Real Analysis Sample Exam, November 6, 2007

Show detailed argument to each problem.

1. A "totally unlucky" number is one that contains no sevens in any decimal expansion. Compute the Lebesgue measure of the totally unlucky numbers in $[0,1]$.

solution:

Among the numbers 0.1...., 0.2...., $\cdots$, 0.9...., the lucky numbers has measure $s = \frac{1}{10}$. Among the numbers 0.11...., 0.12...., $\cdots$, 0.19...., the lucky numbers has measure $\frac{1}{10} \cdot s$. Based on this observation, all of the lucky numbers has measure

$$\begin{aligned}
& s + (1-s)\,s + [1 - s - (1-s)\,s]\,s + [1 - \{s + (1-s)\,s + [1 - s - (1-s)\,s]\,s\}]\,s + \cdots \\
= \;& s + (1-s)\,s + (1-s)^2\,s + (1-s)^3\,s + \cdots \\
= \;& \frac{s}{1-(1-s)} = 1.
\end{aligned}$$

Hence the totally unlucky numbers has measure zero. We are lucky!!! ¤

2. Let $f : [a,b] \to \mathbf{R}$ be a finite **increasing** function. Show that f is a measurable function on $[a,b]$. For any $p \in (a,b)$, evaluate the following limits:

$$\lim_{h\to 0^+} \frac{1}{h} \int_{[p,p+h]} f \qquad \text{(Lebesgue integral)}$$

and

$$\lim_{h\to 0^+} \frac{1}{h} \int_{[p-h,p]} f \qquad \text{(Lebesgue integral).}$$

solution:

If $f : [a,b] \to \mathbf{R}$ is an increasing function, the number of $x \in [a,b]$ such that f is discontinuous at x is at most **countable** (see Rudin, p.96). Hence f is continuous a.e. on $[a,b]$ and so measurable.

We also know that both $f(p+)$ and $f(p-)$ exist for any $p \in (a,b)$. For fixed $p \in (a,b)$, there exists a number A such that for any $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in (p, p+\delta)$ then $|f(x) - A| < \varepsilon$. Hence

$$\frac{1}{h}\int_{[p,p+h]} (A-\varepsilon) - A \le \left(\frac{1}{h}\int_{[p,p+h]} f\right) - A \le \frac{1}{h}\int_{[p,p+h]} (A+\varepsilon) - A$$

for all $0 < h < \delta$. Therefore $\lim_{h\to 0^+} \frac{1}{h}\int_{[p,p+h]} f = A = f(p+)$. Similarly we have $\lim_{h\to 0^+} \frac{1}{h}\int_{[p-h,p]} f = f(p-)$. ¤

3. Let $E_n \subset \mathbf{R}$ be a sequence of measurable sets. Let

$$A = \{x \in \mathbf{R} : \; x \in E_n \text{ for infinitely many } n\}.$$

Is the set measurable or not? Give your reasons.

solution:

We actually have

$$A = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right).$$

Hence the set A is measurable ¤

4. Give an example of a measurable function $h : E \subset \mathbf{R} \to \mathbf{R}$ such that for some measurable set $B \subset \mathbf{R}$ the inverse image $h^{-1}(B)$ is **NOT** measurable.

solution:

Let $f(x) : [0,1] \to [0,1]$ be the Cantor-Lebesgue function and let $g(x) = x + f(x)$. It is easy to see that $g(x) : [0,1] \to [0,2]$ is a strictly increasing continuous function. Hence $g(x)$ is a homeomorphism of $[0,1]$ onto $[0,2]$. On each interval $I_1, I_2, I_3, \ldots$, removed in the construction of the Cantor set, say the interval $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, the function $g(x)$ becomes $g(x) = x + \frac{1}{2}$. Hence $g(x)$ sends I_1 onto an open interval with the same length. Using this observation one can see that

$$\left| g\left(\bigcup_{k=1}^{\infty} I_k \right) \right| = \left| \bigcup_{k=1}^{\infty} g(I_k) \right| = \sum_{k=1}^{\infty} |g(I_k)| = \sum_{k=1}^{\infty} |I_k| = 1$$

which implies $|g(C)| = 2 - 1 = 1$, where C is the Cantor set. Since $g(C)$ has positive measure, there exists a non-measurable set $A \subset g(C)$. Now consider the set $B = g^{-1}(A) \subset C$. It has measure zero, hence it is measurable. Let $h = g^{-1}$. Then it is a measurable function and $h^{-1}(B) = A$ is not measurable. ¤

5. Let E be a measurable set in $\mathbf{R}^n$ with $|E| < \infty$ and f is a measurable function on E. Let

$$E_n = \{x \in E : |f(x)| \geq n\}, \quad n = 0, 1, 2, 3 \ldots.$$

Show that $f \in L(E)$ if and only if $\sum_{n=0}^{\infty} |E_n| < \infty$.

solution:

$(\Longrightarrow)$ Assume $f \in L(E)$. Then f is finite a.e. in E (without loss of generality, we can assume f is finite everywhere in E). It is not hard to see that

$$\lim_{\lambda \to \infty} \int_{\{|f| \geq \lambda\}} |f| = 0$$

which is like the case of an absolutely convergence sequence. This also implies (by Tchebyshev's inequality)

$$\lim_{n \to \infty} n|E_n| \leq \lim_{n \to \infty} \int_{\{|f| \geq n\}} |f| = 0. \tag{0.1}$$

By the relation $E = E_0 \supset E_1 \supset E_2 \supset \cdots$, $|E_0| < \infty$, we can decompose E as

$$E = (E_0 - E_1) \mathring{\cup} (E_1 - E_2) \mathring{\cup} (E_2 - E_3) \cdots \qquad \text{(disjoint union)}$$

and observe that

$$n \cdot |E_n - E_{n+1}| \leq \int_{E_n - E_{n+1}} |f| \leq (n+1) \cdot |E_n - E_{n+1}|, \quad n = 0, 1, 2, 3, \ldots \tag{0.2}$$

Hence

$$\sum_{n=0}^{\infty} n \cdot |E_n - E_{n+1}| \leq \sum_{n=0}^{\infty} \int_{E_n - E_{n+1}} |f| = \int_E |f|$$

where (in below we need to use the fact that $|E| < \infty$, and so $|E_n - E_{n+1}| = |E_n| - |E_{n+1}|$) by (0.1) we have

$$\begin{aligned} \sum_{n=0}^{\infty} n \cdot |E_n - E_{n+1}| &= (|E_1| - |E_2|) + 2(|E_2| - |E_3|) + 3(|E_3| - |E_4|) + \cdots \\ &= |E_1| + |E_2| + |E_3| + |E_4| + \cdots. \end{aligned}$$

Therefore $\sum_{n=0}^{\infty} |E_n| < \infty$.

($\Longleftarrow$) Conversely if $\sum_{n=0}^{\infty} |E_n| < \infty$, then the set $\{x \in E : |f(x)| = \infty\}$ must have measure zero, implying f is finite a.e. in E . Again we assume f is finite everywhere in E. Since f is bounded on $E_n - E_{n+1}$, it is integrable on it. Note that

$$\sum_{n=0}^{\infty} (n+1) \cdot |E_n - E_{n+1}| = 2(|E_1| - |E_2|) + 3(|E_2| - |E_3|) + 4(|E_3| - |E_4|) + \cdots$$
$$= 2|E_1| + |E_2| + |E_3| + |E_4| + \cdots < \infty.$$

By (0.2) we know $|f|$ must be integrable on E. Hence f is integrable on E. ¤

6. Assume $h(x)$ is a differentiable function on $\mathbf{R}$. Show that $h'(x)$ is a measurable function on $\mathbf{R}$.

solution:

Let
$$f_n(x) = \frac{h\left(x + \frac{1}{n}\right) - h(x)}{\frac{1}{n}}, \qquad x \in \mathbf{R}.$$

For each $n = 1, 2, 3, \ldots$, $f_n(x)$ is a finite measurable function on $\mathbf{R}$ with $f_n(x) \to h'(x)$ for all $x \in \mathbf{R}$. Hence $h'(x)$ is a measurable function on $\mathbf{R}$. ¤

7. In the Lebesgue Dominated Convergence Theorem (Theorem 5.36) if we replace the condition "$f_k \to f$ a.e. in E" by "$f_k \to f$ in measure on E", is the theorem still correct or not? Give your reasons.

solution:

The theorem is still correct.

First note that by Theorem 4.22 there exists a subsequence $f_{k_j} \to f$ a.e. on E. This implies that $|f| \le \varphi$ a.e. in E and so $f \in L(E)$ (since $\varphi \in L(E)$). By the usual Lebesgue Dominated Convergence Theorem we have

$$\int_E f_{k_j} \to \int_E f \quad \text{as} \quad j \to \infty. \tag{0.3}$$

We then use contradiction argument to show that $\int_E f_k \to \int_E f$ as $k \to \infty$. Assume not. Then there exists a subsequence of f_k, still denote it as f_{k_j}, $j = 1, 2, 3, \ldots$, so that

$$\left| \int_E f_{k_j} - \int_E f \right| \ge \varepsilon > 0$$

for all j. But then this subsequence has a further subsequence so that (0.3) holds, a contradiction. ¤