Real Analysis Homework 12, due 2007-12-12 in class

1. (10 points) Do Exercise 1 in p. 123.

<u>Solution:</u> f is defined and measurable on $\mathbf{R}^n$. Assume $|f| > 0$ on a set E with $|E| > 0$. We know that there exists $N \in \mathbf{N}$ such that $\left|\left\{|f| > \frac{1}{N}\right\}\right| > 0$. Denote the set $\left\{|f| > \frac{1}{N}\right\}$ by E_N. We have $\frac{1}{N}\chi_{E_N}(x) \leq |f(x)|$ on E_N (and on $\mathbf{R}^n$ also) so

$$\frac{1}{N}\chi^*_{E_N}(x) \leq f^*(x) \quad \text{on} \quad \mathbf{R}^n.$$

Hence by estimate (7.7) in p. 104, there exists a large number d such that

$$\left(\frac{c_1 |E_N|}{N}\right)\frac{1}{|x|^n} \leq \frac{1}{N}\chi^*_{E_N}(x) \leq f^*(x) \quad \text{for all} \quad |x| \geq d.$$

On the compact set $S = \{x \in \mathbf{R}^n : 1 \leq |x| \leq d\}$, since $f^*(x)$ is lower semicontinuous on $\mathbf{R}^n$ with $f^*(x) > 0$ everywhere, it attains its positive minimum on S (see exercise 7, p. 61). Hence there exists a small positive constant c_2 such that $f^*(x) \geq \frac{c_2}{|x|^n}$ on S. Let $c = \min\left\{\frac{c_1|E_N|}{N}, c_2\right\}$. We have

$$f^*(x) \geq \frac{c}{|x|^n} \quad \text{for all} \quad |x| \geq 1.$$

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2. (10 points) Do Exercise 2 in p. 123.

<u>Solution:</u> Assume $|\phi| \leq M$ on $\mathbf{R}^n$ and x is in the Lebesgue set of f. We have

$$\begin{aligned}|(f * \phi_\varepsilon)(x) - f(x)| &\leq \int_{\mathbf{R}^n} |f(x-y) - f(x)|\,|\phi_\varepsilon(y)|\,dy \\ &= \int_{B_\varepsilon(O)} |f(x-y) - f(x)|\,|\phi_\varepsilon(y)|\,dy \leq \frac{M}{\varepsilon^n}\int_{B_\varepsilon(O)} |f(x-y) - f(x)|\,dy\end{aligned}$$

where $B_\varepsilon(O) = \{|y| \leq \varepsilon\}$ has measure $C(n)\varepsilon^n$. Hence

$$|(f * \phi_\varepsilon)(x) - f(x)| \leq C(n)\,M \cdot \frac{1}{|B_\varepsilon(O)|}\int_{B_\varepsilon(O)} |f(x-y) - f(x)|\,dy$$

and we know that

$$\lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(O)|}\int_{B_\varepsilon(O)} |f(x-y) - f(x)|\,dy = 0$$

due to Theorem 7.16. ¤

3. (10 points) Let $C_0(\mathbf{R}^n)$ be the space of all continuous functions on $\mathbf{R}^n$ with compact support. We know that it is dense in the space $L^1(\mathbf{R}^n)$ (Lemma 7.3 of the book). It is also clear that each $g(x) \in C_0(\mathbf{R}^n)$ is uniformly continuous on $\mathbf{R}^n$. Use this dense property to show that if $f \in L^1(\mathbf{R}^n)$, then we have the following property called "Continuity of Translation in L^1":

$$\lim_{y \to 0} \int_{\mathbf{R}^n} |f(x+y) - f(x)|\,dx = 0.$$

Solution: For any $\varepsilon > 0$, choose a function $g(x) \in C_0(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} |f(x) - g(x)|\, dx < \frac{\varepsilon}{3}$. Then for any $y \in \mathbf{R}^n$ we also have

$$\int_{\mathbf{R}^n} |f(x+y) - g(x+y)|\, dx < \frac{\varepsilon}{3}.$$

Let $B_r(O)$ be the ball centered at the origin with large radius $r > 0$ such that it contains the support of g. Then g is uniformly continuous on $B_r(O)$ and so there exists $0 < \delta < 1$ such that if $|y| \leq \delta$, then

$$|g(x+y) - g(x)| \leq \frac{\frac{\varepsilon}{3}}{|B_{r+1}(O)|} \quad \text{for all} \quad x \in \mathbf{R}^n.$$

Now if $|y| < \delta < 1$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} |f(x+y) - f(x)|\, dx \\
&\leq \int_{\mathbf{R}^n} |f(x+y) - g(x+y)|\, dx + \int_{\mathbf{R}^n} |g(x+y) - g(x)|\, dx + \int_{\mathbf{R}^n} |g(x) - f(x)|\, dx \\
&\leq \frac{\varepsilon}{3} + \int_{B_{r+1}(O)} |g(x+y) - g(x)|\, dx + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

□

4. (10 points) There are many applications of the use of convolution in analysis. One easy example is the following. Let

$$h(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ e^{-\frac{1}{t}} & \text{if} \quad t > 0. \end{cases}$$

It is known that $h(t)$ is a C^∞ function on $\mathbf{R}$. Next let $g(x) = h\left(1 - |x|^2\right)$, $x \in \mathbf{R}^n$, then $g(x) \in C_0^\infty(\mathbf{R}^n)$. One can divide it by its integral over $\mathbf{R}^n$ so that the new function $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$ satisfies $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$. For any number $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)$. Then it satisfies $\varphi_\varepsilon(x) \in C_0^\infty(\mathbf{R}^n)$, $\varphi_\varepsilon(x) \geq 0$, $\varphi_\varepsilon(x) > 0 \iff |x| < \varepsilon$, $\int_{\mathbf{R}^n} \varphi_\varepsilon(x)\, dx = 1$. Show that:

(a) If $f \in C(\mathbf{R}^n)$, then $(f * \varphi_\varepsilon)(x)$ converges uniformly to $f(x)$ on compact subsets of $\mathbf{R}^n$ as $\varepsilon \to 0^+$. (It is easy to see that $(f * \varphi_\varepsilon)(x) \in C^\infty(\mathbf{R}^n)$. You do not have to show this.)

(b) If $f \in C_0(\mathbf{R}^n)$, then for any $\varepsilon > 0$, $(f * \varphi_\varepsilon)(x)$ also has compact support.

Solution: (a). We know that f is uniformly continuous on compact subsets of $\mathbf{R}^n$. Let S be a compact subset of $\mathbf{R}^n$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x-y) - f(x)| \leq \varepsilon$$

for all $x \in S$, $|y| \leq \delta$.

For $x \in S$ we have

$$\begin{aligned}
&|(f * \phi_\delta)(x) - f(x)| \\
&= \left| \int_{\mathbf{R}^n} [f(x-y) - f(x)] \cdot \phi_\delta(y)\, dy \right| = \left| \int_{B_\delta(O)} [f(x-y) - f(x)] \cdot \phi_\delta(y)\, dy \right| \\
&\leq \int_{B_\delta(O)} |f(x-y) - f(x)| \cdot \phi_\delta(y)\, dy \leq \varepsilon \int_{B_\delta(O)} \phi_\delta(y)\, dy = \varepsilon
\end{aligned}$$

where $B_\delta(O) = \{|y| \leq \delta\}$. Hence $(f * \varphi_\delta)(x)$ converges uniformly to $f(x)$ on S as $\delta \to 0^+$.

(b). Assume S is the compact support of $f.$ Then

$$(f * \varphi_\varepsilon)(x) = \int_{\mathbf{R}^n} f(x-y) \cdot \phi_\varepsilon(y)\, dy = \int_{B_\varepsilon(O)} f(x-y) \cdot \phi_\varepsilon(y)\, dy.$$

From above we see that if $x \notin S$ with $dist(x, S) > \varepsilon,$ then we also have $x-y \notin S$ for any $y \in B_\varepsilon(O)$. For such $x,$ we have

$$(f * \varphi_\varepsilon)(x) = \int_{B_\varepsilon(O)} f(x-y) \cdot \phi_\varepsilon(y)\, dy = 0.$$

Hence $(f * \varphi_\varepsilon)(x)$ also has compact support. ¤