Real Analysis Homework 2, due 2007-9-25 in class

Show Your Work to Each Problem

1. (10 points) Use Lemma 3.16 to prove Lemma 3.15. Note that in proving Lemma 3.16 we do not have to use Lemma 3.15.

Solution: Let $\{I_k\}_{k=1}^N$ be a finite family of nonoverlapping intervals. It suffices to show that

$$\left|\bigcup_{k=1}^N I_k\right| \geq \sum_{k=1}^N |I_k|.$$

For each k, consider smaller interval $I_k^* \subset I_k$ with $|I_k^*| \geq |I_k| - \varepsilon/2^k$. Since $\{I_k\}_{k=1}^N$ are **nonoverlapping**, the set I_1^* and $\cup_{k=2}^N I_k^*$ has positive distance. We can apply Lemma 3.15 to get

$$\left|\bigcup_{k=1}^N I_k\right| \geq \left|\bigcup_{k=1}^N I_k^*\right| = \sum_{k=1}^N |I_k^*| \geq \sum_{k=1}^N |I_k| - \varepsilon.$$

The proof is done. ¤

2. (10 points)

   (a) (5 points) Assuming the validity of Theorem 3.30 and the existence of non-measurable sets in $\mathbb{R}^n$ at this moment. Show that there exist two nonempty sets E_1 and E_2 in $\mathbb{R}^n$ such that $E_1 \cap E_2 = \varnothing$, but

   $$|E_1 \cup E_2|_e < |E_1|_e + |E_2|_e.$$

   Hence the condition $d(E_1, E_2) > 0$ in Lemma 3.16 can not be replaced by just $E_1 \cap E_2 = \varnothing$.

   (b) (5 points) Construct a sequence of nonempty sets $E_k \subset [0,1]$, $k = 1, 2, 3...$, so that

   $$\limsup E_k = [0,1], \qquad \liminf E_k = \varnothing. \tag{0.1}$$

Solution: (a). This is easy by Theorem 3.30. Choose $E \subset \mathbb{R}^n$ to be a nonmeasurable set. Then there exists some nonempty set $A \subset \mathbb{R}^n$ such that

$$|A|_e < |A \cap E|_e + |A - E|_e \tag{0.2}$$

where $A \cap E$ and $A - E$ are disjoint. One can easily see that if (0.2) holds, then both $A \cap E$ and $A - E$ are nonempty. (b). Choose for example the sequence

$$\begin{aligned}
E_1 &= [0,1] \\
E_2 &= \left[0, \frac{1}{2}\right], \quad E_3 = \left[\frac{1}{2}, 1\right] \\
E_4 &= \left[0, \frac{1}{3}\right], \quad E_5 = \left[\frac{1}{3}, \frac{2}{3}\right], \quad E_6 = \left[\frac{2}{3}, 1\right] \\
E_7 &= \left[0, \frac{1}{4}\right], \quad E_8 = \left[\frac{1}{4}, \frac{2}{4}\right], \quad E_9 = \left[\frac{2}{4}, \frac{3}{4}\right], \quad E_{10} = \left[\frac{3}{4}, 1\right] \\
&\quad \cdots\cdots.
\end{aligned}$$

Then we have (0.1). ¤

3. (10 points) Assuming there exists a non-measurable set contained in $[0,1]$, do Exercise 17 in p. 48.

<u>Solution</u>: Let $E = f^{-1}(A)$, where A is a nonmeasurable set contained in $[0,1]$. Here $f : C$ (Cantor set) $\to [0,1]$ is the Cantor-Lebesgue function in p. 35. It is continuous. Then since $|E| = 0$, E is measurable. But $f(E) = A$ is nonmeasurable. ¤

4. (10 points) Do Exercise 18 in p. 48.

<u>Solution</u>: We first asume that $0 \leq |E|_e < \infty$.

It is clear that for any interval I, $|I_h| = |I|$. For any $\varepsilon > 0$, choose a sequence of intervals I_k, covering E, such that

$$\sum_k |I_k| < |E|_e + \varepsilon.$$

Now the translation $E_h \subset \bigcup_k (I_k)_h$ and so

$$|E_h|_e \leq \sum_k |(I_k)_h| = \sum_k |I_k| < |E|_e + \varepsilon$$

which gives $|E_h|_e \leq |E|_e$. Conversely $|E|_e = \left|(E_h)_{-h}\right|_e \leq |E_h|_e$. We are done.

If $|E|_e = \infty$, then we must have $|E_h|_e = \infty$ also (note that from the above observation, if $|E_h|_e < \infty$, it will force $|E|_e < \infty$).

Assume E is measurable. For any $\varepsilon > 0$, there exists open set $G \supset E$ such that $|G - E|_e < \varepsilon$. This implies

$$|G_h - E_h|_e = |G - E|_e < \varepsilon$$

where G_h is also an open set. Hence E_h is also measurable. ¤