Real Analysis Homework 4, due 2007-10-9 in class

Show Your Work to Each Problem

1. (20 points)

   (a) (10 points) Use definition (do not use Theorem 3.33) to show that the Cantor-Lebesgue function $f(x) : [0,1] \to [0,1]$ is not a Lipschitz continuous function.

   (b) (10 points) Show that the Cantor-Lebesgue function $f(x) : [0,1] \to [0,1]$ satisfies the following
$$|f(x) - f(y)| \le 2|x-y|^{\alpha}, \quad \forall\ x,\ y \in [0,1]$$
where $\alpha \in (0,1)$ is a constant given by $\alpha = \log 2/\log 3$. (Hint: Use the fact that if $x,\ y \in [0,1]$ with $|x-y| \le 3^{-k}$ for some $k \in \mathbb{N}$, then the difference $|f(x) - f(y)|$ is at most 2^{-k}. For arbitrary $x,\ y \in [0,1]$ one can choose an unique $k \in \mathbb{N}$ such that $3^{-k-1} < |x-y| \le 3^{-k}$, which implies $|f(x) - f(y)| \le 2^{-k}$. Rewrite the estimate without involving k.)

Solution: (a). For example, look at the function near the point $x = \frac{2}{9} = \frac{6}{27} = \frac{18}{81}$. We have $f\left(\frac{2}{9}\right) = \frac{1}{4}$. We also have $f\left(\frac{7}{27}\right) - f\left(\frac{2}{9}\right) = \frac{1}{8}$, which gives
$$\frac{f\left(\frac{7}{27}\right) - f\left(\frac{2}{9}\right)}{\frac{7}{27} - \frac{2}{9}} = \left(\frac{3}{2}\right)^3.$$
Keep going to get (note that $\frac{6}{27} = \frac{18}{81},\ \frac{7}{27} = \frac{21}{81}$)
$$\frac{f\left(\frac{19}{81}\right) - f\left(\frac{2}{9}\right)}{\frac{19}{81} - \frac{2}{9}} = \left(\frac{3}{2}\right)^4$$
$\cdots$, *etc.* Thus it is impossible to find a finite constant $M > 0$ such that
$$\left|\frac{f(y) - f(x)}{y - x}\right| \le M \quad \text{for all} \quad x,\ y \in [0,1].$$

(b). **First note that if $x,\ y \in [0,1]$ with $|x-y| \le 3^{-k}$ for some $k \in \mathbb{N}$, then the difference $|f(x) - f(y)|$ is at most 2^{-k}.** For arbitrary $x,\ y \in [0,1]$ one can choose largest $k \in \mathbb{N}$ such that $|x-y| \le 3^{-k}$; therefore
$$3^{-k-1} < |x-y| \le 3^{-k} \tag{0.1}$$
which will gives the best estimate
$$|f(x) - f(y)| \le 2^{-k}. \tag{0.2}$$
(0.1) is equivalent to
$$-(k+1)\log 3 < \log|x-y| \le -k \log 3$$
which gives
$$-k < 1 + \frac{\log|x-y|}{\log 3}.$$

By (0.2) we get

$$\log|f(x)-f(y)| \le -k\log 2 < \log 2 + \log|x-y|^{\alpha}, \quad \alpha = \frac{\log 2}{\log 3} \tag{0.3}$$

and so

$$|f(x)-f(y)| \le 2|x-y|^{\alpha}, \qquad x,\ y \in [0,1].$$

∎

2. (10 points) Do Exercise 20 in p. 48.

Solution: Let E be a non-measurable subset of $[0,1]$, as established in Corollary 3.39 of the book. We know $|E|_e > 0$. Let $r_0 = 0,\ r_1,\ r_2,\ ...,$ be the set of all rationals in $[0,1]$ and let $E_k = E + r_k$ for $k = 0,\ 1,\ 2,\ 3,\ ....$ Each $E_k \subset [0,2]$ for all k with $|E_k|_e = |E|$, and $E_i \cap E_j = \varnothing$ for different i and j. We clearly have

$$\left|\bigcup_k E_k\right|_e \le 2 < \sum_k |E_k|_e = \infty.$$

∎

3. (10 points) Do Exercise 21 in p. 48.

Solution: Following the notation of Ex 20, we let

$$\begin{aligned} A_1 &= \left(\bigcup_k E_k\right) - E_0, \qquad A_2 = \left(\bigcup_k E_k\right) - \left(E_0 \cup E_1\right) \\ A_3 &= \left(\bigcup_k E_k\right) - \left(E_0 \cup E_1 \cup E_2\right) \\ &\cdots \end{aligned}$$

then $A_k \searrow \varnothing$, $|A_k|_e \le 2 < \infty$ (due to $E_k \subset [0,2]$ for all k), and $|A_k|_e \ge |E|_e > 0$ for all k. Hence we have $\lim_{k\to\infty} |A_k|_e > 0$. ∎

4. (10 points) Do Exercise 23 in p. 49.

Solution: For each $n \in \mathbb{N}$ let

$$Z_n = Z \bigcap [-n,n].$$

we have $|Z_n| = 0$ for all n. Also let

$$T_n(x) = \begin{cases} x^2 & \text{if } \ x \in [-n,n] \\ n^2 & \text{otherwise.} \end{cases}$$

Then $T_n : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}$. Hence $|TZ_n| = 0$ for all n. As $TZ \subset \bigcup_n T_n Z_n$, we have $|TZ| = 0$. ∎