Real Analysis Homework 9, due 2007-11-21 in class

1. (10 points) Do Exercise 6 in p. 85.

Solution:

In this problem we assume $\frac{\partial}{\partial x} f(x,y)$ exists on $I = [0,1] \times [0,1]$. we also know that it is a bounded function on I. Let

$$F_n(x,y) = \frac{f\left(x + \frac{1}{n}, y\right) - f(x,y)}{\frac{1}{n}}, \quad n = 1,\ 2,\ 3,\ \ldots.$$

we see that for each fixed x, $F_n(x,y)$ is a sequence of bounded (use mean value theorem to see this) measurable functions of y. By

$$\frac{\partial}{\partial x} f(x,y) = \lim_{n\to\infty} F_n(x,y)$$

we know that for each fixed x, $\frac{\partial}{\partial x} f(x,y)$ is a measurable function of y. Now by the Bounded Convergence Theorem, we obtain

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \lim_{n\to\infty}\int_0^1 F_n(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

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2. (10 points) Do Exercise 9 in p. 85.

Solution:

For any $\varepsilon > 0$ we have

$$|\{|f_k - f|^p > \varepsilon\}| \le \frac{1}{\varepsilon}\int_E |f_k - f|^p \to 0 \quad \text{as} \quad k \to \infty$$

due to the Tchebyshev inequality. Hence f_k converges to f in measure on E. ¤

3. (10 points) Do Exercise 10 in p. 85.

Solution:

By Exercise 9 we know that $f_k \to f$ in measure. In particular, there exists a subsequence f_{k_j} such that it converges to f a.e. on E. Fatou's lemma implies

$$\int_E \liminf_{j\to\infty} \left|f_{k_j}\right|^p = \int_E |f|^p \le \liminf_{j\to\infty} \int_E \left|f_{k_j}\right|^p \le M.$$

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4. (10 points) Do Exercise 20 in p. 85.

Solution:

Case1: If $f(x) = \chi_{E_1}$, $E_1 \subset E$, then LHS of the identity is $|E_1|$, and the RHS of the identity is given by

$$|\det T| \int_{T^{-1}E} \chi_{E_1}(Tx)\,dx = |\det T| \cdot \left|T^{-1}E_1\right|.$$

We see that the identity holds by Theorem 3.35.

Case2: Assume $f \geq 0$. Then there exists a sequence of simple functions $0 \leq s_n \nearrow f$ on E where

$$s_n = a_1 \chi_{E_1} + \cdots + a_{k(n)} \chi_{E_{k(n)}}, \quad k(n) \text{ depends on } n.$$

Now by Case1

$$\begin{aligned}
\int_E s_n(y)\,dy &= a_1 \int_E \chi_{E_1}(y)\,dy + \cdots + a_{k(n)} \int_E \chi_{E_{k(n)}}(y)\,dy \\
&= a_1 |\det T| \int_{T^{-1}E} \chi_{E_1}(Tx)\,dx + \cdots + a_{k(n)} |\det T| \int_{T^{-1}E} \chi_{E_{k(n)}}(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} \left(a_1 \chi_{E_1} + \cdots + a_{k(n)} \chi_{E_{k(n)}} \right)(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} s_n(Tx)\,dx
\end{aligned}$$

and by the Monotone Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_E s_n(y)\,dy = \int_E f(y)\,dy, \qquad \lim_{n\to\infty} |\det T| \int_{T^{-1}E} s_n(Tx)\,dx = |\det T| \int_{T^{-1}E} f(Tx)\,dx.$$

The conclusion follows.

For general f, use

$$\begin{aligned}
\int_E f &= \int_E f^+ - \int_E f^- = |\det T| \int_{T^{-1}E} f^+(Tx)\,dx - |\det T| \int_{T^{-1}E} f^-(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} f(Tx)\,dx.
\end{aligned}$$

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