Schedule　　不再頹廢!!相信你是最強的

2012-11-12.(一)

讀物理、游泳、普物預報

下午✓晚、　✓　✓

2012-11-13(二)

中文作業、讀微積分,　　、要去買頭帶,

✓　　X　晚、重訓 ✓

2012-11-14(三)

讀微積分,上午,下午、游泳 X

✓　✓

2012-11-15(四) 9:00 pm,工一,北機工人

讀微積分,上午、晚、英文作業

2012-11-16(五)

2013-1-4(五)

要去交作業,選課,

2012-11-14

The police

Ch2. Limits and Continuity

◎ Section 2.2 [Definition of limit]

- 名詞. 無理數. 任何不循環的無限小數.

- Def. 2.2.1

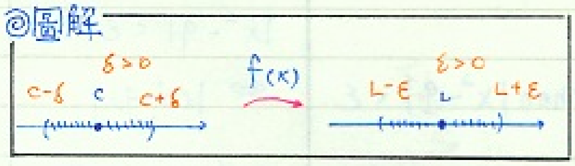
$C \in R, L \in R, p > 0 . p \in R.$

Let $f : (C-p, c) \cup (c, c+p) \to R$ be a function.
"不包含c"

$\lim_{x \to c} f(x) = L$. iff. $\forall \varepsilon > 0, \exists \delta > 0$, such that if $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$

若且唯若 (對所有) (存在) 使得. (s.t.)

◎圖解



$0 < |x-c| < \delta \Leftrightarrow c - \delta < x < c + \delta, x \neq c$

$|f(x) - L| < \varepsilon \Leftrightarrow L - \varepsilon < f(x) < L + \varepsilon$

- eg 1. prove $\lim_{x \to 2} (2x-1) = 3$

○Think. $|2x-1-3| < \varepsilon$

$\overset{\text{iff}}{\Leftrightarrow} |2x-4| < \varepsilon$

$\Leftrightarrow 2|x-2| < \varepsilon$

$\Leftrightarrow |x-2| < \frac{\varepsilon}{2} = \delta$

pf: $c = 2$, $L = 3.$, $f(x) = 2x-1$

we need to prove that

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$

$\forall \varepsilon > 0, \delta = \frac{\varepsilon}{2}$. so if $0 < |x-2| < \delta$

then $|x-2| < \varepsilon/2$. so. $2|x-2| < \varepsilon$

hence $|2x-4| < \varepsilon$ Finally $|2x-1-3| < \varepsilon$

- eg 2. prove $\lim_{x \to -1} (2-3x) = 5$

○Think

$|2-3x-5| < \varepsilon$

$\Leftrightarrow 3|x+1| < \varepsilon$

$\Leftrightarrow |x+1| < \frac{\varepsilon}{3}$

pf. $c = -1$, $L = 5$, $f(x) = 2-3x$

$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$

if $0 < |x+1| < \delta$ then $|x+1| < \frac{\varepsilon}{3}$

Finally $|2-3x-5| < \varepsilon$

made by 101033236

- eg3 prove $\lim_{x\to c} x = c$ [恆等函數].

pf: $c = L$, $f(x) = x$

$\forall \varepsilon > 0$ take $\delta = \varepsilon$ so. if $0 < |x-c| < \delta$ then. $|x-c| < \varepsilon$.

- eg4. prove $\lim_{x\to c} |x| = |c|$

pf. $f(x) = |x|$. $L = |c|$

$\forall \varepsilon > 0$ take $\delta = \varepsilon$ so.

if $0 < |x-c| < \delta$ then. we have $|x-c| < \varepsilon$

Finally we have $\big||x| - |c|\big| \le |x-c| < \varepsilon$

○ Think.

$\big||x| - |c|\big| < \varepsilon$

$|x-c| < \delta$ ↓找關係!!

$\big||x| - |c|\big| \le |x-c|$

- eg6. prove $\lim_{x\to 3} x^2 = 9$

pf. It suffices to prove

$\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t. if $0 < |x-3| < \delta$, then $|x^2 - 9| < \varepsilon$

Take $|x-3| < 1$ then $2 < x < 4$

↑自己取,證明有許多方法

So $|x-3| <$ ⑦ $\frac{\varepsilon}{7}$? So. $\forall \varepsilon > 0$ take $\delta = \min\{1, \frac{\varepsilon}{7}\}$

Hence if $0 < |x-3| < \delta$ then we have

$|x^2 - 9| = |x+3||x-3| < 7|x-3| < 7\delta \le 7(\frac{\varepsilon}{7}) = \varepsilon$

○ Think

$|x^2 - 9| < \varepsilon$

$\Leftrightarrow |(x^2 - 9)|$

$= |(x+3)||(x-3)|$

$< 7|x-3| < \varepsilon$

- eg7. prove $\lim_{x\to 4} \sqrt{x} = 2$

pf. since we have $\sqrt{x}$. we require $x \ge 0$

The since we need to have $|x-4| < \delta$, $4 - \delta < x < 4 + \delta$

So we need to have $4 - \delta \ge 0$ that is $\delta \le 4$

↳ $(x \ge 0)$

$\forall \varepsilon > 0$. take $\delta \min = \{4, \varepsilon\}$

So if $0 < |x-4| < \delta$ then since

we have

$|x-4| = |\sqrt{x} - 2||\sqrt{x} + 2|$

and since $|\sqrt{x} + 2| \ge 2$ we have

$|\sqrt{x} - 2| < |x-4| < \delta \le \varepsilon$

○ Think.

$|x-4|$

$= |\sqrt{x} + 2||\sqrt{x} - 2|$

$(x \ge 0)$

since $|\sqrt{x} + 2| \ge 2$

$|\sqrt{x} - 2| < |x-4| < \varepsilon$

$|x-4| < \delta$

◎ 2.2.7

$\lim\limits_{x\to c} f(x) = L \Leftrightarrow \lim\limits_{x\to c} (f(x) - L) = 0$

Def. 2.2.7 $f: (c-p, c) \to \mathbb{R}$ (left hand side)

is a function $\lim\limits_{x\to c^-} f(x) = L$

iff $\forall \varepsilon > 0\ \exists\ \delta > 0$, s.t. if $c - \delta < x < c$

then $|f(x) - L| < \varepsilon$

◎ 2.2.8

$f: (c, c+p) \to \mathbb{R}$

is a function $\lim\limits_{x\to c^+} f(x) = L$ (right hand side)

iff $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t. if $c < x < c + \delta$

then

$|f(x) - L| < \varepsilon$

- eg 9.

$f(x) \begin{cases} 2x+1 & x \le 0 \\ x^2 - x & x > 0 \end{cases}$ $\lim\limits_{x\to 0^-} f(x) = 1$

極限不存在

$\lim\limits_{x\to 0^+} f(x) = 0$

◎ 2.3.1 <uniqueness>

Pf. suppose not, that is $L \neq M$

Since $L \neq M$, $|L - M| > 0$ "兩不相同的數相減$\neq 0$"

Since $\lim\limits_{x\to c} f(x) = L$ for $\left|\frac{L-M}{2}\right| > 0$, $\exists\ \delta > 0$ s.t.

(1) if $0 < |x - c| < \delta$, then $|f(x) - L| < \left|\frac{L-M}{2}\right|$

moreover since $\lim\limits_{x\to c} f(x) = M$ for $\left|\frac{L-M}{2}\right| > 0$, $\exists\ \delta_2 > 0$

(2) if $0 < |x - c| < \delta_2$ then $|f(x) - M| < \left|\frac{L-M}{2}\right|$

so take $\delta = \min\{\delta_1, \delta_2\}$. so if $0 < |x - c| < \delta$

then by (1) and (2) we have the following

$|L - M| = |L - f(x) + f(x) - M| \le |f(x) - L| + |f(x) - M| < \left|\frac{L-M}{2}\right| + \left|\frac{L-M}{2}\right| = |L - M|$ $(\rightarrow\leftarrow)$

◎Theorem 2.3.2.

$\lim_{x\to c} f(x) = L$ $\lim_{x\to c} g(x) = M$ 極限各別存在

- (a) $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L+M$ (Sum Rule)

(b) $\lim_{x\to c} \alpha f(x) = \alpha \lim_{x\to c} f(x) = \alpha L$ $\alpha \in R$ (constant multiple)

(c) $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = LM$ (Product Rule)

- pf

?! 抄錯?

○ For (a)

It "suffies" to prove $\lim_{x\to c}(f(x)+g(x)) = L+M$

since $\lim_{x\to c} f(x) = L$ $\frac{\varepsilon}{2}>0$, $\exists\, \delta_1 > 0$

s.t. if $0<|x-c|<\delta_1$ then $|f(x)-L|<\frac{\varepsilon}{2}$

Since $\lim_{x\to c} g(x) = M$ · $\frac{\varepsilon}{2}>0$ $\exists\, \delta_2>0$

s.t. if $0<|x-c|<\delta_2$ then $|g(x)-M|<\frac{\varepsilon}{2}$

So if $0<|x-c|<\delta$ then $|f(x)+g(x)-L-M| \le |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

○Think

$|f(x)+g(x)-L-M|$

$\le |f(x)-L|+|g(x)-M|$

$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

○ For (b) $\alpha\in R$

First suppose $\alpha=0$ (b) is true

◎hint

0 × 任意實數 = "0"

Then we suppose $\alpha \ne 0$ $\forall \varepsilon>0$ for $\frac{\varepsilon}{|\alpha|}>0$

$\exists\, \delta>0$ s.t. if $0<|x-c|<\delta$ then

$|f(x)-L|<\frac{\varepsilon}{|\alpha|}$, so we have

$|\alpha f(x) - L\alpha| \le |\alpha|\cdot|f(x)-L| < |\alpha|\frac{\varepsilon}{|\alpha|}=\varepsilon$

○Think

$|\alpha f(x)-\alpha L| \le |\alpha||f(x)-L|<\varepsilon$

$|f(x)-L|<\frac{\varepsilon}{|\alpha|}$

∘ For (c)

Since $\lim_{x\to c} f(x) = L$, for $1 > 0$ $\exists\ \delta_1 > 0$

s.t. if $0 < |x-c| < \delta_1$

then $|f(x) - L| < 1$ so we have

$||f(x)| - |L|| \leq |f(x) - L| < 1$

So we have $|f(x)| \leq |L| + 1$

∘Think

$$|f(x)g(x) - LM|$$
$$= |f(x)g(x) - f(x)M + f(x)M - LM|$$
$$\leq |f(x)||g(x) - M| + |M| \cdot |f(x) - L|$$
$$\leq (|L|+1)|g(x) - M| + (|M|+1)|f(x) - L|$$

Then since $\lim_{x\to c} g(x) = M$, $\frac{\varepsilon}{2(|L|+1)} > 0$.

$\exists\ \delta_2 > 0$, s.t. if $0 < |x-c|\ \delta_2$

then $|g(x) - M| < \frac{\varepsilon}{2(|L|+1)}$ Moreover since $\lim_{x\to c} f(x) = L$

for $\frac{\varepsilon}{2(|M|+1)} > 0$, $\exists\ \delta_3 > 0$

s.t. if $0 < |x-c| < \delta_3$, then.

$|f(x) - L| < \frac{\varepsilon}{2(|M|+1)}$ Finally $\forall \varepsilon > 0$, take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

So if $0 < |x-c| < \delta$ then we have

$$|f(x)g(x) - LM| \leq |f(x)g(x) - f(x)M + f(x)M - LM|$$
$$\leq |f(x)|\,|g(x) - M| + |M||f(x) - L|$$
$$\leq (|L|+1)|g(x) - M| + (|M|+1)|f(x) - L|$$
$$\leq (|L|+1)\frac{\varepsilon}{2(|L|+1)} + (|M|+1)\frac{\varepsilon}{2(|M|+1)} = \varepsilon$$

- $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$.

乘"-1"

$\lim_{x\to c}(f(x) - g(x)) = L - M$. (sum rule ✚ constant multiple)

- eg 1 $\lim_{x\to c} x = c$ $\lim_{x\to c} k = k$ so $\lim p(x) = p(c)$

$$p(x) = \sum_{i=0}^{n} a_i x^i,\ a_i \in R.$$

◎Theorem 2·3·7 (Reciprocal)

If $\lim_{x\to c} g(x) = M$ $\quad M\neq 0$ then. the $\lim_{x\to c}\frac{1}{g(x)} = \frac{1}{M}$

pf. since $\lim_{x\to c} g(x) = M$ for "$\frac{|M|}{2}$" >0 $\hookrightarrow \varepsilon$

$\exists\, \delta_1>0$. s.t. if $0<|x-c|<\delta$, then

$$|g(x)-M| < \frac{|M|}{2}$$

so we have

$$M-\frac{|M|}{2} < g(x) < M+\frac{|M|}{2}$$

(a). $M>0$. $\quad |g(x)|>\frac{|M|}{2}$

(b). $M<0$

$$|g(x)|>\frac{|M|}{2} \quad \left(-\frac{3|M|}{2} < g(x) < \frac{-|M|}{2}\right)$$

$$\Downarrow$$

$$+\frac{3}{2}|M| - g(x) > \frac{|M|}{2}$$

So.

$$\left|\frac{1}{g(x)}\right| < \frac{2}{|M|}$$

○Think

$$\left|\left(\frac{1}{g(x)}-\frac{1}{M}\right)\right| < \varepsilon.$$

$$\Leftrightarrow \left|\frac{g(x)-M}{g(x)M}\right| < \varepsilon$$

key: 令 $\frac{1}{g(x)} < ?$

$$\Leftrightarrow \left|\frac{g(x)-M}{g(x)M}\right| \leq \frac{2}{|M|^2}|g(x)-M|$$

Morever. since $\lim_{x\to c} g(x) = M$. for $\frac{|M|^2}{2}\varepsilon > 0$.

$\exists\,\delta_2>0$ s.t. if $0<|x-c|<\delta$, then $|g(x)-M| < \frac{|M|^2}{2}\varepsilon$.

Finally. $\forall\, \varepsilon>0$, take $\delta=\min\{\delta_1,\delta_2\}$ so if $0<|x-c|<\delta$

then we have $\left|\frac{1}{g(x)}-\frac{1}{M}\right| \leq \left|\frac{g(x)-M}{g(x)M}\right| \leq \frac{2}{|M|^2}|g(x)-M| \leq \frac{2}{|M|^2}\,\frac{|M|^2}{2}\varepsilon =$ "ε"

◎Theorem 2.3.8 (Quotient Rule)

$\lim_{x\to c} f(x) = L$ $\quad \lim_{x\to c} g(x) = M$ "$\neq 0$"

The $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c} f(x) \Big/ \lim_{x\to c} g(x) = L/M$

pf. $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c} f(x)\cdot\frac{1}{g(x)}$

$$= \lim_{x\to c} f(x)\,\lim_{x\to c}\frac{1}{g(x)} = L/M$$

- eg3.(Rational Function)

$R(x) = \frac{P_1(x)}{P_2(x)}$ $P_1(x)$ and $P_2(x)$ are polynomials
《多項式》

then. $\lim_{x \to c} R(x) = \frac{P_1(c)}{P_2(c)}$ " $P_2(c) \neq 0$ "

- eg4. $\lim_{x \to 2} \frac{x+1}{x^2+1} = \frac{3}{5}$ It's trivial ~ ~

◎Theorem. 2.3.10.

$\lim_{x \to c} f(x) = L \neq 0$ $\lim_{x \to c} g(x) = 0$ then. $\lim_{x \to c} \frac{f(x)}{g(x)}$ doesn't.

pf. suppose not. that is $\lim_{x \to c} \frac{f(x)}{g(x)}$ exist.

Let $\lim_{x \to c} \frac{f(x)}{g(x)} = k \in R$. $0 \neq L = \lim_{x \to c} f(x) = \lim_{x \to c} f(x) \frac{g(x)}{g(x)}$

$= \lim_{x \to c} \frac{f(x)}{g(x)} g(x) = \lim_{x \to c} \frac{f(x)}{g(x)} \lim_{x \to c} g(x) = k \cdot "0" = 0$ $(\rightarrow\leftarrow)$

◎ Section 2.4 continuity

- Def. 2.4.1 $f: (c-p, c+p) \to R$ we say f is cont... at c.
連續

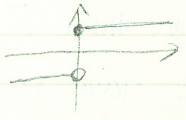
if. $\lim_{x \to c} f(x) = f(c)$, that is. $\forall \varepsilon > 0$ $\exists$ $\delta > 0$ s.t. if $|x-c| < \delta$, then

$|f(x) - f(c)| < \varepsilon$

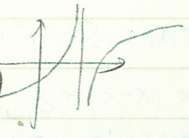
(1) $\lim_{x \to c} f(x)$ exist. but. $\lim_{x \to c} f(x) \neq f(c)$ c is called to be a removable discontinuity

(2) $\lim_{x \to c} f(x)$ doesn't exist. c is to be an essential discontinuity

jump: ex:
(jump discontinuity)



infinity
(infinite discontinuity)



neither $f(x) = \begin{cases} 0 & x \in L \\ 0 & x \notin L \end{cases}$

- eg 5.

$$f(x)=\begin{cases} x^2+1, & x>0 \\ 2, & x=0 \end{cases} \quad \lim_{x\to 0^+} f(x)=1 \quad \lim_{x\to 0^-} f(x)=1 \quad f(0)=2.$$

$f(x)$ 在 0 處不連續.

- eg 6.

$$f(x)=\begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational} \end{cases}$$

Since the limit of $f(x)$ at any point doesn't exist, $f(x)$ has no cont. point. 《連續》

- eg 7.

$\lim_{x\to c} p(x) = p(c)$ $P(x)$ is a polynomial (多項式的任一點 continuous)

〈常數函數 included〉

$p(x)$ is continuous at $c \in R$

- eg 8.

$R(x) = \frac{f(x)}{g(x)}$ $f(x)$ and $g(x)$ are polynomials

$\lim_{x\to c} R(x) = \frac{f(c)}{g(c)}$, $\boxed{g(c) \neq 0.}$ $R(x)$ is continuous

- eg 9. $\lim_{x\to c} |x| = |c|$ $c \in R$ so. $f(x)=|x|$ is cont. at $c \in R$.

- eg 10

$\lim_{x\to c} \sqrt{x} = \sqrt{c}$, $c \geq 0$

pf: we separate in the following cases

∘Think

$\sqrt{x} < \varepsilon$ $0 < x < \delta$

取 ε^2

(1). $c=0$ we need to prove $\lim_{x\to 0^+} \sqrt{x} = 0$

$\forall \varepsilon > 0$, take $\delta = \varepsilon^2 > 0$ so if $0 < x < \delta$

then $0 < x < \varepsilon^2$ so. $\sqrt{x} < \varepsilon$

(2). $c > 0$ We need to prove $\lim_{x\to c} \sqrt{x} = \sqrt{c}$

pf: Since we have $\sqrt{x}$, $x \geq 0$. Morever,

since we have $|x-c| < \delta$, $0 \leq c-\delta < x < c+\delta$

$\forall \varepsilon > 0$, take $\delta = \min\{c, \varepsilon|\sqrt{c}|\}$

So if $0 < |x-c| < \delta$, then we have

$|\sqrt{x}-\sqrt{c}||\sqrt{c}| \leq |\sqrt{x}-\sqrt{c}||\sqrt{x}+\sqrt{c}|$

$= |x-c| < \delta \leq \varepsilon|\sqrt{c}|$

so we have $|\sqrt{x}-\sqrt{c}| < \varepsilon$

∘Think:

$|x-c| = |\sqrt{x}-\sqrt{c}||\sqrt{x}+\sqrt{c}|$

$|\sqrt{x}-\sqrt{c}|\sqrt{c} \leq |\sqrt{x}-\sqrt{c}||\sqrt{x}+\sqrt{c}|$

$= |x-c|$

$|\sqrt{x}-\sqrt{c}| \leq \frac{|x-c|}{\sqrt{c}} < \varepsilon$

$\delta = \varepsilon|\sqrt{c}|$

◎Theorem 2.4.2.

$f(x)$ and $g(x)$ are continuous at $C \in R$

(a) $f \pm g$ is cont- at C.

(b) αf is cont- at C. $\alpha \in R$

(c) $f \cdot g$ is cont- at C

(d) $\frac{f}{g}$ is cont- at C $g(C) \neq 0$

pf: For (a) since $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$

$\lim_{x \to c}(f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = f(c) \pm g(c)$ (by sum Rule of limit)

By the $\underset{類似}{\text{analogous}}$ argument as above we can prove (b), (c), and (d).

- eg 1.

$$f(x) = \overset{C}{3|x|} + \overset{S}{\frac{x^3}{x^2+1}} + 4$$

$\frac{x^3}{x^2+1}$ → Rational

$f(x)$ is cont-

◎ g is cont- at c f is cont- at $g(c)$

then the composite map $h = \underset{後}{f} \cdot \underset{先}{g}$ is cont- at C

pf.

Since f is cont- at $g(c)$.

$\forall \varepsilon > 0, \exists \delta_1 > 0$ s.t. if $|y - g(c)| < \delta_1$

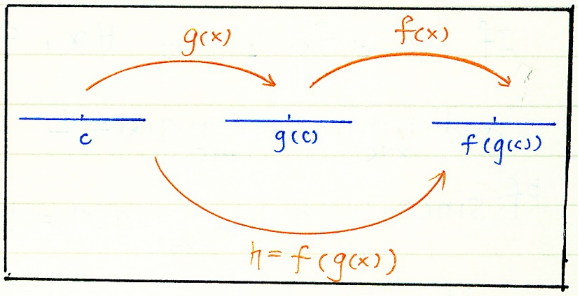
then $|f(y) - f(g(c))| < \varepsilon$

代數變換

Then since g is cont- at c, for $\delta_1 > 0$

$\exists \delta > 0$ s.t. if $|x - c| < \delta_1$ then $|g(x) - g(c)| < \delta_1$

so by the above, we have $|f(g(x)) - f(g(c))| < \varepsilon$



- eg 2.

$$h(x) = \sqrt{\frac{x^2+1}{x^4+2}}$$: 令 $g(x) = \frac{x^2+1}{x^4+2}$ $f(x) = \sqrt{x}$ "$h = f(g(x))$"

◎ Def 2.4.5

f is cont at c from right if $\lim_{x \to c^+} = f(x)$

and f is cont at c from left if $\lim_{x \to c^-} = f(x)$

- eg 1.

close

$f = [a, b] \to R.$ a→右連續 b→左連續

(左無定義) (右無定義)

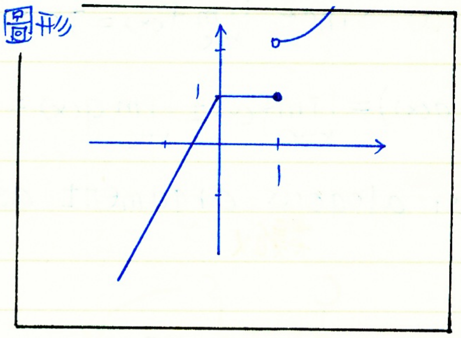
$f(x) = x^2.$

多項式(每一點連續) but

- eg 2.

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ 1, & 0 < x \le 1 \\ x^2+1, & x > 1 \end{cases}$$

圖形



$\lim_{x \to 1^-} f(x) = 1 = f(1)$ $\lim_{x \to 1^+} f(x) = 2.$

f(x) isn't cont at 1

◎ Section 2.5 (Pinching and Trigonometric)

◎ Theorem 2.5.1

If $h(x) \le f(x) \le g(x)$ $\forall x,\ 0 < |x-c| < p$

and $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L$

pf: since $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L$ $\forall \varepsilon > 0, \exists \delta_1 > 0\ \delta_2 > 0$

if $0 < |x-c| < \delta_1$ then $|h(x) - L| < \varepsilon$, and if $0 < |x-c| < \delta_2$

the $|g(x) - L| < \varepsilon$

$\forall \varepsilon > 0$ take $\delta = \min\{p, \delta_1, \delta_2\}$

So if $0 < |x-c| < \delta$ then we have $L - \varepsilon < h(x) < L + \varepsilon$

$L - \varepsilon < g(x) < L + \varepsilon$

So we have

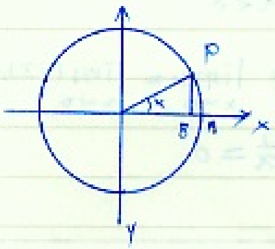
$L - \varepsilon < h(x) \le f(x) \le g(x) < L + \varepsilon$ hence $|f(x) - L| < \varepsilon$

◎Fact 1

$$\lim_{x \to 0} \sin x = 0$$

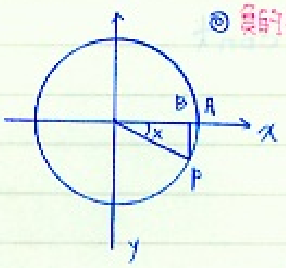
直徑

$$0 < \overline{BP} = \sin x < \overline{AP} < \widehat{AP} = x \quad : \quad \frac{x}{2\pi} \cdot (2 \cdot \pi)$$



$$0 < \sin x < x$$

◎負的時候



$$0 < \overline{BP} = |\sin x| < \widehat{AP} = |x| , x < 0$$

$$0 < |\sin x| < |x| \quad x \in R$$

So by pinching them, Since

抄錯?!

$$\lim_{x \to 0} |x| = 0 \quad \text{and} \quad \lim_{x \to 0} \boxed{0} = 0$$

$$\lim_{x \to 0} |\sin x| = 0 \quad \text{So} \quad \lim_{x \to 0} \sin x = 0$$

◎Fact 2.

$$\lim_{x \to 0} \cos x = 1$$

pf. suppose x is 1-th or 4-th quadrant

?

then we have "$\cos x = \sqrt{1 - \sin^2 x}$"

$$\lim_{x \to 0} \cos x = 1$$

筆記完稿-微積分-6 made by 101033236

補充

$\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$

(constant mul)

Pf: since $-1 \le \sin\frac{1}{x} \le 1$ if $x>0$, $-x \le \sin\frac{1}{x} \le x$ and if $x<0$

注意正負號

$-x \ge x\sin\frac{1}{x} \ge x$. Finally since $\lim\limits_{x\to 0} x = \lim\limits_{x\to 0}(-x) = 0$

by Pinching Then, we have $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$

◎Fact 3. $f(x) = \sin x$. $g(x) = \cos x$. $f(x)$ and $g(x)$ are cont

Pf: To prove $\sin x$ is cont

It suffices prove $\lim\limits_{h\to 0}\sin(C+h) = \sin C$. $C\in R$

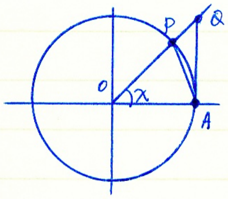
(constant multiple)

Then Since $\sin(C+h) = \sin C \underset{\substack{\downarrow \\ 1}}{\cos h} \underset{\substack{\uparrow \\ \text{(Sum Rule)}}}{+} \cos C \underset{\substack{\downarrow \\ 0}}{\sin h}$

$\lim\limits_{h\to 0}\sin(C+h) = \sin C$

So by the analogous argument as above we can prove $\cos x$ is cont

◎Fact 4. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$



area of $\triangle OAP = \frac{1}{2}\sin x$

area of sector $OAP = \frac{1}{2}x$

扇形

area of $\triangle OAQ = \frac{1}{2}\tan x = \frac{1}{2}\frac{\sin x}{\cos x}$

So $\cancel{\frac{1}{2}}\sin x < \cancel{\frac{1}{2}}x < \cancel{\frac{1}{2}}\frac{\sin x}{\cos x}$ "$x>0$"

$\downarrow$

$\frac{\sin x}{x} < 1 < \frac{\sin x}{x\cos x}$ so we have $\cos x < \frac{\sin x}{x} < 1$ $x\in R$

$\rightarrow$ 同乘 $\frac{x\cos x}{\sin x}$ $\nearrow$

$$\begin{cases} \cos x < \frac{\sin x}{x} < 1 & x>0 \\ \cos(-x) < \frac{\sin(-x)}{(-x)} < 1 & x>0 \end{cases}$$

finally since $\lim\limits_{x\to 0}\cos x = 1$

by Pinching Then, we have

$\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

◎Fact 5

$$\lim_{x\to 0}\frac{1-\cos x}{x}$$

$$\text{"}\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}\text{"} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)}$$

So $\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} = 0$

(by Fact4)

- eg 1.

$$\lim_{x\to 0}\frac{\sin 4x}{3x} = \lim_{x\to 0}\text{"}\frac{4}{3}\text{"}\frac{\sin 4x}{4x} = \frac{4}{3}$$

1. (by Fact 4)

◎ Section 2.6 Two Theorems

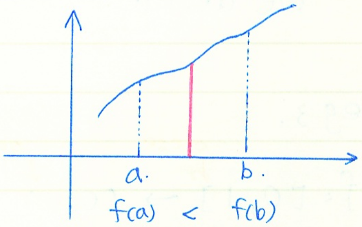
◎Theorem 2.6.1 (intermediate)

suppose $f:[a,b]\to R$ is cont

If $f(a)<k<f(b)$ or $f(b)<k<f(a)$ then $\exists\, c \in (a,b)$

S.t. $f(c)=k$

◎notice $C\neq a,b$



$f(a) < f(b)$

$f(a) < k < f(b)$

◎ Theorem 2.6.2 (Extreme-Value)

Suppose $f:[a,b]\to R$ is cont

Then f has the absolutely maximum and absolutely minimum

that is, $\exists\, m_1, m_2 \in [a,b]$ S.t. $f(m_1)$ is the max and

$f(m_2)$ is the min

- eg 1.

$$f(x) = \begin{cases} x^2 & x \text{ is rational} \\ x^4 & x \text{ is irrational} \end{cases}$$ prove $\lim_{x\to 0} f(x) = 0$

pf: $\forall \varepsilon > 0$, take $\delta = \min\{\sqrt{\varepsilon}. \sqrt[4]{\varepsilon}\}$

so if $0 < |x| < \delta$ than we have $|x| < \sqrt{\varepsilon}$ and $|x| < \sqrt[4]{x}$,

so $|x^2| < \varepsilon$ and $|x^4| < \varepsilon$ so. $|f(x)| < \varepsilon$

- eg 2.

$$f(x) = \begin{cases} \sin x \cdot \sin\frac{1}{x} & x \neq 0 \\ k & x = 0 \end{cases}$$ Find the k such that f(x) is cont at 0

pf:

(同乘 $\sin x$)

since $-1 \leq \sin\frac{1}{x} \leq 1$. if $\sin x \geq 0$ $-\sin x \leq \sin x \cdot \sin\frac{1}{x} \leq \sin x$

and if $\sin x < 0$. $-\sin x \geq \sin x \cdot \sin\frac{1}{x} \geq \sin x$

So by pinching Them, since $\lim_{x\to 0} \sin x = 0$ $\lim_{x\to 0} \sin x \sin\frac{1}{x} = 0$

We define $k = 0$ 極限值 = 函數值 → cont

- eg 3.

$f: \overset{\text{close}}{[0,1]} \to \overset{\text{open}}{(0,1)}$ is a cont map → ? WTF

Then we have $c \in (0,1)$ s.t. $f(c) = c$

pf. Define $F(x) = f(x) - x$

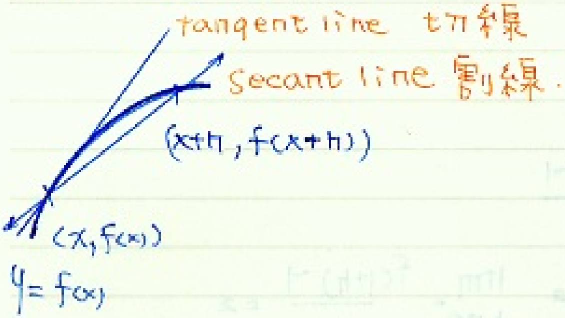
So $F(x)$ is cont on $[0,1]$ then we have $F(1) = f(1) - 1 < 0$

$F(0) = f(0) > 0$ Hence by intermediate Them,

$\exists\, c \in (0,1)$ s.t. $F(c) = 0$, that is $f(c) = c$

Ch3 The Derivative and the Process of Differentiation

◎ Section 3.1



The slope of the secant line

$$= \frac{f(x+h)-f(x)}{h}$$

◎ Def 3.1.1

f is differentiable at x (f在x可微)

if the limit $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Morever, if the above limit exists, then it is called the derivate of f at x, denoted by $f'(x)$,

that is $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(h)}{h}$

- eg1.

$f(x) = mx+b$. $m.b \in R$

prove $f'(x) = m$

pf: $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{m(x+h)+b-mx-b}{h} = \lim\limits_{h\to 0} \frac{mh}{h} = m$

- eg2. $f(x) = x^2$. $f'(x) = 2x$

pf: $f'(x) = \lim\limits_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim\limits_{h\to 0} \frac{2xh+h^2}{h} = 2x$

- eg 3.

$$f(x) = \begin{cases} x^2, & x \le 1 \\ 2x-1, & x > 1 \end{cases}$$

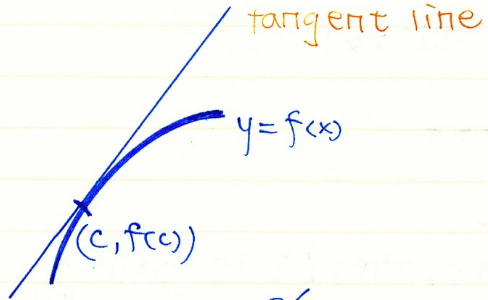
$f'(1) = ?$

pf. $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h) - 1}{h}$

$$\lim_{h\to 0^+} \frac{f(1+h) - 1}{h} = \lim_{h\to 0^+} \frac{\overset{f(1+h)}{(2(1+h)-1)} - 1}{h} = 2 \quad \lim_{h\to 0^-} \frac{f(1+h) - 1}{h} = 2$$

so $f'(1) = 2$

◎ Remark



$y - f(c) = f'(c)(x-c)$?

- eg 1. $f(x) = |x|$

f is cont so f is cont at 0

$f'(0) = ?$

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} \quad \lim_{h\to 0^+} \frac{|h|}{h} = 1, \quad \lim_{h\to 0^-} \frac{|h|}{h} = -1$$

$f'(0)$ doesn't exist

◎ Theorem 3.1.3

f is diff(differentiable) at c, then f is cont at c

pf: since $\lim_{h\to 0} (f(c+h) - f(c))$

$= \lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} \cdot h \right)$ and $f'(c)$ exists

by product rule, $\lim_{h\to 0} (f(c+h) - f(c)) = f'(c) \cdot 0 = 0$

◎ Section 3.2   Diff(action) formulas

Fact 1. $f(x) = \alpha$ $\alpha \in R$, $\forall x \in R$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\alpha - \alpha}{h} = 0.$$

Fact 2. $f(x) = x$ $\forall x \in R$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{x + h - x}{h} = \frac{h}{h} = 1$$

◎ Theorem 3.2.3.

$\alpha \in R$ f and g are diff(able) at x, then. αf and $\alpha f + g$ are diff(able) at x and $(\alpha f)'(x) = \alpha f'(x)$ $(f \pm g)'x = f'(x) \pm g'(x)$

pf. $(\alpha f)(x) = \lim_{h \to 0} \frac{1}{h}(\alpha f(x+h) - \alpha f(x))$

$$= \lim_{h \to 0} \alpha \frac{1}{h}(f(x+h) - f(x)) = \alpha f'(x)$$ (by constant mulp)

$$(f+g)'(x) = \lim_{h \to 0} \frac{1}{h}((f+g)(x+h) - (f+g)(x))$$

$$= \lim_{h \to 0} \frac{1}{h}(f(x+h) - f(x) + g(x+h) - g(x))$$

$$= \lim_{h \to 0} \left(\frac{1}{h} f(x+h) - f(x)\right) + \left(\frac{1}{h} g(x+h) - g(x)\right)$$ ← (by sum rule)

$$= \lim_{h \to 0} \frac{1}{h}(f(x+h) - f(x)) + \lim_{h \to 0} \frac{1}{h}(g(x+h) - g(x)) = f'(x) + g'(x)$$

◎ Theorem 3.2.6

f and g are diff at x, then $f \cdot g$ is diff at x, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

pf: since we have $\lim_{h \to 0} \frac{1}{h}((fg)(x+h) - (fg)(x))$

$= \lim_{h \to 0} \frac{1}{h} (f(x+h) g(x+h) - f(x) g(x+h) + f(x) g(x+h - f(x) g(x)))$

$= \lim_{h \to 0} [(g(x+h) \frac{1}{h} (f(x+h) - f(x)) + f(x) \frac{1}{h} (g(x+h) - g(x)))]$

$\downarrow$

$g(x)$·可微!! $h \to 0$ $g(x+h)$ 連續 → $g(x+h) = g(x)$

✡ and since $\lim_{h \to 0} g(x+h) = g(x)$ (since g is diff at x)

by constant mulp, product rule, sum rule,

we have $f'(x) g(x) + f(x) g'(x)$

• eg1. $f(x) = x$ $f'(x) = 1$ $f(x) = x^2 = x \cdot x$. $f'(x) = x + x = 2x$

(by product rule)

$f(x) = x^n, n \in N$ $f'(x) = n x^{n-1}$ $n \in N$ (power formula)

Power formula

$f(x) = x^n$ $n \in N$ then $f'(x) = n x^{n-1}$

pf: prove by induction on n.

First when $n=1$ $f(x) = x$ so $f'(x) = 1$

Hence the formula is true for $n=1$

So this starts the Induction, so we assume the formula is ture for $n-1$ (inductive hypothesis),

then we are going to prove the formula is true of n

so we have

$$f(x) = x^n = x^1 \cdot x^{n-1}$$

by inductive hypothesis and product rule

$$f'(x) = x(n-1)x^{n-2} + x^{n-1}$$
$$= n x^{n-1}$$

◎ Theorem 3.2.9 (Reciprocal)

g is diff. at x ; and $g(x) \neq 0$, then $(\frac{1}{g})'(x) = \frac{-g'(x)}{g^2(x)}$

pf: since g is diff at x g is cont at x that is,

$\lim_{h\to 0} g(x+h) = g(x)$ so $\lim_{h\to 0} \frac{1}{g(x+h)} = \frac{1}{g(x)}$.

Then $(\frac{1}{g})'(x) = \lim_{h\to 0} \frac{1}{h} (\frac{1}{g(x+h)} - \frac{1}{g(x)})$

$= \lim_{h\to 0} \frac{1}{h} (- \frac{g(x+h) - g(x)}{g(x+h)\, g(x)}) = \frac{-g'(x)}{g^2(x)}$

◎Theorem 3.2.11 (Quotient Rule)

f and g are diff at x $g(x) \neq 0$ the f/g is diff and

$(f/g)'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{g^2(x)}$

pf: $(f/g)'(x) = (f \cdot \frac{1}{g})'(x)$

$= f'(x) \frac{1}{g(x)} + f(x) \frac{-g(x)}{g^2(x)} = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{g^2(x)}$

Remark.

$y = f(x)$ $f'(x) = \frac{dy}{dx} = \frac{df(x)}{dx}$ differentiate y with respect to x

$f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f(x)}{dx^2}$

•eg1. $y = f(x) = x^2+1$

$f'(x) = \frac{dy}{dx} = \frac{d(f(x))}{dx} = \frac{d(x^2+1)}{dx} = 2x$

◎ Section 3.5 (Chain Rule)

$y = f(u) \qquad u = g(x)$

$y = F(g(x))$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (Chain Rule)

- eg1. $y = f(u) = u^2+1 \quad \overset{u}{\cancel{y}} = g(x) = x+1$

  $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u = 2(x+1)$

- eg2. $y = f(x) = (x^3+1)^{100}$

  $f'(x)$

  $y = g(u) = u^{100} \quad u = h(x) = x^3+1$

  $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 100u^{99}(3x^2)$

  $= 100(x^3+1)^{99}(3x^2)$

Fact. $\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx} \quad y=u^n \quad u = g(x).$

- eg1. $y = (x^4+2)^5 \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4(4x^3)$

  $= 5(x^4+2)^4(4x^3)$

  $\begin{cases} y = f(u) = u^5 \\ u = g(x) = x^4+2 \end{cases}$

$y = f(u), \; u = g(x), \; x = h(t)$

$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt}$

$\frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt} = 3u^2(-1x^{-2})2t$

◎ Section 3.6 . Diff of Trig Function

$y = f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{1}{h} (\sin(x+h) - \sin x)$$

$$= \lim_{h\to 0} \frac{1}{h} (\sin x \cosh + \cos x \cdot \sinh - \sin x)$$

$$= \lim_{h\to 0} \left(\sin x \underset{\downarrow\, 0}{\frac{(\cosh - 1)}{h}} + \cos x \underset{\downarrow\, 1}{\frac{\sinh}{h}} \right) = \cos x$$

<<詳見P6 Fact 4>>

$y = f(x) = \cos x$

$$f'(x) = \lim_{h\to 0} \left(\frac{\cos(x+h) - \cos x}{h} \right)$$

$$= \lim_{h\to 0} \left(\frac{\cos x \cosh - \sin x \sinh - \cos x}{h} \right)$$

$$= \lim_{h\to 0} \cos x \underset{\downarrow = 0}{\frac{[\cosh - 1]}{h}} - \underset{\downarrow = 1}{\frac{\sin x \sin h}{h}} = -\sin x$$

$y = f(x) = \tan x = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{\cos x^2 - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

◎ Fact

$$\frac{d \sin u}{dx} = \underset{\left(\frac{dy}{du}\right)}{\cos u} \frac{du}{dx} \qquad \frac{d \cos u}{dx} = -\sin u \frac{du}{dx}$$

- eg 1

$y = f(x) = \sin(x^2+1)$

$f'(x) = \cos(x^2+1) \cdot (2x)$

- eg 2.

$y = f(x) = \sin^3(x^3+2)$

$y = \sin^3 u$

$u = x^3+2$

$$\boxed{\frac{dy}{du} = \frac{dy}{dv}\cdot\frac{dv}{du} \quad \begin{cases} y = f(v) = v^3 \\ v = \sin u \end{cases} \quad \frac{dy}{du} = 3v^2\cos u \\ = 3\sin^2 u\cos u}$$

$f'(x) = 3\sin^2 u \cos u\,(3x^2)$ $\quad \frac{dy}{du}\frac{du}{dx}$

$= 3\sin^2(x^3+2)\cos(x^3+2)(3x^2)$

◎ Section 3.7 Implicit Diff and Rational power

- eg 1. $y = \sqrt{1-x^2}$ $\quad$ notice "$-1 \le x \le 1$" 求 $\frac{dy}{dx} = ?$

$y^2 = 1 - x^2 \xrightarrow{\text{兩邊對}x\text{微分}} 2y\frac{dy}{dx} = -2x \Longrightarrow \frac{dy}{dx} = \frac{-x}{y} \leftarrow y = \sqrt{1-x^2}$

- eg 2. $y^3 + 2\sin(x^2+1) = 2.$

$3y^2\frac{dy}{dx} + 2\cos(x^2+1)(2x) = 0$ $\quad \frac{dy}{dx} = \frac{-[2\cos(x^2+1)(2x)]}{3y^2}$

- eg 3.

$\sin y^2 = 4x^5 + \tan x$

$\cos y^2 \underbrace{(2y)}_{\text{Don't forget}} \frac{dy}{dx} = 20x^4 + \sec^2 x$

◎ Power formula

$y = f(x) = x^n$, $n = \frac{p}{q}$, $p, q \in \mathbb{Z}$ (整). $q \neq 0$

$\frac{dy}{dx} = f'(x) = nx^{n-1}$

pf. First $n = -m$, $m \in \mathbb{N}$

$f(x) = x^{-m} = \frac{1}{x^m}$

$f'(x) = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1}$

?

Secondly $n=\frac{1}{q} \cdot q\in z \cdot q\neq 0$

$y=f(x)=x^{1/q}$

$y^{q}=x \quad q y^{q-1}\frac{dy}{dx}=1 \quad \frac{dy}{dx}=\frac{1}{q}y^{1-q}=\frac{1}{q}(x^{1/q})^{1-q}=\frac{1}{q}x^{\frac{1}{q}-1}$

Finally $n=\frac{p}{q} \quad p,q\in z \quad q\neq 0$

$y=f(x)=x^{p/q}=(x^{1/q})^{p}$

$f'(x)=p(x^{1/q})^{p} \quad f'(x)=p(x^{1/q})^{p-1}\frac{1}{q}x^{\frac{1}{q}-1}=\frac{p}{q}x^{p/q-1}$

- eg1. [Section 3.7 eg1., again]

$y=(1-x^{2})^{1/2}$

$y'=\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^{2}}}(-2x)$

- eg2. $y=\sin^{1/2}(x^{2}+1)$

$$\frac{dy}{dx}=\frac{1}{2}\sin^{-1/2}(x^{2}+1)\cos(x^{2}+1)(2x)$$

額外例題

※eg1. Suppose $f'(a)$ exist then compute $\lim\limits_{x\to a}\frac{xf(a)-af(x)}{x-a}$ [台大考題]

pf. key!!

$$\lim_{x\to a}\frac{1}{x-a}(\underbrace{xf(a)-xf(x)}+\underbrace{xf(x)-af(x)})$$

$$=\lim_{x\to a}\left(x\frac{-(f(x)-f(a))}{x-a}+f(x)\right)$$

$$=-af'(a)+f(a)$$

$$\boxed{\begin{aligned}f'(a)&=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\&=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.\end{aligned}}$$

※ eg2. Suppose $f'(0) = a$ compute $\lim\limits_{x\to 0} \frac{f(3x) - f(\sin x)}{x}$

pf: $\lim\limits_{x\to 0} \frac{1}{x}\left(f(3x) = f(0) + f(0) - f(\sin x)\right)$

$= \lim\limits_{x\to 0}\left(3 \cdot \frac{f(3x) - f(0)}{3x} - \frac{\sin x}{x} \frac{f(\sin x) - f(0)}{\sin x}\right)$

$= \lim\limits_{x\to 0}\left(3 \frac{\overset{(u\to 0)}{f(u) - f(0)}}{u} - \underbrace{\frac{\sin x}{x}}_{\hookrightarrow = 1} \frac{\overset{(v\to 0)}{f(v) - f(0)}}{v}\right)$ $u = 3x$ $v = \sin x$

$= 3a - a = 2a$

※ eg3.

$f\left(\frac{x-1}{x^2+1}\right) = x^2$ $f'(0) = ?$

$f'\left(\frac{x-1}{x^2+1}\right) \frac{(x^2+1) - 2x(x-1)}{(x^2+1)^2} = 2x$ $f'(0) = 4$

$f'\left(\frac{x-1}{x^2+1}\right) = 2x \cdot \frac{(x^2+1)^2}{(x^2+1) - 2x(x-1)}$ $f'(0) \Rightarrow x=1$ 代入

※ eg4.

$y = f(x) = \sqrt{x + \sqrt{x}}$

$f'(x) = \frac{1}{2} \frac{1}{\sqrt{x + \sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)$

Ch 4. Mean-Value Thm

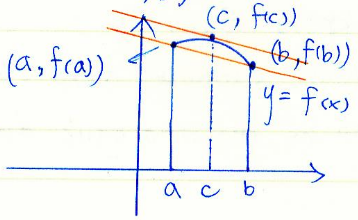
Application of derivatives.

◎Section 4.1 Mean-Value Thm

◎Theorem 4.1.1 (Mean-Value)

f is diff on (a, b) and cont on $[a, b]$ then $\exists\, c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



◎Theorem 4.1.2

f is diff at x_0. then

(1) If $f'(x_0) > 0$, then $\exists\, \delta > 0$ s.t. if $0 < h < \delta$. then $f(x_0 - h) < f(x_0) < f(x_0 + h)$

(2) If $f'(x_0) < 0$, then $\exists\, \delta > 0$ s.t. if $0 < h < \delta$ then $f(x_0 - h) > f(x_0) > f(x_0 + h)$

pf:

- For (1) Since $\lim\limits_{k \to 0} \overset{f(x)}{\frac{f(x_0 + k) - f(x_0)}{k}} = \overset{(L)}{f'(x_0)} > 0$

for $\underset{(\varepsilon)}{f'(x)} > 0$ $\exists\, \delta > 0$ s.t. if $0 < |k| < \delta$. then $\left|\overset{f(x) - L}{\frac{f(x_0 + k) - f(x_0)}{k} - f'(x_0)}\right| < \overset{\varepsilon}{f'(x_0)}$

So

$$-f'(x_0) < \frac{f(x_0 + k) - f(x_0)}{k} - f'(x_0) < f'(x_0)$$

$$0 < \frac{f(x_0 + k) - f(x_0)}{k} < 2f'(x_0) \quad \text{So. } \frac{f(x_0 + k) - f(x_0)}{k} > 0 \quad 0 < |k| < \delta$$

So if $0 < h < \delta$ then $0 < |h| < \delta$. $0 < |-h| < \delta$, and

$$\frac{f(x_0 + h) - f(x_0)}{h} > 0 \text{ and } \frac{\overset{+(-h)}{f(x_0 - h)} - f(x_0)}{\underset{(h)}{-h}} > 0 \quad \text{so } f(x_0 - h) < f(x_0 < f(x_0 + h)$$

◎Theorem (Rolle's) 4.1.3

f is diff on (a, b) and cont on $[a, b]$ and $f(a) = f(b) = 0$ then $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$

pf: First suppose $f(x) = 0$ $\forall x \in [a, b]$ $\overset{?}{\text{The}}$ $f'(x) = 0$ $\forall x \in (a, b)$

hence the theorem is true for this case

Secondary Suppose $f(x)>0$ for some $x \in [a,b]$

By Them 2.6.2 $\exists\, c \in [a,b]$ s.t. $f(c)$ is the absolutely maximum

Then since $f(x)>0$ for some $x \in [a,b]$, $f(c)>0$. So $c \in (a,b)$

Hence $f'(c)$ exists. Then by Them 4.1.2

We know $f'(c)=0$

$c \neq a,b$ ∵ $f(c)>0$ but $f(a)=f(b)=0$

Finally suppose $f(x)<0$ for some $x \in [a,b]$

Then by the analogus argument as in the second case

Them 4.1.1

pf. Define

$g(x) = f(x) - [\frac{f(b)-f(a)}{b-a}(x-a) + f(a)]$ Then g is diff on (a,b)

and is cont on $[a,b]$. Moreover, $g(a)=g(b)=0$

By Rolle's Them, $\exists\, c \in (a,b)$ s.t. $g'(c)=0$

Then by above, we have $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

hence $0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$

- eg 1.

prove $|\sin b - \sin a| \leq |b-a|$ $a \in R$ $b \in R$

pf. Let $f(x)=\sin x$

The by Mean-value Them, for $a<b$ we have that $\exists\, c \in (a,b)$ s.t

$$f(c) = \frac{f(b)-f(a)}{b-a} = \cos c$$

so $\left|\frac{f(b)-f(a)}{b-a}\right| = |\cos c| \leq 1$

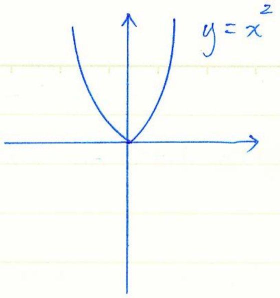
◎ Section 4.2 Increasing and Decreasing

Def 4.2.1 I is an interval

(1) f is increasing on I if $x_1, x_2 \in I$
$x_1 < x_2$, and $f(x_1) < f(x_2)$

(2) f is decreasing on I if $x_1, x_2 \in I$ $x_1 < x_2$ and
$f(x_1) > f(x_2)$

- eg 1. $y = f(x) = x^2$ $\begin{cases} f\uparrow \text{ on } [0, \infty) \\ f\downarrow \text{ on } (-\infty, 0] \end{cases}$



◎ Theorem 4.2.2

I is an open interval, and f is diff on I, then

(1) $f'(x) > 0$, $\forall x \in I$, then $f\uparrow$ on I

(2) $f'(x) < 0$ $\forall x \in I$, then $f\downarrow$ on I

(3) $f'(x) = 0$, $\forall x \in I$, then f is constant

Pf. suppose $x_1, x_2 \in I$ $x_1 < x_2$. so f is diff on (x_1, x_2) and f is cont on $[x_1, x_2]$

By Mean-Value Them, $\exists$ $c \in (x_1, x_2)$ s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

So we have the following

(1) since $f'(c) > 0$, $f(x_2) > f(x_1)$

(2) since $f'(c) < 0$, $f(x_2) < f(x_1)$

(3) since $f'(c) = 0$, $f(x_2) = f(x_1)$

- eg 1. $y = f(x) = x^2$

  $f'(x) = 2x$

  (1) $f'(x) > 0$, $\forall x \in (0, \infty)$, $f\uparrow$ on $(0, \infty)$

  (2) $f'(x) < 0$, $\forall x \in (-\infty, 0)$, $f\downarrow$ on $(-\infty, 0)$

◎ Theorem 4.2.3

↙扣除邊界 !!

I is an interval, f is diff on the Interior of I, and f is count on I ↑Including 邊界

◎[邊界點連續,但不一定可微]

(1) $f'(x) > 0$, $\forall x$ x is in the interior of I, then $f\uparrow$ on I
[遞增, Increasing]

(2) $f'(x) < 0$, $\forall x$ x is in the interior of I, then $f\downarrow$ on I
[遞減 decreasing]

(3) $f'(x) = 0$, $\forall x$. x is in the interior of I, then f is constant on I
↑including 邊界

pf. $x_1 < x_2$ $x_1 \in I$. $x_2 \in I$

f is diff on (x_1, x_2) → [皆在內 (interior) 可能是邊界].

f is cont on $[x_1, x_2]$

By Mean-value Them,

$\exists\ c \in (x_1, x_2)$ s.t. $f(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

- eg 1.

$y = f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 1$

$f'(x) = x^2 - 3x + 2$

$= (x-2)(x-1)$

f ↑ on $(-\infty, 1]$ f在 $-\infty \to 1$ 連續 包括1. ↩因為是多項式

f ↓ on $[1, 2]$.

f ↑ on $[2, +\infty)$

- eg 2.

$$f(x) = \begin{cases} x^3, & x < 1 \\ \frac{1}{2}x + 2. & x \geq 1 \end{cases}$$

pf: $\lim_{x \to 1^-} f(x) = 1$ $\lim_{x \to 1^+} f(x) = 5/2$ f(1) doesn't exist

$$f'(x) \begin{cases} 3x^2. & x < 1 \\ \frac{1}{2} & x > 1 \end{cases}$$

↳ notice z→x

f ↑ on $(-\infty, 0]$ ← 0點連續

f ↑ on $[0, 1)$

f ↑ on $[1, \infty)$

@ Section 4.3 Local extreme values

Def 4.3.1 c is Interior point (內點) of the domain

(1) c is a local maximum of f if $\exists\ \delta > 0$ s.t. $f(c) \geq f(x)$, $\forall\ x \in (c-\delta, c+\delta)$

(2) c is a local minimum of f is $\exists\ \delta > 0$ s.t. $f(c) \leq f(x)$, $\forall\ x \in (c-\delta, c+\delta)$

Both are called local extreme point

◎ Def. 4.3.2

C is an interior point of the domain

If c is a local extreme point $f'(c)=0$ or $f'(c)$ doesn't exist

pf. suppose $f'(c)$ exist, Then by Them 4.1.2.

$f'(c)=0$

$f'(c)$ doesn't exist

◎ Def. 4.3.3

c is an interior point of the domain, and $f'(c)=0$ or $f'(c)$ doesn't exist, then C is called a critical point

- eg1. $y=f(x)=3-x^2$

  $f'(x)=-2x=0$ $x=0$ is the only critical pt. (point)

- eg2. $f(x)=\frac{1}{x-1}$

  $f'(x)=\frac{-1}{(x-1)^2}$ no critical pt. (1不在定義域)

◎ Def. 4.3.4. First derivative test

C is a critical point, f is cont at c then $\exists\, \delta>0$ s.t.

(1) $f'(x)>0$ $\forall\, x\in(c-\delta,c)$ and $f'(x)<0$ $\forall\, x\in(c,c+\delta)$

then c is a local maximum

(2) $f'(x)<0$ $\forall\, x\in(c-\delta,c)$ and $f'(x)>0$ $\forall\, x\in(c,c+\delta)$

then C is a local minimum

(3) $f'(x)$ keeps the constant sign on $(c-\delta,c)\cup(c,c+\delta)$

then C is not a local extreme point.

pf: for (1), since $f'(x)>0$ $\forall\, x\in(c-\delta,c)$

f is cont on $(c-\delta,c]$ by them 4.2.3

$f\uparrow$ on $(c-\delta,c]$ Then by the analogous argument

$f\downarrow$ $[c,c+\delta)$ so, $f(c)\geq f(x)$, $\forall\, x\in(c-\delta,c+\delta)$

So. C is a local maximum point

- eg1 $f(x) = x^4 - 2x^3$

$f'(x) = 4x^3 - 6x^2$

$= 2x^2(2x-3)$ $\quad \underset{\hookrightarrow \text{nothing}}{x=0.} \quad x = \frac{3}{2}$ (local minimum)

◎ Theorem 4.3.5 (Second derivative test)

$f'(c)=0$, and $f''(c)$ exists, then

(1) $f''(c)>0$, then c is a local minimum

(2) $f''(c)<0$, then c is a local maximum

Pf. For (1) suppose $f''(c)>0$

By Theorem 4.1.2, $\exists \delta>0$ s.t. if $0<h<\delta$

$$f'(c-h) < \underset{0}{\underset{\downarrow}{f'(c)}} < f'(c+h)$$

Then since $f'(c)=0$, $c-h > c-\delta$, and $c+h < c+\delta$

$$f'(x)<0 \quad \forall\, x \in (\underset{\hookrightarrow [c-h]}{c-\delta}, c) \text{ and } f'(x)>0 \ \forall x \in (c, c+\delta)$$

so by first derivative test, c is a local minimum

- eg1. $f(x) = 2x^3 - 3x^2 - 12x + 5$ $\qquad x=2, x=-1$ critical pt

$f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$ $\qquad f''(x) = 12x - 6 \quad f''(2) > 0$ mini

$= 6(x-2)(x+1)$ $\qquad f''(-1) < 0$ maxi

期中考補充.

- eg1.

$y = f(x) = 2x^{5/3} + 5x^{2/3}$

$f'(x) = \frac{10}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$

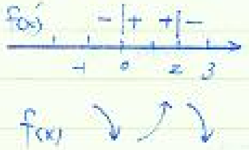
$= \frac{10}{3}x^{-1/3}(x+1)$

◎ Section 4.4. Endpoint extreme and absolute extreme

- eg 1.

$y = f(x) = 1 + 4x^2 - \frac{1}{2}x^4 \quad x \in [-1, 3]$

$f'(x) = 8x - 2x^3 = 2x(4 - x^2) = 2x(2-x)(2+x)$



$f(-1) = \frac{9}{2}$ $f(3) = \frac{-7}{2}$

endpt maximum endpt minimum

by Thm 2.6.2

絕對極大 絕對極小

$f(0) = 1$ $f(2) = 9$

local minimum local maximum

$$\begin{cases} f(2) = 9 \rightarrow \text{absolute maximum} \\ f(3) = \frac{-7}{2} \rightarrow \text{absolute minimum} \end{cases}$$

◎ Remark

(1) $\lim_{x\to\infty} f(x) = \infty$ [極限不存在] iff $\forall M > 0$ $\exists k > 0$ s.t. if $x \geq k$,

then $f(x) \geq M$

(2) $\lim_{x\to\infty} f(x) = -\infty$ [極限不存在] iff $\forall -M < 0$ $\exists k > 0$. s.t. if $x \geq k$

then $f(x) \leq M$

- eg 1.

$y = f(x) = x^2$. then

$\lim_{x\to\infty} f(x) = \infty$

抄錯了

- Think

$x^2 \geq M$

$x \geq M$

pf.

$\forall M > 0$, take $k = \sqrt{M}$ So if $x \geq \sqrt{M}$, then $x^2 \geq \sqrt{M}$

- eg 2. (exercise)

$p(x)$ is polynomial then $\lim_{x\to\infty} p(x) = \pm\infty$

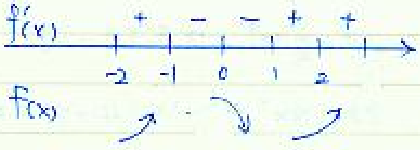
• eg 3.

↗ thm 8.6.2 不適用，因為非 close interval

$f(x) = \frac{1}{4}(x^3 - \frac{3}{2}x^2 - 6x + 2)$, $x \in [-2, \infty)$

$f'(x) = \frac{1}{4}(3x^2 - 3x - 6)$

$= \frac{3}{4}(x-2)(x+1)$



$f(-2) = 0$ end pt. minimum

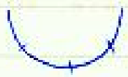
$f(-1) = \frac{11}{8}$ local max

$f(2) = -2$ local min (ab min)

◎ Section 4.6 concavity and point of inflection

◎ Def 4.6.1 f is diff on the open interval I

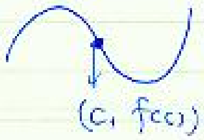
(1) f is concave up if f' ↑ on I

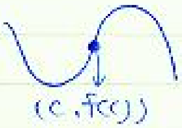


(2) f is concave down if f' ↓ on I

◎ Def 4.6.2

f is cont at c then c is called to be a point of inflection if ∃ δ > 0 s.t. f is concave in one sence on (c−δ, c) and f is concave in the opposite sence on (c, c+δ)





◎ Theorem 4.6.3

(二次皆可微)

f is twice diff on the open interval I

(1) $f''(x) > 0$ ∀ x ∈ I then f' ↑ on I then f is concave up

(2) $f''(x) < 0$ ∀ x ∈ I then f' ↓ on I then f is concave down

pf. By 4.2.2 (key. 視 $f'(x)$ 為 $g(x)$，則 $f''(x)$ 為 $g'(x)$)

◎ Theorem 4.6.4

C is a point of inflection of f the $f''(c)=0$ or $f''(c)$ doesn't exist.

Pf. Suppose $f''(c)$ exist, then we need to prove $f''(c)=0$

Since C is a pt of inflection, $\exists\ \delta>0$ $f'\uparrow$ on $(c-\delta, c)$ and $f'\downarrow$ on $(c, c+\delta)$, then since $f''(c)$ exists, f' is cont at c.

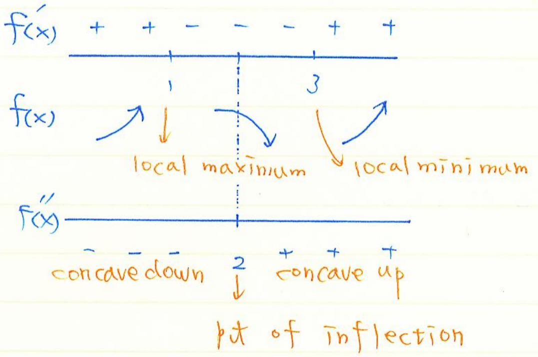
So $f'\uparrow$ on $(c-\delta, c]$ and $f'\downarrow$ on $[c, c+\delta)$ Hence c is a local max of f'

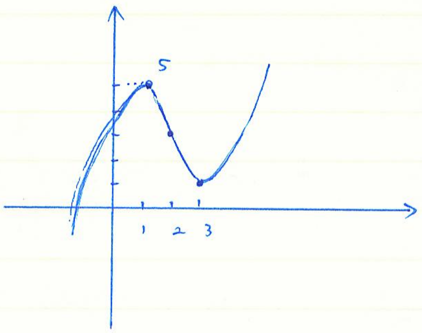
- eg 1.

$y = f(x) = x^3-6x^2+9x+1$ $f(1)=5$. $f(3)=1$ $f(2)=3$

$f'(x) = 3x^2-12x+9 = 3(x-1)(x-3)$

$f''(x) = 6x-12 = 6(x-2)$





◎ Def. 4.7.1

(1) $\lim_{x\to c^+} f(x)=\infty$ $\forall M \geq 0$ $\exists\ \delta>0$ s.t. if $c<x<c+\delta$, then $f(x)\geq M$

(2) $\lim_{x\to c^-} f(x)=\infty$ $\forall M>0$ $\exists\ \delta>0$ s.t. if $c-\delta<x<c$ then $f(x)\geq M$

By the analogous difinitions as above

we can define $\lim_{x\to c^+} f(x)=-\infty$ $\lim_{x\to c^-} f(x)=-\infty$

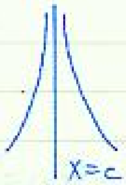
In any case of the above cases $x=c$ is a Vertical asymptote of f

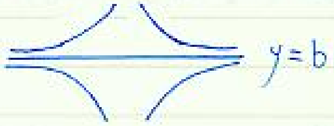
◎ Def 4.7.2

(1). $\lim_{x\to\infty} f(x)=b$ iff $\forall \varepsilon>0$, $\exists k\geq 0$ s.t. if $x\geq k$ then $|f(x)-b|<\varepsilon$

(2). $\lim_{x\to-\infty} f(x)=b$ iff $\forall \varepsilon>0$. $\exists k\leq 0$ s.t. if $x\leq k$ then $|f(x)-b|<\varepsilon$

In any case of the above two cases $y=b$ is a horizontal asymptote





• eg 1.

$y=f(x)=\frac{1}{x-2}$

$f'(x)=\frac{-1}{(x-2)^2}$ f ↓ on $(-\infty,2)\cup(2,\infty)$

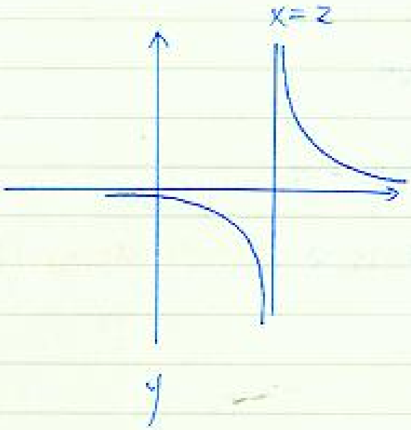
$f''(x)=\frac{2}{(x-2)^3}$ f ∩ on $(-\infty,2)$ and f ∪ on $(2,\infty)$

$\lim_{x\to 2^+} f(x)=\infty$ $\lim_{x\to 2^-} f(x)=-\infty$

x=2 Vertical Asym

$\lim_{x\to\infty} f(x)=0=\lim_{x\to-\infty} f(x)$

y=0 Horizontal Asym.



Ch5. Integration

◎ Section 5.1 and 5.2 Definitions

◎ Def. 5.1.1 $\{a_n\}_{n=1}^{\infty}$ is a sequence(數列) $a_n \in R$ $n \in N$

$\lim_{n\to\infty} a_n = L$ iff $\forall \varepsilon > 0$, $\exists k \in N$ s.t. iff. $n \ge k$, then

$|a_n - L| < \varepsilon$

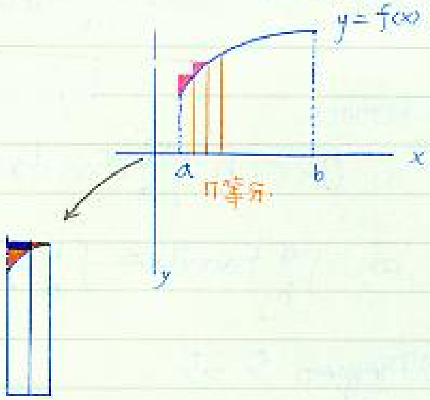
• eg1. Prove $\lim_{n\to\infty} \frac{1}{n} = 0$. $(a_n = \frac{1}{n} \cdot L = 0)$

pf:

$\forall \varepsilon > 0.$, $\exists K \in N$ s.t. $1 < K\varepsilon$ So $\frac{1}{k} < \varepsilon$ Hence $\forall n \ge k$ we have $\frac{1}{n} \le \frac{1}{k} < \varepsilon$. so $|\frac{1}{n}| < \varepsilon$

◎ Def 5.1.2 $f: [a,b] \to R$ is a function

Let $S_n = \frac{b-a}{n} f(a + \frac{b-a}{n}) + \frac{b-a}{n} f(a + 2 \cdot \frac{b-a}{n}) + \cdots + \frac{b-a}{n} f(a + n \cdot \frac{b-a}{n})$



Define $A = \lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$

If the above limit exists, then we say f is integrable and $\int_a^b f(x)\,dx$ is called the definite integral of f.

(定積分)

• eg1.

Compute $\int_0^1 3\,dx$, $f(x) = 3$ $(b=1, a=0)$

sol. $S_n = (\frac{1}{n} \cdot 3) \cdot \underset{\text{等分}}{n} = 3$ $\lim_{n\to\infty} S_n = 3$

• eg2. Compute $\int_1^2 x\,dx$ $f(x)=x$ $(b=2.\ a=1)$

Sol. $S_n=\frac{1}{n}f(1+\frac{1}{n})+\frac{1}{n}f(1+2\cdot\frac{1}{n})+\frac{1}{n}f(1+3\cdot\frac{1}{n})+\cdots+\frac{1}{n}f(1+n\cdot\frac{1}{n})$

So $S_n=\frac{1}{n}(\underset{n個1}{n}+\frac{1}{n}(1+2+3+\cdots+n))=\frac{1}{n}(n+\frac{1}{n}(\frac{n(n+1)}{2}))=1+\frac{n(1+n)}{2n^2}$

提出 n個1

$\lim_{n\to\infty}S_n=\frac{3}{2}$

• eg3. Compute $\int_1^3 x^2\,dx$ $f(x)=x^2$ $(b=3.\ a=1)$

Sol. $S_n=\frac{2}{n}f.(1+\frac{2}{n})+\frac{2}{n}f(1+\cdot2\cdot\frac{2}{n})+\frac{2}{n}f(1+3\cdot\frac{2}{n})+\cdots$

◎ Section 5.3 : $F(x)=\int_a^x f(t)\,dt$

◎ Remark:

(1) $a<c<b.\ \int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$

(2) $\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$

※ ◎ Theorem 5.3.5

對此積分"x"為常數, 故用 t 不用 x

f is cont on $[a,b]$. define $F(x)=\int_a^x f(t)\,dt$ $\forall x\in[a,b]$

Then F is cont on $[a,b]$. is diff on (a,b). and $F(x)=f(x)$. $x\in(a,b)$

pf: First for $x\in[a,b)$, we will $\lim_{h\to0^+}\frac{F(x+h)-F(x)}{h}=f(x)$ ← (1)

Since $x\in[a,b)$. take $h>0$ s.t. $a\le x<x+h<b$

So $\int_a^{x+h}f(t)\,dt=\int_a^x f(t)\,dt+\int_x^{x+h}f(t)\,dt$

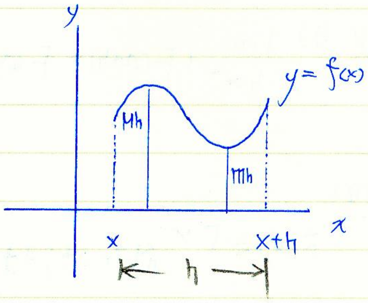
Hence $F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$

Then since f is cont on $[x, x+h]$ by Them 2.6.2 on $[x, x+h]$.

f has the absolute maximum and minimum, and denoted by M_h and m_h respectively.

Hence we have

$$h \cdot m_h \le \int_x^{x+h} f(t)\,dt \le h \cdot M_h$$ (圖解→)



Since $h>0$, we have

$$m_h \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le M_h$$

Then since f is cont at x

$$f(x) = \lim_{h\to 0^+} f(x+h) = \lim_{h\to 0^+} M_h = \lim_{h\to 0^+} m_h$$, So by pinching them,

We know

$$\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$ Hence $\lim_{h\to 0^+} \lim \frac{F(x+h) - F(x)}{h} = f(x)$

Secondary for $x \in (a,b]$, we will prove

(2) → $\lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h} = f(x)$ (This is just by the analogus argument)

For $x \in (a,b)$ by (1) and (2), we have

$$\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = f(x) = \lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h}$$

So. $F'(x) = f(x)$, $\forall\, x \in (a,b)$

To finish the proof it suffices to prove F is cont at a from right and at b from left
右連續 左連續

Since $F(a+h) - F(a) = \frac{F(a+h) - F(a)}{h} \cdot h$

by (1) $\lim_{h \to 0^+} (F(a+h) - F(a)) = f(a) \cdot 0 = 0$

• eg1.

$F(x) = \int_{-1}^{x} (t^3 + 3t)\,dt$, $F'(x) = ?$

$F'(x) = x^2 + 3x$

• eg2.

$G(x) = \int_{0}^{x^2} (3t^3 + 4t)\,dt$, $G'(x) = ?$

$u = g(x) = x^2 \quad y = G(x) = F(u) = F(g(x))$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$\frac{dy}{dx} = (3u^3 + 4u)(2x = (3x^6 + 4x^2)(2x)$$

◎ Section 5.4 Fundational Them of Calculus

◎ Def. 5.4.1 f is cont on [a, b], G is called an antiderivative of f if G is cont on [a, b] is diff on (a, b), and $G'(x) = f(x) \quad \forall x \in (a, b)$

◎ Theorem 5.4.2 (Fundational Them of calculus)

f is cont on [a, b], and G is any antiderivative of f. then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Pf.

let $F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a, b]$

By Them 5.3.5, F is cont on $[a, b]$ is diff on (a, b), and

$F'(x) = f(x) \ \forall x \in (a, b)$

So $F - G$ is cont on $[a, b]$ is diff on (a, b)

and $F'(x) - G'(x) = 0 \quad \forall x \in (a, b)$. Hence by Them 4.2.3

$F - G$ is constant on $[a, b]$, that is $F(x) - G(x) = C \ \forall x \in [a, b]$

so $F(a) - G(a) = C$, Then since $F(a) = 0$, $C = -G(a)$

Finally $F(b) - G(b) = C = -G(a)$

$$\int_a^b f(x)\,dx = F(b) = G(b) - G(a)$$

- eg 1.

$\int_1^4 x^2\,dx = ? \qquad f(x) = x^2$

$G(x) = \frac{1}{3}x^3 \quad$ so $\int_1^4 x^2\,dx = \frac{1}{3}x^3 \Big]_1^4 = \frac{64}{3} - \frac{1}{3} = \frac{63}{3} = 21$

- eg 2.

$\int_0^{\frac{\pi}{2}} \sin x\,dx = ? \quad f(x) = \sin x$

$G(x) = -\cos x \quad$ so $\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x \Big|_0^{\frac{\pi}{2}} = 0 - (-1) = 1$

◎

Fun.	Anti
x^r	$\frac{1}{r+1}x^{r+1}$, $r \neq -1$
$\sin x$	$\cos x$
$\sec^2 x$	$\tan x$

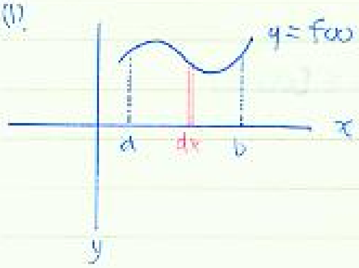
◎Remark

(1) $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

(2) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

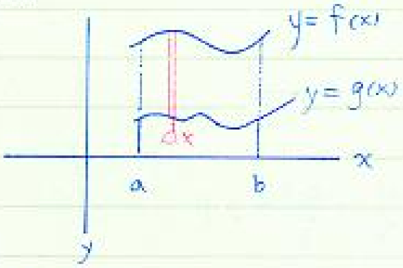
◎Section 5.5 Some area problem

(1).



$A = \int_a^b f(x)\,dx$

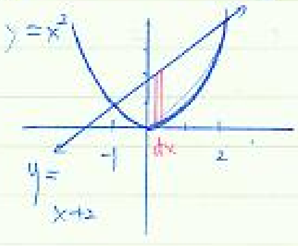
(2).



$A = \int_a^b (f(x)-g(x))\,dx$

•eg1. Find the area bouned by

$y = f(x) = x^2 \quad y = g(x) = x+2$



$\begin{cases} y = x^2 \\ y = x+2 \end{cases}$

$x^2 - x - 2 = 0$

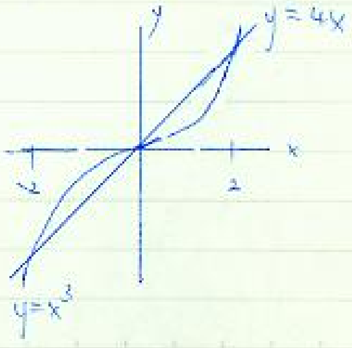
$x = 2$ or $x = -1$

$\int_{-1}^{2} (x+2-x^2)\,dx$

$= \frac{x^2}{2} + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2} =$

•eg2. Find the area bouned by

$y = 4x \quad y = x^3$



$y = x^3$

$\begin{cases} y' = 3x^2 \\ y'' = 6x \end{cases}$

$\int_0^2 (4x - x^3)\,dx + \int_{-2}^{0} (x^3 - 4x)\,dx$

$=$

◎ Section 5.6 and 5.7 : Indefinite integral and u-substitution

$\int f(x)dx = G(x) + C$, G is anti of f

- eg1.

$\int x^2 dx = \frac{1}{3}x^3 + C$ $\qquad \int \sec^2 x\, dx = \tan x + C$

$\int \sin x\, dx = -\cos x + C$

- eg2.

$\int (x^2-1)^4 x\, dx$

$u = x^2-1 \quad du = 2x\,dx \quad x\,dx = \frac{1}{2}du$

$$\int (x^2-1)^4 x\, dx = \int u^4 \tfrac{1}{2} du$$
$$= \frac{1}{2}\int u^4 du$$
$$= \frac{1}{10}u^5 + C = \frac{1}{10}(x^2-1)^5 + C$$

- eg3.

$\int \sin^2 x \cos x\, dx$

$u = \sin x \quad du = \cos x\, dx$

$$\int \sin^2 x \cos x\, dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$$

- eg4.

$\int 2x^2 \sin(x^3+1)\, dx$

$u = x^3+1 \quad du = 3x^2 dx \quad x^2 dx = \frac{1}{3}du$

$$\frac{1}{3}\int 2\sin u\, du = \frac{2}{3}\sin u\, du = \frac{-2}{3}\cos u + C$$

- eg5.

$\int x^2\sqrt{4+x^3}\, dx$

$u = 4+x^3 \quad du = 3x^2 dx$

$$\int \underline{x^2}\sqrt{4+x^3}\, \underline{dx} = \frac{1}{3}\int u^{1/2} du$$
$$= \frac{2}{9}u^{3/2} + C$$
$$= \frac{2}{9}(4+x^3)^{3/2} + C$$

- eg 6.

$\int \sec^3 x \tan x \, dx$ $u = \sec x$, $du = \sec x \tan x \, dx$

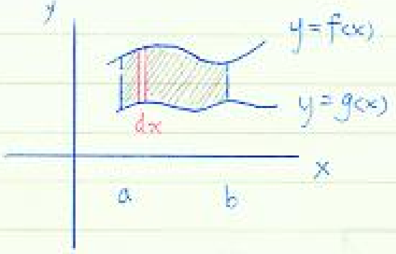
$= \int \sec^2 x \, d\sec x$ $\int \sec^3 x \tan x \, dx = \int u^2 du$

$= \frac{1}{3} \sec^3 x + C$

Ch6 Applications of Integral

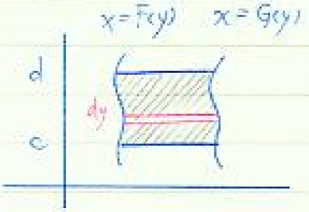
◎Section 6.1 6.2 and 6.3 : Area and Volume

Area:



$▯ = (f(x) - g(x)) \cdot dx$

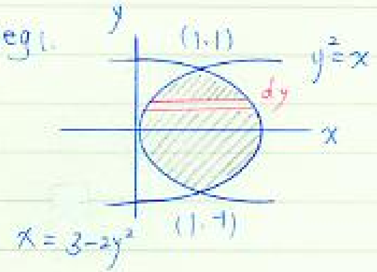
$A = \int_a^b (f(x) - g(x)) \cdot dx$



$▭ = (G(y) - F(y))\, dy$

$A = \int_c^d (G(y) - F(y))\, dy$

•eg1.

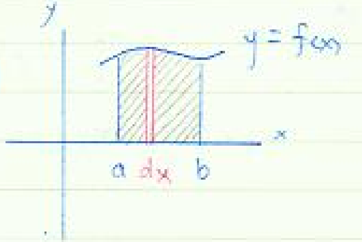


$▭ = (3-2y^2-y^2)\, dy$

$A = \int_{-1}^{1} (3-2y^2-y^2)\, dy = 4$

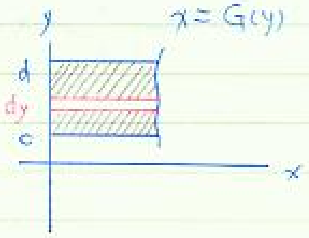
Volume

Disk. Method



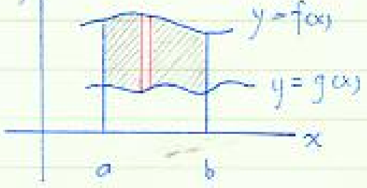
$▯ = \pi f^2(x) dx$ $(\pi r^2 \times h)$

$V = \int_a^b \pi f^2(x) dx$



$▭ = \pi G^2(y) dy$ $(\pi \cdot r^2 \cdot h)$

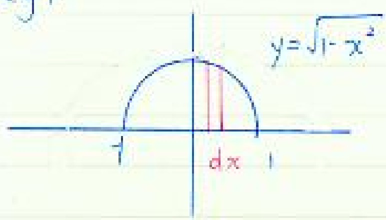
$V = \int_c^d \pi G^2(y) dy$



$▯ = \pi (f^2(x) - g^2(x))\, dx$ $(\pi [r_1^2 - r_2^2] \cdot h)$

$V = \int_a^b \pi (f^2(x) - g^2(x))\, dx$

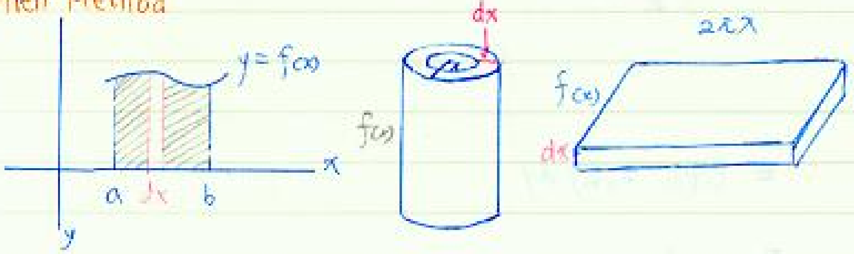
- eg1.



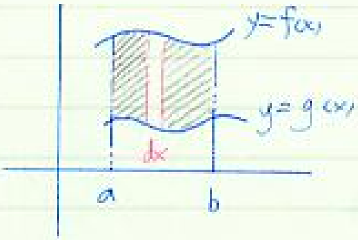
$$▯ = \pi\left[\sqrt{(1-x^2)}\right]^2 dx$$

$$\int_{-1}^{1} \pi\left[\sqrt{(1-x^2)}\right]^2 dx$$

Shell Method

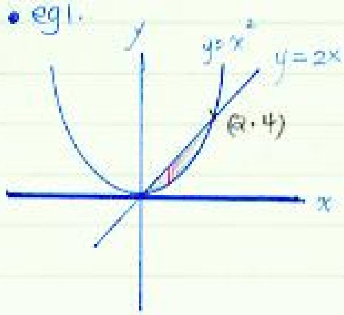


$$V=\int_a^b 2\pi x f(x)\,dx$$



$$V=\int_a^b 2\pi x(f(x)-g(x))\,dx$$

- eg1.



About x-axis.

Disk.

$$V=\int_0^2 \left[(2x)^2-(x^2)^2\right]\pi\ dx$$

$$= \frac{64}{15}$$

Shell
$$\begin{cases} x=y^{\frac{1}{2}} \\ x=\frac{1}{2}y \end{cases}$$

$$V=\int_0^4 2\pi y\left(y^{\frac{1}{2}}-\frac{1}{2}y\right)dy$$

About y-axis

Disk

$$V=\int_0^4 \pi\left[(y^{\frac{1}{2}})^2-\left(\frac{1}{2}y\right)^2\right]\ dy$$

Shell

$$V=\int_0^2 2\pi x(2x-x^2)\,dx$$

Ch7. Transcendental Function

◎Section 7.1 Inverse Fun

$f: A \to B$ is a function

(1) f is 1-1 (injective) if $f(x_1)=f(x_2)$, then $x_1 = x_2 \in A$

(2) f is onto (surjective) $\forall y \in B, \exists x \in A$ s.t $f(x)=y$

Inverse Fun:

If $f: A \to B$ is 1-1 and onto, then $\exists g: B \to A$, which is called

the inverse function of f, s.t.

$$\begin{cases} g(f(x)) = x & \forall x \in A \\ f(g(x)) = y & \forall y \in B \end{cases}$$

We usually denote g by f^{-1} that is $g=f^{-1}$

- eg 1.

$y=f(x)=x^3 \quad f: R \to R$

prove f is 1-1 and onto

pf: $f'(x)=3x^2, \; x \in R \quad (-\infty, 0] \, [0, \infty)$

By Them 4.2.3 $f \uparrow$ on R so f is 1-1

◎Theorem 7.1.8

f is 1-1 and onto f is diff at a $f'(a)=0$, and $f(a)=b$

Then. f^{-1} is diff at b. and $(f^{-1})'(b) = \dfrac{1}{f'(a)}$

pf: Since $f^{-1}(f(x)) = x$, $y=f(x)$

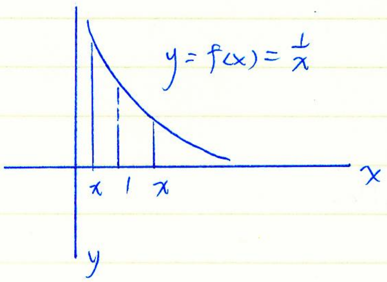
let $u=f^{-1}(y)$ so $\dfrac{du}{dx}=1$ hence $\dfrac{du}{dy}\dfrac{dy}{dx}=1$ so $(f^{-1})'(y) f'(x)=1$

so $(f^{-1})(b) f'(a)=1$ so $(f^{-1})'(b) = \dfrac{1}{f'(a)}$

◎Section 7.2 Natural logarithm fun

Define

$$L(x)=\int_1^x \frac{1}{t}\,dt \quad x>0 \quad x\in(0,\infty)$$



(1) By Them 5.3.5 $L'(x)=\frac{1}{x}$ $x>0$ So $L'(x)>0$ $x>0$ Hence by Them 4.2.2.

L is increasing on $(0,\infty)$ so L is 1-1

(2) $L(x)<0$, $0<x<1$ $L(1)=0$ $L(x)>0$ $x>1$

◎Theorem 7.2.1

$a>0$. $b>0$, $L(ab)=L(a)+L(b)$ $\quad L(x)=\int_1^x \frac{1}{t}\,dt$ $x>0$ $x\in(0,\infty)$

pf: $b>0$., $x>0$

$$\frac{dL(xb)}{dx} \xrightarrow[\text{chain rule}]{} =\overset{dy}{\underset{du}{\frac{1}{xb}}}\cdot\overset{du}{\underset{dx}{b}}=\frac{1}{x}=\frac{dL(x)}{dx}\ .\ x\in(0,\infty)$$

By Them 4.2.2, $L(xb)-L(x)=C$, $\forall x>0$

Replace x by 1. $L(b)-\underset{0}{\underset{\|}{L(1)}}=C$ $\quad C=L(b)$

Finally replace x by a $L(ab)=L(a)+L(b)$

◎Fact 7.2.2

(1) $b>0$ $L(\frac{1}{b}) = -L(b)$

(2). $a>0$. $b>0$. $L(\frac{a}{b}) = L(a) - L(b)$

pf:

$$0 = L(1) = L(\frac{b}{b}) = L(b\cdot\frac{1}{b}) \overset{\text{Them 7.2.1}}{=} L(b) + L(\frac{1}{b})$$

◎Fact 7.2.3

$a>0$, $\underset{\text{有理數}}{\frac{p}{q} \in Q}$, $L(a^{\frac{p}{q}}) = \frac{p}{q}L(a)$

pf. $x>0$

$$\frac{dL(x^{\frac{p}{q}})}{dx} = \frac{1}{x^{p/q}}\cdot\frac{p}{q}\cdot x^{\frac{p}{q}-1} = \frac{p}{q}x^{-1} = \frac{d\frac{p}{q}L(x)}{dx}$$

◎hint

$x>0$ Chain Rule, Power Formula, Them 7.2.1

$$\frac{dL(x^{\frac{p}{q}})}{dx} = \frac{1}{x^{p/q}}\frac{p}{q}x^{\frac{p}{q}-1}$$

$$= \frac{p}{q}\frac{1}{x} = \frac{d\frac{p}{q}L(x)}{dx}$$

So by Them 4.2.2.

$L(x^{\frac{p}{q}}) - \frac{p}{q}L(x) = c$, $x>0$

Replace x by 1. $c=0$

So $L(x^{\frac{p}{q}}) = \frac{p}{q}L(x)$

◎ Fact 7.2.4

$L: (0, \infty) \to R = (-\infty, \infty)$ L is 1-1 and onto

pf: since $L'(x) = \frac{1}{x} > 0$, $x>0$. by Them 4.2.2 L is increasing on $(0, \infty)$

So L is 1-1

To prove L is onto, it suffices to prove that for any $M>0$,

$\forall k \in [-M, M]$ $\exists x \in (0, \infty)$ s.t. $L(x) = k$

①

Since $M>0$, and $L(2) = \int_1^2 \frac{1}{t}dt > 0$, $\exists n \in N$, s.t. $nL(2) > M$

↓同乘"−"

$-n\,L(2) < -M$ so. $L(2^n) > M$, and $L(2^{-n}) < -M$

Hence $[-M, M] \leq [L(2^{-n}), L(2^n)]$

and $L(2^{-n}) < k < L(2^n)$ ---- by ①

Then since $L: [2^{-n}, 2^n] \to R$ is cont, and $L(2^{-n}) < k < L(2^n)$

by Intermediate Them $\exists\, x \in (2^{-n}, 2^n)$ s.t. $L(x) = k$
(Them. 2.6.1)

◎ Remark

Since L is onto for $1 \in R$ $\exists$ an unique $e \in (0, \infty)$ s.t. $L(e) = 1$

that is $\int_1^e \frac{1}{t}\,dt = 1$ $e = 2.1782\ldots\ldots$

From now on, we denote $L(x)$ by $\ln x$, $x > 0$ and we call $\ln x = y$ to be the natural logarithm function

◎ Remark

(1) $\ln(1) = 0$

(2) $\ln(ab) = \ln a + \ln b$

(3) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

(4) $\ln(a^{p/q}) = \frac{p}{q} \ln a$

(5) $\ln e = 1$

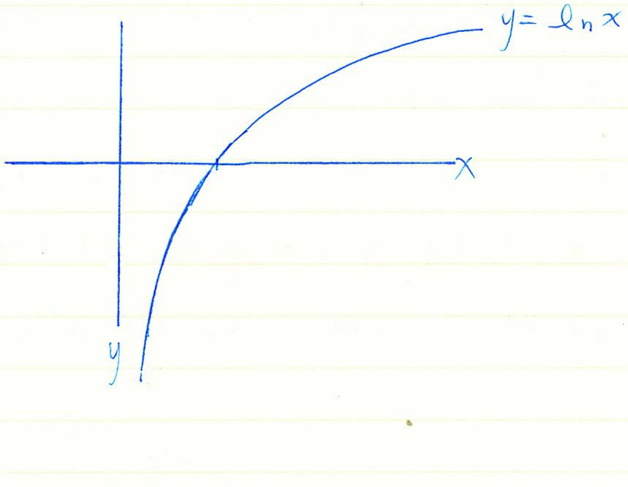
- eg1

$y = \ln x$, $x > 0$

$y' = \frac{1}{x} > 0$, $x > 0$. ↗ $y'' = -\frac{1}{x^2} < 0$ $x > 0$ ∩

$\lim_{x \to 0^+} \ln x = -\infty$

$\lim_{x \to \infty} \ln x = \infty$



◎Section 7.3 : logarithm (II)

$\frac{d\ln x}{dx} = \frac{1}{x}$, $x > 0$

• eg1.

$y = f(x) = \ln(1+x^2)$ $x \in R$

$f'(x) = \frac{1}{1+x^2}(2x)$

• eg 2.

$y = f(x) = \ln(x\sqrt{x^2+1})$ $x > 0$

$y = f(x) = \ln x + \frac{1}{2}\ln(1+x^2)$ $y' = \frac{1}{x} + \frac{1}{2\cdot(1+x^2)}(2x)$

• eg3 $y = f(x) = \ln|x|$ $x \neq 0$

(1) $x>0$ $y = \ln x$, $y' = \frac{1}{x}$

(2) $x<0$ $y = \ln \underset{\overset{\downarrow}{u}}{(-x)}$ $y' = \frac{1}{-x}\underset{\text{chain rule}}{(-1)} = \frac{1}{x}$ $f'(x) = \frac{1}{x}$

◎Fact

$\int \frac{1}{x}dx = \ln|x| + C$

• eg1. $y = f(x) = \ln|1-x^3|$

$y' = \frac{1}{1-x^3}(-3x^2)$

• eg 2. $\int \frac{x^2}{1-x^3}dx$

$u = 1-x^3$ $du = -3x^2dx$ $\int \frac{x^2}{1-x^3}dx = \frac{-1}{3}\int\frac{du}{u} = -\frac{1}{3}\ln|u| + C$

• eg 3.

$\int \frac{\ln x}{x}dx = \int u\cdot du = \frac{1}{2}u^2 + C$

$u = \ln x$. $du = \frac{1}{x}dx$

• eg4.

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C \overset{\prime\prime}{=} \ln|\sec x| + C''$$

$u = \cos x \quad du = -\sin x\, dx$

• eg5.

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{du}{u} = \ln|u| + C$$

$u = \sec x + \tan x \quad du = (\sec x \tan x + \sec^2 x)\, dx$

◎ Section 7.4. : Natural exponential fun.

Since $L = (0,\infty) \to R$ is 1-1 and onto, $\exists\, E: R \to (0,\infty)$, which is inverse function of L, s.t.

(1). $L(E(x)) = x$, $x \in R$

(2). $E(L(x)) = x$, $x > 0$

(3) $E(0) = 1$, $E(1) = e$

(4) $E(x) > 0$. $\forall x \in R$

◎ Theorem 7.4.7

$a \in R$. $b \in R$

$E(a+b) = E(a)E(b)$

pf. since

$$L(\underline{E(a+b)}) = a+b = L(E(a)) + L(E(b))$$
$$= L(\underline{E(a)E(b)}).$$

and since L is 1-1, $E(a+b) = E(a)E(b)$

◎ Remark

$E(a) = e^a$ $a \in R$

pf. First $a \in R$. we discuss the following cases:

(1).

$a = n \in N$

$$E(n) = E(1+1+\cdots+1) = \underbrace{E(1)E(1)E(1)\cdots E(1)}_{n個} = e^n$$

(2). $a = \frac{1}{n}$ $n \in N$

$$e = E(1) = E(\underbrace{\tfrac{1}{n}+\tfrac{1}{n}+\cdots+\tfrac{1}{n}}_{n個}) = E(\tfrac{1}{n})\,E(\tfrac{1}{n})\,E(\tfrac{1}{n})\cdots E(\tfrac{1}{n})$$ So $E(\frac{1}{n}) = e^{\frac{1}{n}}$

$[E(\frac{1}{n})]^n = e$ $E(\frac{1}{n}) = e^{\frac{1}{n}}$

(3). $a = \frac{m}{n}$ $n \in N$, $m \in N$

$$E(\tfrac{m}{n}) = E(\underbrace{\tfrac{1}{n}+\tfrac{1}{n}+\cdots\tfrac{1}{n}}_{m個}) = E(\tfrac{1}{n})E(\tfrac{1}{n})E(\tfrac{1}{n})\cdots E(\tfrac{1}{n}) = e^{\frac{1}{n}}e^{\frac{1}{n}}\cdots e^{\frac{1}{n}} = e^{\frac{m}{n}}$$

◎ Remark.

$$\boxed{\begin{array}{l} E(-b) = \frac{1}{E(b)} \quad b \in R \\ 1 = E(0) = E(b-b) = E(b)E(-b) \end{array}}$$

(4). $\underset{有理數}{a \in Q}$ $a > 0$

$$E(-a) = \frac{1}{E(a)} = \frac{1}{e^a} = e^{-a}$$

So far, we have $E(a) = e^a$, $a \in Q$

Secondary a is irrational, So $\exists$ a sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in Q$

S.t. $\lim_{n\to\infty} a_n = a$ Hence we have $\underline{e^a} = \lim_{n\to\infty} e^{a_n} = \lim_{n\to\infty} E(a_n) = \underline{E(a)}$

◎ Remark

$y = e^x$ iff $\ln y = x$ iff $\log_e y = x$

pf. ($\Rightarrow$) if $y = e^x$, then $\ln y = \ln e^x = x$

($\Leftarrow$) if $\ln y = x$, then $e^{\ln y} = e^x = y$

◎Theorem 7.4.9

$y = e^x$ then $\frac{dy}{dx} = e^x$

pf: Since $\ln e^x = x$ $\frac{1}{e^x}\frac{de^x}{dx} = 1$. So $\frac{de^x}{dx} = e^x$

隱函數微分

$\frac{de^u}{dx} = e^u \frac{dy}{dx}$, $\frac{d\ln x}{dx} = \frac{1}{u}\frac{du}{dx}$

- eg1. $y = f(x) = e^{\sqrt{x}}$ $x \geq 0$

$$\frac{dy}{dx} = e^{\sqrt{x}} \frac{1}{2}\frac{1}{\sqrt{x}}$$

- eg2. $y = e^{-x^2}$ $y' = e^{-x^2}(-2x)$

◎Fact: $\int e^x dx = e^x + C$

- eg3

$$\int \frac{e^{3x}}{e^{3x}+1} dx$$

$u = e^{3x} + 1$ $du = 3e^{3x} dx$

$$\int \frac{e^{3x}}{e^{3x}+1} dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$$

◎Section 7.5 Arbitrary Powers

$x > 0$, $r \in \mathbb{R}$

$x^r = e^{r\ln x}$

Remark

$e^{r\ln x} = e^{\ln x^r} = x^r$, $r \in Q$

◎Fact 1 $r \in R \cdot s \in R \quad x > 0$

(1) $x^{r+s} = x^r \cdot x^s$

(2) $x^{r-s} = \frac{x^r}{x^s}$

(3). $(x^r)^s = x^{rs}$

pf.

For (1)

$$x^{r+s} = e^{(r+s)\ln x} = e^{r\ln x + s\ln x} = e^{r\ln x} \cdot e^{s\ln x} = x^r \cdot x^s$$

For (2)

$$x^{r-s} = e^{(r-s)\ln x} = e^{r\ln x - s\ln x} = \frac{e^{r\ln x}}{e^{s\ln x}} = \frac{x^r}{x^s}$$

For (3)

$$(x^r)^s = e^{s\ln x^r} = e^{s\ln e^{\underline{r\ln x}}} = e^{sr\ln x} = x^{rs}$$

◎Fact 2 · Power formula

$x > 0 \quad r \in R$

$$\frac{dx^r}{dx} = rx^{r-1}$$

pf: $\frac{dx^r}{dx} = \frac{de^{r\ln x}}{dx} = e^{\ln x} r \cdot \frac{1}{x} = x^r \cdot r\frac{1}{x} = rx^{r-1}$

- eg 1.

$y = f(x) = x^{\sqrt{2}} \qquad y' = \sqrt{2}\, x^{\sqrt{2}-1}$

$p > 0 \quad x > 0$

$$\log_p x = \frac{\ln x}{\ln p}$$