Real Analysis Homework 12, due 2007-12-12 in class

1. (10 points) Do Exercise 1 in p. 123.

2. (10 points) Do Exercise 2 in p. 123.

3. (10 points) Let $C_0(\mathbf{R}^n)$ be the space of all continuous functions on $\mathbf{R}^n$ with compact support. We know that it is dense in the space $L^1(\mathbf{R}^n)$ (Lemma 7.3 of the book). It is also clear that each $g(x) \in C_0(\mathbf{R}^n)$ is uniformly continuous on $\mathbf{R}^n$. Use this dense property to show that if $f \in L^1(\mathbf{R}^n)$, then we have the following property called "Continuity of Translation in L^1":
$$\lim_{y\to 0}\int_{\mathbf{R}^n} |f(x+y) - f(x)|\,dx = 0.$$

4. (10 points) There are many applications of the use of convolution in analysis. One easy example is the following. Let
$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-\frac{1}{t}} & \text{if } t > 0. \end{cases}$$
It is known that $h(t)$ is a C^∞ function on $\mathbf{R}$. Next let $g(x) = h\left(1 - |x|^2\right)$, $x \in \mathbf{R}^n$, then $g(x) \in C_0^\infty(\mathbf{R}^n)$. One can divide it by its integral over $\mathbf{R}^n$ so that the new function $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$ satisfies $\int_{\mathbf{R}^n}\varphi(x)\,dx = 1$. For any number $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$. Then it satisfies $\varphi_\varepsilon(x) \in C_0^\infty(\mathbf{R}^n)$, $\varphi_\varepsilon(x) \ge 0$, $\varphi_\varepsilon(x) > 0 \iff |x| < \varepsilon$, $\int_{\mathbf{R}^n}\varphi_\varepsilon(x)\,dx = 1$. Show that:

   (a) If $f \in C(\mathbf{R}^n)$, then $(f * \varphi_\varepsilon)(x)$ converges uniformly to $f(x)$ on compact subsets of $\mathbf{R}^n$ as $\varepsilon \to 0^+$. (It is easy to see that $(f * \varphi_\varepsilon)(x) \in C^\infty(\mathbf{R}^n)$. You do not have to show this.)

   (b) If $f \in C_0(\mathbf{R}^n)$, then $(f * \varphi_\varepsilon)(x)$ also has compact support.