Real Analysis Homework 14, due 2008-1-2 in class

1. (10 points) Let $E \subset \mathbf{R}^n$ be a measurable set ($|E| < \infty$ or not). If for any $0 < p < \infty$ we have $f \in L^p(E)$ and $\|f\|_p \le K$, where K is a constant independent of p. Show that $f \in L^\infty(E)$ and $\|f\|_\infty \le K$ also.

Solution:

We first assume $|E| < \infty$. In this case, we have (see Theorem 8.1) $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. Hence $f \in L^\infty(E)$ and $\|f\|_\infty \le K$.

If $|E| = \infty$, decompose $E = \bigcup_{k=1}^\infty E_k$, where each $|E_k| < \infty$. On each E_k, we have $f \in L^\infty(E_k)$ and $\|f\|_{\infty,E_k} \le K$. Since the constant K is independent of the set E_k, we have $f \in L^\infty(E)$ and $\|f\|_{\infty,E} \le K$. $\square$

2. (10 points) Do Exercise 5 in P. 143.

Solution:

We have $1 \le p < \infty,\ 0 < |E| < \infty$, and

$$N_p[f] = \left(\frac{1}{|E|}\int_E |f|^p\right)^{1/p} = \frac{1}{|E|^{1/p}}\|f\|_p .$$

If $p_1 < p_2,\ 1 \le p_1,\ p_2 < \infty$, then set $s = \frac{p_2}{p_1} \in (1,\infty)$. Its conjugate exponent is $t = \frac{s}{s-1} = \frac{p_2}{p_2-p_1} \in (1,\infty)$. By Hölder inequality we get

$$\begin{aligned}\|f\|_{p_1}^{p_1} &= \int_E |f(x)|^{p_1}\cdot 1dx\\ &\le \left(\int_E |f(x)|^{p_1 s}dx\right)^{1/s}\left(\int_E 1^t dx\right)^{1/t} = \left(\int_E |f(x)|^{p_2}dx\right)^{p_1/p_2}|E|^{(p_2-p_1)/p_2}.\end{aligned}$$

Taking $\frac{1}{p_1}$ power on both sides gives the conclusion.

Also by Minkowski inequality we have

$$N_p[f+g] = \frac{1}{|E|^{1/p}}\|f+g\|_p \le \frac{1}{|E|^{1/p}}\left[\|f\|_p + \|g\|_p\right] = N_p[f] + N_p[g].$$

Furthermore, Hölder inequality implies

$$\frac{1}{|E|}\int_E |fg| \le \frac{1}{|E|^{1/p}}\|f\|_p \cdot \frac{1}{|E|^{1/p'}}\|g\|_{p'} = N_p[f]\cdot N_{p'}[g], \qquad \frac{1}{p}+\frac{1}{p'} = 1.$$

Finally since $0 < |E| < \infty$, we have

$$\lim_{p\to\infty} N_p[f] = \lim_{p\to\infty}\left(\frac{1}{|E|^{1/p}}\|f\|_p\right) = \lim_{p\to\infty}\|f\|_p = \|f\|_\infty .$$

$\square$

3. (10 points) Prove the converse of Hölder inequality (Theorem 8.8) for the case $p = 1$ and $p = \infty$.

Solution:

Assume $p = 1$. Clearly we have
$$\|f\|_1 \geq \sup \int_E fg \tag{1}$$
for all $g \in L^\infty(E)$ with $\|g\|_\infty \leq 1$.

Conversely, take $g = sign\ f \in L^\infty(E)$. Then $\|g\|_\infty \leq 1$ and
$$\|f\|_1 = \int_E |f| = \int_E fg$$
which implies RHS of (1) $\geq$ LHS of (1).

For $p = \infty$, again we have
$$\|f\|_\infty \geq \sup \int_E fg$$
for all $g \in L^1(E)$ with $\|g\|_1 \leq 1$.

Conversely, if $\|f\|_\infty = 0$, then it is clear. For $0 < \|f\|_\infty < \infty$, we may assume $\|f\|_\infty = 1$. Let
$$E_n = \left\{ x \in E : |f(x)| > 1 - \frac{1}{n}, \quad n \in \mathbf{N} \right\}.$$
Then $|E_n| > 0$ for all n. On each E_n one can choose $g_n(x)$ satisfying $g_n \geq 0$, $\int_{E_n} g_n = 1$, and let $g_n = 0$ outside E_n. Now
$$\int_E |f| g_n = \int_{E_n} |f| g_n \geq \left(1 - \frac{1}{n}\right) \int_{E_n} g_n = 1 - \frac{1}{n}, \qquad \int_E g_n = 1$$
and so
$$\|f\|_\infty = \sup \int_E |f| g$$
for all $g \in L^1(E)$ with $\|g\|_1 \leq 1$. Finally it is easy to see that $\sup \int_E |f| g = \sup \int_E fg$.

For the case $\|f\|_\infty = \infty$, just repeat the above process with
$$E_n = \{x \in E : |f(x)| > n, \quad n \in \mathbf{N}\}.$$

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4. (10 points) Assume $1 \leq p < \infty$. Do Exercise 12 in P. 144.

Solution:

In this problem, we assume $1 \leq p < \infty$ (in fact, as long as $0 < p < \infty$, we have the same conclusion).

($\Longrightarrow$) By Minkowski inequality we have
$$\left| \|f\|_p - \|f_k\|_p \right| \leq \|f - f_k\|_p \to 0 \quad \text{as} \quad k \to \infty.$$
Hence we have $\|f_k\|_p \to \|f\|_p$ as $k \to \infty$.

($\Longleftarrow$) We assume that $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$ as $k \to \infty$. By the inequality
$$2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p \geq 0$$
we have (by Fatou's Lemma)
$$\int_E \liminf_k \left[2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right] \leq \liminf_k \int_E \left[2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right]$$

where

$$\int_E \liminf_k \left[2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right] = \int_E \left[2^p |f|^p + 2^p |f|^p\right]$$

and

$$\liminf_k \int_E \left[2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right] = \int_E \left[2^p |f|^p + 2^p |f|^p\right] - \limsup_k \int_E |f - f_k|^p .$$

Since we assume $f \in L^p$, the integral $\int_E [2^p |f|^p + 2^p |f|^p]$ is finite (this is essential). Hence we conclude

$$\limsup_k \int_E |f - f_k|^p \leq 0.$$

The proof is done. ¤

Remark 1 When $p = \infty$, the conclusion in $(\Longleftarrow)$ fails. Just take $f = 1$ on **R** and $f_k = \chi_{(-k,k)}$.