Real Analysis Homework 8, due 2007-10-31 in class

1. (10 points) Let $f: E \to \mathbf{R} \bigcup \{\pm\infty\}$ be a nonnegative measurable function such that $\int_E f < \infty$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable subset $E_1 \subset E$ with $|E_1| < \delta$ we have $\int_{E_1} f < \varepsilon$.

   Solution: Let

$$f_k(x) = \begin{cases} f(x), & \text{if} \quad f(x) < k \\ k, & \text{if} \quad f(x) \geq k \end{cases}, \quad x \in E.$$

Then $0 \leq f_k(x) \nearrow f(x)$ on E and by the Monotone Convergence Theorem we have

$$\lim_{k\to\infty} \int_E f_k dx = \int_E f < \infty$$

and so for any $\varepsilon > 0$ there exists N such that $\int_E f - \int_E f_N < \varepsilon/2$. Note that $f_N \leq N$ on E and so if $E_1 \subset E$ with $|E_1| < \delta := \frac{\varepsilon}{2N}$ we would have $\int_{E_1} f_N \leq N|E_1| \leq \varepsilon/2$. Therefore for any $E_1 \subset E$ with $|E_1| < \delta$ we get

$$\int_{E_1} f = \int_{E_1} f - \int_{E_1} f_N + \int_{E_1} f_N < \varepsilon.$$

$\square$

2. (10 points) Do Exercise 3 in p. 85.

   Solution: Since $f_k \leq f$ a.e. on E (both are nonnegative), we have $\int_E f_k \leq \int_E f$ for all k. On the other hand, by Fatou's lemma we get

$$\int_E \liminf_{k\to\infty} f_k = \int_E f \leq \liminf_{k\to\infty} \int_E f_k$$

which implies

$$\int_E f \leq \liminf_{k\to\infty} \int_E f_k \leq \limsup_{k\to\infty} \int_E f_k dx \leq \int_E f.$$

Hence we have $\lim_{k\to\infty} \int_E f_k = \int_E f$. $\square$

3. (10 points) Let $f_k: E \to \mathbf{R} \bigcup \{\pm\infty\}$ be a sequence of nonnegative measurable function satisfying $\int_E f_k \to 0$ as $k \to \infty$. Show that $f_k \to 0$ **in measure** as $k \to \infty$.

   Solution: For any $\varepsilon > 0$ by Tchebyshev's inequality we have

$$|\{x \in E : |f_k - 0| > \varepsilon\}| = |\{x \in E : f_k > \varepsilon\}| \leq \frac{1}{\varepsilon} \int_E f_k.$$

Letting $k \to \infty$, the conclusion follows. $\square$

Remark 1 (be careful) $\int_E f_k \to 0$ as $k \to \infty$ does not, in general, imply that $f_k \to 0$ a.e. on E.

4. (10 points) Compute the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx$$

   and justify your answer.

   Solution: Let

$$f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n e^{x/2}, & \text{if} \quad x \in [0, n] \\ 0, & \text{if} \quad x > n. \end{cases}$$

One can check that $f_n(x) \nearrow f(x) = e^{-x/2}$ on $E = [0, \infty)$. By Monotone Convergence Theorem we have

$$\lim_{n\to\infty} \int_E f_n = \int_E f = \int_0^\infty e^{-x/2} dx = 2.$$

$\square$