Real Analysis Homework 13, due 2007-12-26 in class

1. (10 points) Do Exercise 21 in P. 144.

Solution:

Let $f \in L^p(\mathbf{R}^n)$, where $0 < p < \infty$ is a constant. We first recall the following elementary calculus inequality:

Lemma 1 For any $0 < p < \infty$, there exist positive constants $c(p)$ and $C(p)$ depending only on p such that

$$c(p)(a^p + b^p) \leq (a+b)^p \leq C(p)(a^p + b^p) \tag{1}$$

for all $a > 0,\ b > 0$.

Let $\{r_k\}_{k=1}^{\infty}$ be the set of all rational numbers. For any k and any $0 < p < \infty$, by (1) the function $|f(y) - r_k|^p$ is clearly locally integrable on $\mathbf{R}^n$. By Theorem 7.11, for each k we have

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|^p\, dy = |f(x) - r_k|^p \tag{2}$$

for a.e. $x \in \mathbf{R}^n$. Let Z_k be the set such that (2) is not valid, $|Z_k| = 0$, and set $Z = \bigcup_{k=1}^{\infty} Z_k$, $|Z| = 0$. If $x \notin Z$, then by the inequality (in below, Q is centered at x and k is arbitrary)

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(y) - f(x)|^p\, dy &\leq \frac{1}{|Q|}\int_Q \{|f(y) - r_k| + |r_k - f(x)|\}^p\, dy \\
&\leq \frac{C(p)}{|Q|}\int_Q |f(y) - r_k|^p\, dy + \frac{C(p)}{|Q|}\int_Q |r_k - f(x)|^p\, dy \\
&= \frac{C(p)}{|Q|}\int_Q |f(y) - r_k|^p\, dy + C(p)\,|r_k - f(x)|^p.
\end{aligned}$$

Hence

$$\limsup_{Q \searrow x} \frac{1}{|Q|}\int_Q |f(y) - r_k|^p\, dy \leq 2C(p)\,|r_k - f(x)|^p \quad \text{for all} \quad x \notin Z.$$

By choosing r_k approximating $f(x)$ (note that $f(x)$ is finite almost everywhere in $\mathbf{R}^n$; without loss of generality, we can assume it is finite everywhere), we obtain

$$\limsup_{Q \searrow x} \frac{1}{|Q|}\int_Q |f(y) - f(x)|^p\, dy = 0 \quad \text{for all} \quad x \notin Z$$

for any $0 < p < \infty$. ¤

2. (10 points) Prove the following more general version of the Tchebyshev inequality: Assume $f \geq 0$ is measurable on E satisfying $\int_E f^p dx < \infty$, where $0 < p < \infty$ is a constant. Then for any $\alpha > 0$ we have

$$|\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha^p} \cdot \int_E f^p dx.$$

Solution:

Over the set $S := |\{x \in E : f(x) > \alpha\}|$ we have $f^p \geq \alpha^p$ and so

$$\int_E f^p dx \geq \int_S f^p dx \geq \alpha^p\,|S|.$$

The conclusion follows. ¤

3. (10 points) Let $H(x)$ be the Heaviside function given by

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Find the set of those $x \in \mathbf{R}$ such that

$$\lim_{h \to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(\theta)\, d\theta = f(x). \tag{3}$$

Determine if the point $x = 0$ is in the Lebesgue set of f or not.

Solution:

Clearly for any $x \in \mathbf{R}$ with $x \neq 0$ we have (3). For $x = 0$, we have

$$\lim_{h \to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(\theta)\, d\theta = \lim_{h \to 0^+} \frac{1}{2h} \int_{-h}^{h} f(\theta)\, d\theta = \lim_{h \to 0^+} \frac{1}{2h} \int_{0}^{h} f(\theta)\, d\theta = \frac{1}{2} = f(0).$$

Hence the set of those $x \in \mathbf{R}$ such that (3) holds is the collection of all real numbers.

To determine if $x = 0$ is in the Lebesgue set of f or not, we compute

$$\lim_{h \to 0^+} \frac{1}{2h} \int_{-h}^{h} |f(\theta) - f(0)|\, d\theta = \lim_{h \to 0^+} \frac{1}{2h} \int_{-h}^{h} \left| f(\theta) - \frac{1}{2} \right| d\theta = \frac{1}{2} \neq 0.$$

Hence $x = 0$ is **not** in the Lebesgue set of f. ¤

4. (10 points) Let $g(x)$ be the function given by

$$g(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine if the point $x = 0$ is in the Lebesgue set of g or not. Also let $G(x) = \int_0^x g(s)\, ds$, $x \in \mathbf{R}$. Do we have $G'(0) = g(0)$ or not.

Solution:

(This solution is provided by TA Yu-Chu Lin) We claim that $x = 0$ is **not** in the Lebesgue set of g. Compute

$$\lim_{h \to 0^+} \frac{1}{2h} \int_{-h}^{h} |g(\theta) - g(0)|\, d\theta = \lim_{h \to 0^+} \frac{1}{2h} \int_{-h}^{h} \left| \sin \frac{1}{\theta} \right| d\theta = \lim_{h \to 0^+} \frac{1}{h} \int_{0}^{h} \left| \sin \frac{1}{\theta} \right| d\theta$$
$$= \lim_{q \to \infty} \left(q \int_q^\infty \frac{|\sin y|}{y^2} dy \right).$$

We look at $q = 2p\pi,\ p \in \mathbf{N}$. Over the intervals

$$I_1 = \left(2p\pi + \frac{\pi}{4},\ 2p\pi + \frac{3\pi}{4} \right), \quad I_2 = \left(2p\pi + 2\pi + \frac{\pi}{4},\ 2p\pi + 2\pi + \frac{3\pi}{4} \right)$$
$$I_3 = \left(2p\pi + 4\pi + \frac{\pi}{4},\ 2p\pi + 4\pi + \frac{3\pi}{4} \right), \cdots$$

we have $|\sin y| \geq \frac{\sqrt{2}}{2}$, which implies

$$\begin{aligned}
&\int_{2p\pi}^{\infty} \frac{|\sin y|}{y^2} dy \geq \frac{\sqrt{2}}{2} \int_{I_1 \cup I_2 \cup I_3 \cup \cdots} \frac{1}{y^2} dy \\
&= \frac{\sqrt{2}}{2} \left\{ \left(\frac{1}{2p\pi + \frac{\pi}{4}} - \frac{1}{2p\pi + \frac{3\pi}{4}} \right) + \left(\frac{1}{2p\pi + 2\pi + \frac{\pi}{4}} - \frac{1}{2p\pi + 2\pi + \frac{3\pi}{4}} \right) + \cdots \right\} \\
&\geq \frac{\sqrt{2}}{2} \frac{\pi}{2} \left\{ \frac{1}{\left[2p\pi + \frac{3\pi}{4}\right]^2} + \frac{1}{\left[2(p+1)\pi + \frac{3\pi}{4}\right]^2} + \frac{1}{\left[2(p+2)\pi + \frac{3\pi}{4}\right]^2} + \cdots \right\}.
\end{aligned}$$

To estimate the sum of the series (denote the series by S) we note that (by comparing areas)

$$S \geq \int_{p}^{\infty} \frac{1}{\left(2x\pi + \frac{3\pi}{4}\right)^2} dx = \frac{1}{2\pi \left(2p\pi + \frac{3\pi}{4}\right)}$$

and so

$$2p\pi \int_{2p\pi}^{\infty} \frac{|\sin y|}{y^2} dy \geq \frac{\sqrt{2}}{2} \frac{\pi}{2} \frac{2p\pi}{2\pi \left(2p\pi + \frac{3\pi}{4}\right)} \to \frac{\sqrt{2}}{8} \quad \text{as} \quad p \to \infty$$

which implies that

$$\lim_{q \to \infty} \left(q \int_{q}^{\infty} \frac{|\sin y|}{y^2} dy \right) \neq 0.$$

The claim is proved.

Next we estimate

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h \sin \frac{1}{\theta} d\theta = \lim_{q \to \infty} \left(q \int_q^{\infty} \frac{\sin y}{y^2} dy \right)$$

where

$$\int_q^{\infty} \frac{\sin y}{y^2} dy = \int_q^{\infty} \frac{1}{y^2} d(-\cos y) = \frac{\cos q}{q^2} - 2 \int_q^{\infty} \frac{\cos y}{y^3} dy$$

and so

$$\lim_{q \to \infty} \left(q \int_q^{\infty} \frac{\sin y}{y^2} dy \right) = \lim_{q \to \infty} \left(\frac{\cos q}{q} - 2q \int_q^{\infty} \frac{\cos y}{y^3} dy \right) = 0.$$

Similarly we have

$$\lim_{h \to 0^-} \frac{1}{h} \int_0^h \sin \frac{1}{\theta} d\theta = 0$$

and so $G'(0) = g(0)$. ¤