Real Analysis Homework 3, due 2007-10-3 in class

Show Your Work to Each Problem

1. (20 points) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. Define the collection of sets $\sum$ as

$$\sum = \left\{ B \subset \mathbf{R} : \ f^{-1}(B) \ \text{is measurable} \right\}.$$

   Does $\sum$ form a σ-algebra? If $B \subset \mathbf{R}$ is a Borel set, does it follow that $f^{-1}(B)$ is a Borel set? Give your reasons.

2. (20 points) Do Exercise 12 in p. 48. Hint: You can use Theorem 3.29.

3. (10 points) It has been proved in class that if $E \subset \mathbf{R}^n$ is an arbitrary measurable set ($|E| = \infty$ is allowed). We have

$$|E| = \inf_{G \supset E,\ G \text{ open in } \mathbf{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbf{R}^n} |F|.$$

   Show that if $E \subset \mathbf{R}^n$ is an arbitrary set satisfying the following

$$\begin{cases} |E|_e < \infty \\ \inf_{G \supset E,\ G \text{ open in } \mathbf{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbf{R}^n} |F| \end{cases}$$

   then E must be measurable. Use an example to explain that the condition $|E|_e < \infty$ is necessary. That is, there exists a set E with $|E|_e = \infty$, $\inf_{G \supset E,\ G \text{ open in } \mathbf{R}^n} |G| = \sup_{F \subset E,\ F \text{ closed in } \mathbf{R}^n} |F|$, but it is not measurable.