Real Analysis Homework 3, due 2007-10-3 in class

Show Your Work to Each Problem

1. (20 points) Let $f:\mathbb{R}^n\to\mathbb{R}$ be a continuous function. Define the collection of sets $\sum$ as

$$\sum=\left\{B\subset\mathbb{R}:\ f^{-1}(B)\ \text{is measurable}\right\}.$$

Does $\sum$ form a σ-algebra? If $B\subset\mathbb{R}$ is a Borel set, does it follow that $f^{-1}(B)$ is a Borel set? Give your reasons.

Solution: Clearly $\varnothing,\ \mathbb{R}\in\sum$. If $E\in\sum$, by

$$f^{-1}(E^c)=\left(f^{-1}(E)\right)^c \tag{0.1}$$

we know that $E^c\in\sum$ also. Similarly if $\{E_k\}_{k=1}^{\infty}$ is a collection of subsets of $\sum$, then by

$$f^{-1}\left(\bigcup_k E_k\right)=\bigcup_k f^{-1}(E_k)\quad\text{and}\quad f^{-1}\left(\bigcap_k E_\alpha\right)=\bigcap_k f^{-1}(E_\alpha) \tag{0.2}$$

we know that $\cup_k E_k\in\sum$ and $\cap_k E_\alpha\in\sum$. Hence $\sum$ is a σ-algebra.

Similarly we can show that the set

$$\Lambda=\left\{B\subset\mathbb{R}:\ f^{-1}(B)\ \text{is Borel measurable}\right\}$$

forms a σ-algebra. For any open set $O\subset\mathbb{R}$, the set $f^{-1}(O)$ is also open. Hence $f^{-1}(O)\in\Lambda$. The same for closed set. Hence all open sets and closed sets are contained in Λ. Since Λ is a σ-algebra, if $B\subset\mathbb{R}$ is a Borel set, then $f^{-1}(B)$ must also be a Borel set. $\square$

2. (20 points) Do Exercise 12 in p. 48. Hint: You can use Theorem 3.29.

Solution: We first show that $E_1\times E_2\subset\mathbb{R}^2$ is measurable.

Write $E_1=H_1-Z_1,\ H_1$ is G_δ set and $|Z_1|=0$. The same for $E_2=H_2-Z_2$. Then

$$E_1\times E_2=H_1\times H_2-Z_1\times H_2-H_1\times Z_2$$

where

$$H_1\times H_2=\left(\bigcap_{n=1}^{\infty}O_n\right)\times\left(\bigcap_{n=1}^{\infty}\tilde{O}_n\right)=\bigcap_{n=1}^{\infty}\left(O_n\times\tilde{O}_n\right)$$

is a G_δ set in $\mathbb{R}^2$ (O_n and $\tilde{O}_n$ are open set in $\mathbb{R}^1$). We see that $H_1\times H_2\subset\mathbb{R}^2$ is measurable.

Assume first that $|H_2|<\infty$. One can find open sets $G_1\supset Z_1,\ G_2\supset H_2$ such that $|G_1|<\varepsilon,\ |G_2|<|H_2|+\varepsilon$. Write

$$G_1=\bigcup_{j=1}^{\infty}I_j\quad\text{(nonoverlapping closed intervals in }\mathbb{R}^1)$$

$$G_2=\bigcup_{k=1}^{\infty}\tilde{I}_k\quad\text{(nonoverlapping closed intervals in }\mathbb{R}^1)$$

to see that

$$Z_1 \times H_2 \subset G_1 \times G_2 = \bigcup_{j,\ k=1}^{\infty} \left(I_j \times \tilde{I}_k \right) \qquad \text{(nonoverlapping closed intervals in } \mathbb{R}^2\text{)}$$

with

$$|G_1 \times G_2| = \sum_{j,k} \left| I_j \times \tilde{I}_k \right| = \sum |I_j| \cdot \sum \left| \tilde{I}_k \right| = |G_1| \cdot |G_2| < \varepsilon \left(|H_2| + \varepsilon \right) \tag{0.3}$$

which implies $|Z_1 \times H_2| = 0$.

If $|H_2| = \infty$, decompose $H_2 = \bigcup_{n=-\infty}^{\infty} (H_2 \cap [n, n+1))$, disjoint union, and see that $|Z_1 \times H_2| = 0$. The same for $|H_1 \times Z_2| = 0$. Hence $E_1 \times E_2 \subset \mathbb{R}^2$ is measurable.

Next we show that $|E_1 \times E_2| = |E_1| \times |E_2|$

Case 1: $|E_1| < \infty,\ |E_2| < \infty$.

From (0.3) we see that $|G_1 \times G_2| = |G_1| \times |G_2|$ for any two open sets in $\mathbb{R}^1$.

Choose open sets $G_1 \supset E_1,\ G_2 \supset E_2$ with $|G_1 - E_1| < \varepsilon$, $|G_2 - E_2| < \varepsilon$. Then $E_1 \times E_2 \subset G_1 \times G_2$ and so

$$|E_1 \times E_2| \le |G_1 \times G_2| = |G_1| \times |G_2| \le (|E_1| + \varepsilon) \cdot (|E_2| + \varepsilon)$$

which implies that

$$|E_1 \times E_2| \le |E_1| \times |E_2| . \tag{0.4}$$

Write $G_1 = \bigcup_{j=1}^{\infty} I_j = S \cup N_1$, where

$$S = \bigcup_{j=1}^{M} I_j, \qquad N_1 = \bigcup_{j=M+1}^{\infty} I_j \qquad (M \text{ is some large number})$$

satisfies

$$|E_1| - \varepsilon \le |S| \le |E_1| + \varepsilon, \qquad |N_1| < \varepsilon. \tag{0.5}$$

Then one can express E_1 as

$$E_1 = (S \cup N_1) \ \text{(open set)} \ - N_2, \quad \text{where} \quad N_2 = G_1 - E_1, \quad |N_2| < \varepsilon.$$

Do the same for G_2 and E_2 to get

$$E_2 = \left(\tilde{S} \cup \tilde{N}_1 \right) \ \text{(open set)} \ - \tilde{N}_2, \quad \tilde{N}_2 = G_2 - E_2, \quad \left| \tilde{N}_2 \right| < \varepsilon$$

with

$$|E_2| - \varepsilon \le \left| \tilde{S} \right| \le |E_2| + \varepsilon, \qquad \left| \tilde{N}_1 \right| < \varepsilon. \tag{0.6}$$

Now

$$E_1 \times E_2 = (S \cup N_1) \times \left(\tilde{S} \cup \tilde{N}_1 \right) - N_2 \times \left(\tilde{S} \cup \tilde{N}_1 \right) - (S \cup N_1) \times \tilde{N}_2 := A - B$$

where A and B are measurable sets with $B \subset A,\ |B| < \infty$. Hence

$$|E_1 \times E_2| = |A| - |B|$$

with

$$|A| = |S \cup N_1| \times \left|\tilde{S} \cup \tilde{N}_1\right| = (|S| + |N_1|) \cdot \left(\left|\tilde{S}\right| + \left|\tilde{N}_1\right|\right) \tag{0.7}$$

and by (0.4)

$$|B| = \left|N_2 \times \left(\tilde{S} \cup \tilde{N}_1\right)\right| + \left|(S \cup N_1) \times \tilde{N}_2\right| \leq C\varepsilon \tag{0.8}$$

for some finite constant C. By (0.4), (0.7), (0.8), (0.5)and (0.6), we get the equality

$$|E_1 \times E_2| = |E_1| \times |E_2| .$$

Case 2: $|E_1| = \infty,\ |E_2| = 0$.

Consider $E_1^{(n)} := E_1 \cap [n, n+1),\ n \in \mathbb{Z}$, then we have $|E_1 \times E_2| = 0 = |E_1| \times |E_2|$ (we view $\infty \cdot 0 = 0$).

Case 3: $|E_1| = \infty,\ |E_2| = \lambda > 0$.

Consider $E_1^{(n)} := E_1 \cap [n, n+1)$, then we have $|E_1 \times E_2| = \left(\sum_{n=-\infty}^{\infty} \left|E_1^{(n)}\right|\right) \cdot |E_2| = \infty = |E_1| \times |E_2|$.

Case 4: $|E_1| = \infty,\ |E_2| = \infty$.

Easy to see that $|E_1 \times E_2| = \infty = |E_1| \times |E_2|$. ¤

3. (10 points) It has been proved in class that if $E \subset \mathbb{R}^n$ is an arbitrary measurable set ($|E| = \infty$ is allowed). We have

$$|E| = \inf_{G \supset E,\ G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbb{R}^n} |F| .$$

Show that if $E \subset \mathbb{R}^n$ is an arbitrary set satisfying the following

$$\begin{cases} |E|_e < \infty \\ \inf_{G \supset E,\ G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbb{R}^n} |F| \end{cases}$$

then E must be measurable. Use an example to explain that the condition $|E|_e < \infty$ is necessary. That is, there exists a set E with $|E|_e = \infty,\ \inf_{G \supset E,\ G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbb{R}^n} |F|$, but it is not measurable.

Solution: By the assumption for any fixed $\varepsilon > 0$ one can choose open set $G \supset E$ and closed set $F \subset E$ so that

$$|E|_e - \varepsilon < |F| \leq |G| < |E|_e + \varepsilon. \tag{0.9}$$

Since $|F| \leq |E|_e < \infty$ and then (0.9) gives

$$|G - F| = |G| - |F| < 2\varepsilon.$$

In particular we have $|G - E| \leq |G - F| < 2\varepsilon$ and so E is measurable.

Let $E = (-\infty, 0) \cup A$, where $A \subset (1, 2)$ is some nonmeasurable set. Clearly E satisfies

$$|E|_e = \infty, \qquad \inf_{G \supset E,\ G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbb{R}^n} |F| = \infty$$

but it is nonmeasurable. ¤