Real Analysis Midterm Exam, November 13, 2007

Show detailed argument to each problem.

1. (10 points) Suppose E is a Lebesgue measurable subset of $\mathbf{R}$ with $|E| < \infty$. Prove that

$$|E| = \sup\{|K| : K \subset E \text{ and } K \text{ is compact}\}. \tag{0.1}$$

solution:

Let $E_n = E \bigcap [-n,n]$, $n \in \mathbf{N}$. We have $E_n \nearrow E$ and so $|E - E_n| \to 0$ as $n \to \infty$ (note that $|E| < \infty$). For any $\varepsilon > 0$ one can find $m \in \mathbf{N}$ so that $|E - E_n| < \varepsilon/2$ for all $n \geq m$. For E_m one can find a closed subset $F \subset E_m$ such that $|E_m - F| < \varepsilon/2$. Hence $|E - F| \leq |E - E_m| + |E_m - F| < \varepsilon$. In particular F is compact since E_m is bounded. We also have

$$|E| \leq |E - F| + |F| < \varepsilon + |F|$$

where $F \subset E$ and F is compact. (0.1) is proved. ¤

2. (10 points) Assume $E(t)$ is a continuously differentiable **increasing** function on $[0,\infty)$ such that $0 \leq E(t) \leq C$ for all $t \in [0,\infty)$, where C is some positive constant. Show that for any $\varepsilon > 0$, we have

$$\left|\left\{t \in [0,\infty) : E'(t) > \varepsilon\right\}\right| \leq \frac{C}{\varepsilon}.$$

solution:

By **Tchebyshev inequality** we have

$$\left|\left\{t \in (0,\infty) :\ E'(t) > \varepsilon\right\}\right| \leq \frac{1}{\varepsilon}\int_0^\infty E'(t)\,dt \leq \frac{1}{\varepsilon}\left[\lim_{t\to\infty} E(t) - E(0)\right] \leq \frac{C}{\varepsilon}$$

since $0 \leq E(t) \leq C$ for all $t \in [0,\infty)$. ¤

3. (10 points) Let $f_n : E \to \mathbf{R}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbf{R}^n$. Let

$$A = \left\{x \in E :\ \lim_{n\to\infty} f_n(x) \text{ exists}\right\}.$$

Is A a measurable set or not? Give your reasons.

solution:

Let $F^*(x) = \limsup_{n\to\infty} f_n(x)$ and $F_*(x) = \liminf_{n\to\infty} f_n(x)$. We know that both functions are measurable on E. Hence the sets

$$\begin{cases} S_1 := \{x \in E : F^*(x) = \infty \quad \text{and} \quad F_*(x) = \infty\} \\ S_2 := \{x \in E : F^*(x) = -\infty \quad \text{and} \quad F_*(x) = -\infty\} \end{cases}$$

are all measurable. Let $S = S_1 \bigcup S_2$. Now $E - S$ is also measurable, and $F^* - F_*$ is a measurable function on $E - S$. By the relation

$$A = \{x \in E - S : F^*(x) - F_*(x) = 0\}$$

we know that A is a measurable set. ¤

4. (10 points) Give an example of a sequence of measurable functions $\{f_k\}$ defined on a measurable set $E \subset \mathbf{R}^n$ such that the following strict inequalities hold:

$$\int_E \liminf_{k\to\infty} f_k dx < \liminf_{k\to\infty} \int_E f_k dx < \limsup_{k\to\infty} \int_E f_k dx < \int_E \limsup_{k\to\infty} f_k dx.$$

solution:

On the interval $[0,1]$, let

$$f_k(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad k = 1,3,5,7,....$$

and

$$f_k(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ 2, & x \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad k = 2,4,6,8,....$$

then

$$0 = \int_E \liminf_{k\to\infty} f_k dx < \liminf_{k\to\infty} \int_E f_k dx = \frac{1}{2}$$

and

$$1 = \limsup_{k\to\infty} \int_E f_k dx < \int_E \limsup_{k\to\infty} f_k dx = \frac{3}{2}.$$

□

5. (15 points) Assume $f \in L[a,b]$ and let $h(x) = \int_a^x f,\ x \in [a,b]$. Is the function $h(x)$ a measurable function on $[a,b]$? Give your reasons.

<u>solution:</u>

By looking at f^+ and f^-, without loss of generality, we may assume that $f \geq 0$ on $[a,b]$. Now there exists a sequence of simple functions $0 \leq f_k \nearrow f$ a.e. on E, where $f_k \in L[a,b]$ also for all k. Monotone Convergence Theorem implies

$$\int_a^x f_k \to \int_a^x f = h(x) \quad \text{as} \quad k \to \infty$$

for all $x \in [a,b]$. As f_k is a bounded function on $[a,b]$, we know that $\int_a^x f_k$ is a continuous function of $x \in [a,b]$. Hence $h(x)$ is a measurable function on $[a,b]$.

Another solution:

We have $h(x) = \int_a^x f^+ - \int_a^x f^-$ and both $\int_a^x f^+$ and $\int_a^x f^-$ are increasing functions on $[a,b]$. We know that an increasing function on $[a,b]$ is continuous a.e. on $[a,b]$. Hence $h(x)$ is a measurable function on $[a,b]$. □

6. (15 points) Suppose $E \subseteq \mathbf{R}$ is measurable with $|E| = \lambda > 0$, where λ is a finite number. Show that for any t with $0 < t < \lambda$, there exists a subset A of E such that A is measurable and $|A| = t$. That is, the Lebesgue measure $|\cdot|$ on $\mathbf{R}$ satisfies the Intermediate Value Theorem.

<u>solution:</u>

Define $f(x) = \left|(-\infty, x) \bigcap E\right|,\ x \in \mathbf{R}$. Then for $x < y$ we have $f(x) \leq f(y)$ and

$$f(y) - f(x) = \left|(-\infty, y) \bigcap E\right| - \left|(-\infty, x) \bigcap E\right| = \left|(x, y) \bigcap E\right| \leq y - x.$$

This means that $f(x)$ is a continuous function on $\mathbf{R}$ with $\lim_{x\to-\infty} f(x) = 0,\ \lim_{x\to\infty} f(x) = \lambda$. For any $0 < t < \lambda$, by the mean value theorem of continuous functions, we have $f(s) = t$ for some $s \in \mathbf{R}$. Hence the set $A = (-\infty, s) \bigcap E$ satisfies $|A| = t$. □

7. (15 points) Let $y = Tx$ be a nonsingular linear transformation of $\mathbf{R}^n$. If $\int_E f(y)\,dy$ exists, show that we have the following change of variables formula:

$$\int_E f(y)\,dy = |\det T| \int_{T^{-1}E} f(Tx)\,dx. \tag{0.2}$$

solution:

Case1: If $f(x) = \chi_{E_1},\ E_1 \subset E$, then LHS of (0.2) is $|E_1|$, and the RHS is given by

$$|\det T| \int_{T^{-1}E} \chi_{E_1}(Tx)\,dx = |\det T| \cdot \left|T^{-1}E_1\right|.$$

We see that (0.2) holds by Theorem 3.35.

Case2: Assume $f \geq 0$. Then there exists a sequence of simple functions $0 \leq s_n \nearrow f$ on E where

$$s_n = a_1\chi_{E_1} + \cdots + a_{k(n)}\chi_{E_{k(n)}}, \quad k(n) \text{ depends on } n.$$

Now by Case1

$$\begin{aligned}
\int_E s_n(y)\,dy &= a_1 \int_E \chi_{E_1}(y)\,dy + \cdots + a_{k(n)} \int_E \chi_{E_{k(n)}}(y)\,dy \\
&= a_1 |\det T| \int_{T^{-1}E} \chi_{E_1}(Tx)\,dx + \cdots + a_{k(n)} |\det T| \int_{T^{-1}E} \chi_{E_{k(n)}}(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} \left(a_1\chi_{E_1} + \cdots + a_{k(n)}\chi_{E_{k(n)}}\right)(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} s_n(Tx)\,dx
\end{aligned}$$

and by the Monotone Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_E s_n(y)\,dy = \int_E f(y)\,dy, \qquad \lim_{n\to\infty} |\det T| \int_{T^{-1}E} s_n(Tx)\,dx = |\det T| \int_{T^{-1}E} f(Tx)\,dx.$$

The conclusion follows.

For general f, use

$$\begin{aligned}
\int_E f &= \int_E f^+ - \int_E f^- = |\det T| \int_{T^{-1}E} f^+(Tx)\,dx - |\det T| \int_{T^{-1}E} f^-(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} f(Tx)\,dx.
\end{aligned}$$

¤

8. (15 points) Let $p > 0$ be a constant and let $f,\ f_k,\ k = 1, 2, 3, ...,$ be measurable functions on E. If $\int_E |f_k - f|^p \to 0$ as $k \to \infty$ and $\int_E |f_k|^p \leq M$ ($M > 0$ is a constant) for all k, show that $\int_E |f|^p \leq M$ also.

solution:

For any $\varepsilon > 0$ we have

$$|\{|f_k - f|^p > \varepsilon\}| \leq \frac{1}{\varepsilon} \int_E |f_k - f|^p \to 0 \quad \text{as} \quad k \to \infty$$

due to the Tchebyshev inequality. Hence f_k converges to f **in measure** on E. In particular, there exists a subsequence f_{k_j} such that it converges to f a.e. on E. Fatou's lemma implies

$$\int_E \liminf_{j\to\infty} \left|f_{k_j}\right|^p = \int_E |f|^p \leq \liminf_{j\to\infty} \int_E \left|f_{k_j}\right|^p \leq M.$$

The proof is done. ¤

Remark 1 (be careful) $\int_E |f_k - f|^p \to 0$ as $k \to \infty$ does **not**, in general, imply that $|f_k - f|^p \to 0$ a.e. on E as $k \to \infty$. Also for $p > 0$ the inequality

$$|f|^p \leq |f_k - f|^p + |f_k|^p$$

is wrong in general. It holds only for $0 < p \leq 1$.