Real Analysis Homework 6, due 2007-10-24 in class

1. (10 points) Do Exercise 7 in p. 62.

Solution: For any $x_0 \in E$ (compact) and $M_{x_0} > f(x_0)$ one can choose $\delta_{x_0} > 0$ such that

$$f(x) < M_{x_0} \quad \text{for all} \quad x \in E \text{ with } |x - x_0| < \delta_{x_0}.$$

The collection $\{x \in E : B(x_0; \delta_{x_0})\}$ forms an open cover of E and so there exist a finite cover $B(x_1; \delta_{x_1}), \cdots, B(x_n; \delta_{x_n})$ of E. Set $M = \max\{M_{x_1}, \cdots, M_{x_n}\}$. We have $f(x) < M$ for all $x \in E$. Hence f is bounded above.

Let $\tilde{M} = \sup_{x \in E} f(x)$. For $k = 1, 2, 3, ...$ choose $x_k \in E$ such that

$$\tilde{M} - \frac{1}{k} < f(x_k) < \tilde{M}. \tag{0.1}$$

As E is compact, by passing to a subsequence if necessary, we may assume that $x_k \in E$ converges to some $x_0 \in E$. Assume $f(x_0) < \tilde{M}$. Choose M with $f(x_0) < M < \tilde{M}$. Since f is usc at x_0, there exists some $\delta > 0$ such that

$$f(x) < M \quad \text{for all} \quad x \in B(x_0; \delta) \cap E. \tag{0.2}$$

For k large enough, we have $x_k \in B(x_0; \delta) \cap E$. (0.1) will contradict to (0.2). Hence we must have $f(x_0) = \tilde{M}$, i.e., the maximum is attained. ¤

2. (10 points) Show that the limit of a **decreasing** sequence of functions (with common domain E) usc at $x_0 \in E$ is also usc at x_0. Give an example of a decreasing sequence of functions continuous at $x_0 \in E$ but its limit is not continuous at x_0 (by the first part of the problem we know that the limit is at least usc at x_0).

Solution: Denote the decreasing sequence of functions by $f_k(x)$ and the limit by $f(x)$. We have

$$f_1(x) \geq f_2(x) \geq \cdots \geq f_k(x) \geq \cdots, \quad x \in E$$

and each $f_k(x)$ is usc at x_0. Since we have $f(x) \leq f_k(x)$ on E for each k, we have

$$\limsup_{x \to x_0; x \in E} f(x) \leq \limsup_{x \to x_0; x \in E} f_k(x) \leq f_k(x_0) \quad \text{for each } k \in \mathbf{N}.$$

As k is arbitrary. Letting $k \to \infty$ gives the conclusion.

Let $E = [0, 1]$ and let $\{x_k : k = 0, 1, 2, 3, ...\}$ be the set of all rationals in E. Set $f_0(x) \equiv 1$ and set for each $k \in \mathbf{N}$ the function

$$f_k(x) = \begin{cases} 0 & \text{at} \quad x = x_1, x_2, \cdots, x_k \\ 1 & \text{otherwise.} \end{cases}$$

Then $f_k(x)$ is a decreasing sequence of functions on E; all are continuous at $\sqrt{2}/2$. But the limit $f(x)$ is not continuous at $\sqrt{2}/2$. However it is usc at $\sqrt{2}/2$. ¤

3. (10 points) Do Exercise 11 in p. 62.

Solution: We only show that $h(x)$ is usc on $\mathbf{R}^n$. The proof of the other case is similar.

For any $x_0 \in \mathbf{R}^n$, we first assume that $h(x_0) < \infty$ (otherwise we are done). It suffices to show that for any $M > h(x_0) = \inf\{f(y) : y \in B(x_0)\}$ there exists $\sigma > 0$ such that $h(x) < M$ for all $x \in B(x_0; \sigma)$.

For any $M > h(x_0)$ choose $y_0 \in B(x_0)$ such that $f(y_0) < h(x_0) + \varepsilon < M$. Set $\eta = |y_0 - x_0| < r$ and $\delta = \frac{1}{2}(r - \eta) > 0$. Then for all x with $|x - x_0| < \delta$ we have

$$|x - y_0| \le |x - x_0| + |y_0 - x_0| \le \delta + \eta < r.$$

Hence $y_0 \in B(x)$ and by definition we get $h(x) = \inf\{f(y) : y \in B(x)\} \le f(y_0) < M$. Thus $h(x)$ is usc at x_0.

For the case of using closed balls, consider in $\mathbf{R}^2$ the function

$$f(p) = \begin{cases} 0 & \text{if } p = (1,0) \\ 1 & \text{if } p \neq (1,0) \end{cases}, \quad p \in \mathbf{R}^2$$

and take $r = 1$, $B(x) = \{p \in \mathbf{R}^2 : |p - x| \le 1\}$. Then $h(0,0) = 0$ and for any $\delta > 0$ we have $h(-\delta, 0) = 1$. Hence h is not usc at the point $(0,0)$. ¤

4. (10 points) Do Exercise 12 in p. 62.

Solution: Assume $f(x) : [a,b] \to \mathbf{R}$ is continuous a.e. on $[a,b]$. Let Γ_k be a sequence of partitions of $[a,b]$ with norms tending to zero. We also assume that each Γ_{k+1} is a refinement of Γ_k. For each k, if $x_1^{(k)} < x_2^{(k)} < \cdots$ are the partitioning points of Γ_k, let $l_k(x)$ and $u_k(x)$ be defined in each semi-open interval $[x_i^{(k)}, x_{i+1}^{(k)})$ as the inf and sup of f on $[x_i^{(k)}, x_{i+1}^{(k)})$. Note that $l_k(x) \le f(x) \le u_k(x)$ for all $x \in [a,b)$ and all k. It is easy to see that for each k, $l_k(x)$ and $u_k(x)$ are measurable functions on $[a,b)$ (even if it is possible that $l_k(x) = -\infty$ or $u_k(x) = +\infty$ on some intervals).

Let $x_0 \in (a,b)$ at which $f(x)$ is continuous. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon$$

and when k is large enough, the interval in Γ_k containing x_0 must lie inside $(x_0 - \delta, x_0 + \delta)$. This implies

$$|l_k(x_0) - f(x_0)| < \varepsilon \quad \text{and} \quad |u_k(x_0) - f(x_0)| < \varepsilon$$

for all k large enough. Hence $\lim_{k\to\infty} l_k(x) = f(x)$ a.e. on $[a,b]$ (we also have $\lim_{k\to\infty} u_k(x) = f(x)$ a.e. on $[a,b]$). By Theorem 4.12 of the book, we know that $f(x)$ is measurable on $[a,b]$. ¤

Remark 1 (be careful) If $g(x)$ is a continuous function on $[a,b]$ and $f(x) = g(x)$ a.e. on $[a,b]$, it does not, in general, imply that $f(x)$ is continuous a.e. on $[a,b]$. For example, take $g(x) = 1$ and let

$$f(x) = \begin{cases} 1, & x \text{ is irrational in } [0,1] \\ 0, & x \text{ is rational in } [0,1]. \end{cases}$$

We see that $f(x) = g(x)$ a.e. on $[0,1]$, but $f(x)$ is discontinuous everywhere on $[0,1]$.