-232.

[**10**] J. Denef and F. Loeser, *Motivic integration, quotient singularities and the Mckay correspondence*, Compositio Math. 131 (2002), 267-290.

[**11**] E. Friedlander, *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. **77** (1991), 55-93.

[**12**] E. Friedlander, *Algebraic cocycles on normal, quasi-projective varieties*, Compositio Mathematica 110 (1998), 127-162.

[**13**] E. Friedlander and O. Gabber, *Cycles spaces and intersection theory*, in Topological Methods in Modern Mathematics (1993), 325-370.

[**14**] E. FRIEDLANDER AND H.B. LAWSON, *A theory of algebraic cocycles*, Annals of Math. **136** (1992), 361-428.
[**15**] E. FRIEDLANDER AND H.B. LAWSON, *Moving algebraic cycles of bounded degree*, Invent. Math. **132** (1998), 91-119.
[**16**] E. FRIEDLANDER AND H.B. LAWSON, *Duality relating spaces of algebraic cocycles and cycles*, Topology **36** (1997), 533-565.
[**17**] E. FRIEDLANDER AND B. MAZUR, *Filtration on the homology of algebraic varieties*, Memoir of the A.M.S., no. 529 (1994).
[**18**] P. GRIFFITHS AND J. HARRIS, *Principles of algebraic geometry*, Wiley & Sons, 1994.
[**19**] WENCHUAN HU, *Birational invariants defined by Lawson homology*, arXiv, math.AG/0511722.
[**20**] H.B. LAWSON, *Algebraic cycles and homotopy theory*, Annals of Math. **129** (1989), 253-291.
[**21**] J. LEWIS, *A survey of the Hodge conjecture*, 1999, Amer. Math. Soc., CRM 10.
[**22**] F. LOESER, *Seattle lecture notes on motivic integration*, http://www.dma.ens.fr /∼ loeser/notes_seattle_17_01_2006.pdf.
[**23**] C. MCCRORY, A. PARUSINSKI, *Virtual Betti numbers of real algebraic varieties*, Comptes Rendus Acad. Sci. Paris, Ser. I, 336 (2003), 763-768.
[**24**] JYH-HAUR TEH, *A homology and cohomology theory for real projective varieties*, arXiv.org, math.AG/0508238.
[**25**] JYH-HAUR TEH, *The Grothendieck standard conjectures, morphic cohomology and the Hodge index theorem*, arXiv.org, math.AG/0512232.
[**26**] M. WALKER, *The morphic Abel-Jacobi map*, Compositio, **143,** (2007), 909-944.
[**27**] C.-L WANG, *On the topology of birational minimal models*, J. Diff. Geo., 50 (1998), 129-146.