Real Analysis Homework 7, due 2007-10-31 in class

1. (10 points)

   (a) (7 points) Do Exercise 15 in p. 62.

   (b) (3 points) Use Exercise 15 in p. 62 to prove the following: Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function where $|E| < \infty$ and $|f| < \infty$ a.e. on E. Show that for any $\varepsilon > 0$ there exists a constant $M > 0$ and a closed set $F \subset E$ such that $|E - F| < \varepsilon$ and

   $$|f(x)| \leq M \quad \text{for all} \quad x \in F.$$

   This says that a finite function is, up to a set of small measure, a bounded function.

<u>Solution</u>: For (a). For each $n = 1, 2, ...$, let

$$E_n = \{x \in E : |f_k(x)| \leq n \ \text{ for all } \ k\}.$$

Then by $|f_k(x)| \leq M_x < \infty$ for all k and all $x \in E$, we have $E_n \nearrow E$ as $n \to \infty$ with $\lim_{n\to\infty} |E_n| = |E|$. Since $|E| < \infty$, we also have $\lim_{n\to\infty} |E - E_n| = 0$. Choose M such that $|E - E_M| < \varepsilon/2$ and choose a closed set $F \subset E_M$ such that $|E_M - F| < \varepsilon/2$. Then we have $|E - F| < \varepsilon$ and the following holds

$$|f_k(x)| \leq M \quad \text{for all } x \in F \text{ and all } k.$$

For (b). By (a), if we choose $f_k(x) = f(x)$ for each $k \in \mathbb{N}$, then we are done. ¤

2. (10 points) Do Exercise 16 in p. 63.

<u>Solution</u>:

$(\Longrightarrow)$. By definition, for any $\varepsilon > 0$ we have

$$\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| = 0.$$

Hence for the same $\varepsilon > 0$ we have

$$|\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| < \varepsilon$$

if $k \geq K$, for some large $K > 0$.

$(\Longleftarrow)$. Fixed an arbitrary $\varepsilon > 0$ first. We want to show that for any $\delta > 0$ there exists $K > 0$ such that if $k \geq K$ we have

$$|\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| < \delta. \tag{0.1}$$

Now for any $\delta > 0$, if $\delta \geq \varepsilon$, then by the assumption we automatically have the existence of a $K > 0$ such that if $k \geq K$ we have

$$|\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| < \varepsilon \leq \delta.$$

Hence we assume $\delta < \varepsilon$. Again by the assumption we have the existence of a $L > 0$ such that if $k \geq L$ we have

$$|\{x \in E : |f(x) - f_k(x)| > \delta\}| < \delta.$$

But the set $\{x \in E : |f(x) - f_k(x)| > \delta\} \subset \{x \in E : |f(x) - f_k(x)| > \varepsilon\}$, and so we have (0.1).

The Cauchy criterion is: For any $\varepsilon > 0$ there exists $K > 0$ such that if $m, n \geq K$ we have

$$|\{x \in E : |f_m(x) - f_n(x)| > \varepsilon\}| < \varepsilon. \tag{0.2}$$

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3. (10 points) Do Exercise 18 in p. 63.

Solution: Given $f : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable and let

$$\omega_f(a) = |\{f > a\}|, \quad \text{where } -\infty < a < \infty.$$

As a function of a, $\omega_f(a)$ is decreasing on $(-\infty, \infty)$. If $f_k \nearrow f$ on E, then set

$$E_k = \{f_k > a\}, \quad k = 1, 2, 3, \ldots$$

We have $E_1 \subset E_2 \subset E_3 \subset \cdots$ and if $f(x) > a$, we will have $f_k(x) > a$ for all k large enough (since $f_k \nearrow f$ on E). Thus

$$\{f > a\} = \bigcup_{k=1}^{\infty} E_k$$

and so $\omega_{f_k}(a) \nearrow \omega_f(a)$ for all $a \in (-\infty, \infty)$.

If $f_k \to f$ in measure on E, given $\varepsilon > 0$ let

$$A_k^1 = \{|f - f_k| > \varepsilon\}, \qquad A_k^2 = \{|f - f_k| \le \varepsilon\}.$$

We have $E = A_k^1 \bigcup A_k^2$ (disjoint union). Hence for each fixed $a \in \mathbb{R}$ we have

$$E_k = \left(E_k \bigcap A_k^1\right) \bigcup \left(E_k \bigcap A_k^2\right), \quad E_k = \{f_k > a\}$$

where $\lim_{k\to\infty} \left|E_k \bigcap A_k^1\right| = 0$ (due to convergence in measure) and

$$E_k \bigcap A_k^2 = \{f_k > a\} \bigcap \{|f - f_k| \le \varepsilon\} \subset \{f > a - \varepsilon\}.$$

We have

$$\omega_{f_k}(a) = |E_k| = \left|E_k \bigcap A_k^1\right| + \left|E_k \bigcap A_k^2\right| \le \left|E_k \bigcap A_k^1\right| + \omega_f(a - \varepsilon)$$

and so

$$\limsup_{k\to\infty} \omega_{f_k}(a) \le \omega_f(a - \varepsilon) \quad \text{for any} \quad \varepsilon > 0.$$

Similarly we have

$$\{f > a + \varepsilon\} \bigcap \{|f - f_k| \le \varepsilon\} \subset \{f_k > a\}$$

which gives

$$\liminf_{k\to\infty} \omega_{f_k}(a) \ge \omega_f(a + \varepsilon) \quad \text{for any} \quad \varepsilon > 0.$$

We conclude

$$\omega_f(a + \varepsilon) \le \liminf_{k\to\infty} \omega_{f_k}(a) \le \limsup_{k\to\infty} \omega_{f_k}(a) \le \omega_f(a - \varepsilon) \quad \text{for any} \quad \varepsilon > 0.$$

Thus if $\omega_f(x)$ is continuous at $x = a$, we have $\lim_{k\to\infty} \omega_{f_k}(a) = \omega_f(a)$. ¤

4. (20 points) Do Exercise 19 in p. 63.

Solution: Let $S = [0,1] \times [0,1]$. The idea is to separate the x variable from the y variable. For each $n = 1, 2, 3, \ldots$., define

$$f_n(x, y) = \sum_{k=1}^{n-1} f(r_k, y)\, \varkappa_{\left[\frac{k-1}{n}, \frac{k}{n}\right)}(x) + f(r_n, y)\, \varkappa_{\left[\frac{n-1}{n}, 1\right]}(x), \quad (x, y) \in S$$

where for each k, $r_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$ and $r_k \in \left[\frac{n-1}{n}, 1\right]$ are arbitrary constants.

For each k, the function $h_k(x,y) := f(r_k,y)\varkappa_{\left[\frac{k-1}{n},\frac{k}{n}\right)}(x) : S \in \mathbb{R}$ satisfies

$$h_k(x,y) = 0 \quad \text{on} \quad \left\{[0,1] - \left[\frac{k-1}{n},\frac{k}{n}\right)\right\} \times [0,1]$$

and on $\left[\frac{k-1}{n},\frac{k}{n}\right) \times [0,1]$ we have

$$\begin{aligned} & \left\{(x,y) \in \left[\frac{k-1}{n},\frac{k}{n}\right) \times [0,1] : h_k(x,y) > a\right\} \\ = & \left[\frac{k-1}{n},\frac{k}{n}\right) \times \{y \in [0,1] : f(r_k,y) > a\} \end{aligned}$$

which is a measurable set (due to Exercise 12, p. 48). Hence we can conclude that $h_k(x,y)$ is a measurable function on S for each k. As a consequence the function $f_n(x,y)$ is also measurable on S.

For each $(x_0,y_0) \in S$ we have (assume that $x_0 \in \left[\frac{k-1}{n},\frac{k}{n}\right)$)

$$|f_n(x_0,y_0) - f(x_0,y_0)| = |f(r_k,y_0) - f(x_0,y_0)|, \qquad r_k \in \left[\frac{k-1}{n},\frac{k}{n}\right)$$

and so $f_n(x_0,y_0) \to f(x_0,y_0)$ as $n \to \infty$. Hence $f(x,y)$ is a measurable function on S. ¤