Real Analysis Homework 4, due 2007-10-9 in class

1. (20 points)

   (a) (10 points) Use definition (do not use Theorem 3.33) to show that the Cantor-Lebesgue function $f(x) : [0,1] \to [0,1]$ is not a Lipschitz continuous function.

   (b) (10 points) Show that the Cantor-Lebesgue function $f(x) : [0,1] \to [0,1]$ satisfies the following
   $$|f(x) - f(y)| \le 2|x - y|^\alpha, \quad \forall \, x, \, y \in [0,1]$$
   where $\alpha \in (0,1)$ is a constant given by $\alpha = \log 2 / \log 3$. (Hint: Use the fact that if $x, \, y \in [0,1]$ with $|x - y| \le 3^{-k}$ for some $k \in \mathbf{N}$, then the difference $|f(x) - f(y)|$ is at most 2^{-k}. For arbitrary $x, \, y \in [0,1]$ one can choose an unique $k \in \mathbf{N}$ such that $3^{-k-1} < |x - y| \le 3^{-k}$, which implies $|f(x) - f(y)| \le 2^{-k}$. Rewrite the estimate without involving k.)

2. (10 points) Do Exercise 20 in p. 48.

3. (10 points) Do Exercise 21 in p. 48.

4. (10 points) Do Exercise 23 in p. 49.