Real Analysis Homework 5, due 2007-10-16 in class

1. (10 points) Do Exercise 2 in p. 61.

Solution: Assume f takes distinct values $a_1, \cdots, a_N$ on disjoint sets $E_1, \cdots, E_N$. We can express the function $f(x)$ as

$$f(x) = a_1 \varkappa_{E_1}(x) + \cdots + a_N \varkappa_{E_N}(x), \qquad x \in E = \bigcup_{k=1}^{N} E_k.$$

If each E_k is measurable, then the characteristic function $\varkappa_{E_k}(x) : E \to \mathbb{R}$ is a measurable function on E. Hence if $E_1, \cdots, E_N$ are all measurable, so is $f(x)$.

Conversely, assume $f(x)$ is measurable. Then by definition we know that (we may assume $a_1 < a_2 < \cdots < a_N$) the set $\{f > a_{N-1}\}$ is measurable. Since $\{f > a_{N-1}\} = E_N$, the set E_N is measurable. Similarly by

$$E_{N-1} = \{f > a_{N-2}\} - E_N$$

we know that E_{N-1} is measurable. Keep going to conclude that $E_1, \cdots, E_N$ are all measurable. ¤

2. (10 points) Do Exercise 3 in p. 61.

Solution: Let $F(x) = (f(x), g(x)),\ x \in \mathbb{R}^n$. $F : \mathbb{R}^n \to \mathbb{R}^2$.

($\Longrightarrow$) Assume F is measurable. For any open set $G_x \subset \mathbb{R}$, let $G = G_x \times \mathbb{R}$. Then $G \subset \mathbb{R}^2$ is open and by the identity

$$F^{-1}(G) = f^{-1}(G_x)$$

we know that $f^{-1}(G_x)$ is measurable for any open set $G_x \subset \mathbb{R}$. Hence $f : \mathbb{R}^n \to \mathbb{R}$ is measurable. The same for $g : \mathbb{R}^n \to \mathbb{R}$.

($\Longleftarrow$) Assume $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are both measurable. Let $G \subset \mathbb{R}^2$ be an arbitrary open set. We can express G as a countable union of nonoverlapping closed intervals $G = \bigcup_{k=1}^{\infty} I_k$, where $I_k = [a_k, b_k] \times [c_k, d_k]$. Note that

$$F^{-1}(I_k) = f^{-1}[a_k, b_k] \bigcap g^{-1}[c_k, d_k] \quad \text{(both sets are measurable)}$$

and so $F^{-1}(I_k)$ is measurable. By

$$F^{-1}(G) = \bigcup_{k=1}^{\infty} F^{-1}(I_k)$$

the set $F^{-1}(G)$ is measurable in $\mathbb{R}^n$. Hence F is a measurable function. ¤

3. (10 points) Do Exercise 4 in p. 61.

Solution: For any $a \in \mathbb{R}$, we have

$$\begin{aligned} &\{x \in \mathbb{R}^n : f(Tx) > a\} = \{x \in \mathbb{R}^n : f(Tx) \in (a, \infty)\} \bigcup \{x \in \mathbb{R}^n : f(Tx) = \infty\} \\ &= \left\{x \in \mathbb{R}^n : T^{-1}\left(f^{-1}(a, \infty)\right)\right\} \bigcup \left\{x \in \mathbb{R}^n : T^{-1}\left(f^{-1}(\{\infty\})\right)\right\}. \end{aligned} \tag{0.1}$$

As $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear and Lipschitz continuous, (0.1) is a measurable set. Hence $f(Tx)$ is a measurable function. ¤

4. (20 points) Do Exercise 5 in p. 61.

<u>First solution:</u> We first prove the following

Lemma 0.1 The Cantor-Lebesgue function $f(x) : [0,1] \to [0,1]$ satisfies $f(x_1) = f(x_2)$, where x_1 and x_2 are in C, if and only if both x_1 and x_2 are endpoints of some interval removed.

Proof. The direction $(\Longleftarrow)$ is trivial.

$(\Longrightarrow)$ Assume at least one of $x_1,\ x_2$ is not endpoint, say x_2. On the interval (x_1, x_2), $x_1 < x_2$, $x_1,\ x_2 \in C$, there exists some $p \notin C$. Let $I = (y_1, y_2)$ be the maximal open interval containing p such that $I \cap C = \varnothing$ (note that the complement of C is open). We now have $y_1,\ y_2 \in C$ and $y_2 < x_2$ (otherwise if $y_2 = x_2$, then x_2 must be an endpoint, impossible). Similarly one can find an open interval $J = (z_1, z_2)$ such that $J \subset (y_2, x_2)$ with $y_2 < z_1$ (otherwise y_2 is an isolated point of C, impossible). Now the open interval J is on the right hand side of the open interval I with a positive distance away. These two distinct intervals must be exactly equal to some removed intervals in the process of constructing the Cantor set. Hence $f(I) < f(J)$, which gives $f(x_1) < f(x_2)$. ¤

Corollary 0.2 If $x \in C$ but x is not endpoint of some interval removed (say $x = \frac{1}{4}$), then there is no $\tilde{x} \in C,\ \tilde{x} \neq x$, such that $f(\tilde{x}) = f(x)$.

Corollary 0.3 Let $\tilde{C} = C - \{$all right endpoints of the removed intervals$\}$, where C is the Cantor set. Then $f : \tilde{C} \to [0,1]$ is **1-1** and **onto**, and is **strictly increasing** on $\tilde{C}$. Here f is the Cantor-Lebesgue function.

By the above corollary we have $g(y) : [0,1] \to \tilde{C} \subset [0,1]$, **strictly increasing** on $[0,1]$, which is the inverse of $f : \tilde{C} \to [0,1]$. For any $a \in [0,1]$ the set

$$E_a := \{y \in [0,1] : g(y) \geq a\} = \{y \in [0,1] : y \geq f(a)\}$$

is measurable. Hence g is a measurable function on $[0,1]$.

Let $A \subset [0,1]$ be a **nonmeasurable** set. Its image under g has measure zero, hence measurable. Let

$$\varphi = \varkappa_{g(A)} : [0,1] \to \mathbb{R}$$

then φ is a measurable function on $[0,1]$. But the composite function $\varphi \circ g : [0,1] \to \mathbb{R}$ is not measurable since the set

$$\left\{ y \in [0,1] : \varphi(g(y)) > \frac{1}{2} \right\} = A$$

is not measurable. ¤

<u>Second solution (much easier):</u>

Let $f(x) : [0,1] \to [0,1]$ be the Cantor-Lebesgue function and let $g(x) = x + f(x)$. It is easy to see that $g(x) : [0,1] \to [0,2]$ is a strictly increasing continuous function. Hence $g(x)$ is a homeomorphism of $[0,1]$ onto $[0,2]$. We denote its continuous inverse by $h : [0,2] \to [0,1]$. On each interval $I_1,\ I_2,\ I_3,\ \ldots$, removed in the construction of the Cantor set, say the interval

$I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, the function $g(x)$ becomes $g(x) = x + \frac{1}{2}$. Hence $g(x)$ sends I_1 onto an open interval with the same length. Using this observation one can see that

$$\left| g\left(\bigcup_{k=1} I_k \right) \right| = \left| \bigcup_{k=1} g(I_k) \right| = \sum_{k=1} |g(I_k)| = \sum_{k=1} |I_k| = 1$$

which implies $|g(C)| = 2 - 1 = 1$, where C is the Cantor set. Since $g(C)$ has positive measure, by Corollary 3.39 in the book, there exists a non-measurable set $B \subset g(C)$. Now consider the set $A = h(B) \subset C$. It has measure zero, hence A is measurable. Let

$$\varphi = \varkappa_A : [0,1] \to \mathbb{R}, \qquad |A| = 0$$

then φ is a measurable function on $[0,1]$. We now have two measurable functions $h : [0,2] \to [0,1]$ (continuous) and $\varphi : [0,1] \to \mathbb{R}$. But the composite function $\varphi \circ h : [0,2] \to \mathbb{R}$ is not measurable since the set

$$\left\{ \theta \in [0,2] : \varphi(h(\theta)) > \frac{1}{2} \right\} = A$$

is not measurable. $\square$