Real Analysis Homework 11, due 2007-12-5 in class

1. (10 points) Do Exercise 4 in p. 96.

Solution: Choose $a = x,\ b = -x$ and integrate over $[0,1]$ to get

$$\int_0^1 \int_0^1 |f(t+x) - f(t-x)|\, dt dx \leq c.$$

Hence the function $F(t,x) = |f(t+x) - f(t-x)|$ is integrable on $E = [0,1] \times [0,1]$. Consider the linear transformation $\xi = t + x,\ \eta = t - x$. We have

$$\iint_E F(t,x)\, dt dx = \frac{1}{2} \iint_{E^*} F\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) d\xi d\eta = \frac{1}{2}\iint_{E^*} |f(\xi) - f(\eta)|\, d\xi d\eta$$

where E^* is the diamond-shaped region in (ξ,η) space with vertices $(0,0)$, $(1,1)$, $(1,-1)$, $(2,0)$. In particular we know that $|f(\xi) - f(\eta)| \in L(E^*)$. By Fubini Theorem, there exists $\xi_0 \in \left(\frac{1}{2}, \frac{3}{2}\right)$ such that as a function of η we have

$$|f(\xi_0) - f(\eta)| \in L\left(E^*_{\xi_0}\right), \quad E^*_{\xi_0} = \{\eta : (\xi_0, \eta) \in E^*\}, \quad \left|E^*_{\xi_0}\right| > 1.$$

By $|f(\eta)| \leq |f(\xi_0) - f(\eta)| + |f(\xi_0)|$, we see that $|f(\eta)| \in L\left(E^*_{\xi_0}\right)$. Since $f(\eta)$ is periodic with period 1 and $\left|E^*_{\xi_0}\right| > 1$, we know that $f \in L(0,1)$. ¤

2. (10 points) Do Exercise 6 in p. 97.

Solution: By definition, we have (assume $f \in L^1(\mathbf{R})$)

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) \cos xt dt - i \int_{-\infty}^{\infty} f(t) \sin xt dt, \quad x \in \mathbf{R}.$$

We claim that if $f,\ g \in L^1(\mathbf{R})$, then the function $f(t-y)\, g(y) \in L^1\left(\mathbf{R}^2\right)$ (a a function of (t,y)). To see this, note that

$$[f(t-y)\, g(y)]^+ \leq |f(t-y)|\, |g(y)|, \quad \forall\ (t,y) \in \mathbf{R}^2$$

and by Tonelli's Theorem we have

$$\iint_{\mathbf{R}^2} [f(t-y)\, g(y)]^+\, dt dy \leq \iint_{\mathbf{R}^2} |f(t-y)|\, |g(y)|\, dt dy = \int_{\mathbf{R}} |g(y)|\, dy \int_{\mathbf{R}} |f(\theta)|\, d\theta < \infty.$$

Similarly $\iint_{\mathbf{R}^2} [f(t-y)\, g(y)]^- \, dt dy < \infty$. Hence the claim is true. Now

$$\begin{aligned}
(f \hat{*} g)(x) &= \int_{-\infty}^{\infty} (f*g)(t) \cos xt dt - i \int_{-\infty}^{\infty} (f*g)(t) \sin xt dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-y)\, g(y)\, dy\right) \cos xt dt - i \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-y)\, g(y)\, dy\right) \sin xt dt \\
&= I_1 + i I_2
\end{aligned}$$

where by Fubini Theorem

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} f(t-y) \cos xt dt\right) dy = \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} f(\theta) \cos x(y+\theta)\, d\theta\right) dy \\
&= \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} f(\theta) [\cos xy \cos x\theta - \sin xy \sin x\theta]\, d\theta\right) dy \\
&= \left(\int_{-\infty}^{\infty} f(\theta) \cos x\theta d\theta\right)\left(\int_{-\infty}^{\infty} g(y) \cos xy dy\right) - \left(\int_{-\infty}^{\infty} f(\theta) \sin x\theta d\theta\right)\left(\int_{-\infty}^{\infty} g(y) \sin xy dy\right) \\
&= \operatorname{Re}\left[\hat{f}(x) \cdot \hat{g}(x)\right].
\end{aligned}$$

Similarly we have

$$\begin{aligned}I_2 &= -\int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} f(t-y)\sin xt\,dt\right)dy\\ &= -\int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} f(\theta)\left[\sin xy\cos x\theta+\cos xy\sin x\theta\right]d\theta\right)dy\\ &= -\left(\int_{-\infty}^{\infty} f(\theta)\cos x\theta d\theta\right)\left(\int_{-\infty}^{\infty} g(y)\sin xy dy\right)-\left(\int_{-\infty}^{\infty} f(\theta)\sin x\theta d\theta\right)\left(\int_{-\infty}^{\infty} g(y)\cos xy dy\right)\\ &= \operatorname{Im}\left[\hat{f}(x)\cdot\hat{g}(x)\right].\end{aligned}$$

We conclude the identity

$$\widehat{(f*g)}(x)=\hat{f}(x)\cdot\hat{g}(x),\quad\forall\, x\in\mathbf{R}.$$

¤

3. (10 points) Do Exercise 10 in p. 97.

Solution: Let $V_n(r)$ be the volume of the ball in $\mathbf{R}^n$ with radius $r>0$. For convenience, denote $V_n(1)=V_n$. We have

Lemma 0.1 There holds the formula

$$V_n=\int_{-1}^{1} V_{n-1}\left(\sqrt{1-\theta^2}\right)d\theta.$$

Proof. Let $S=\left\{(x_1,\cdots,x_n): x_1^2+\cdots+x_n^2\leq 1\right\}$. Then

$$V_n=\int_S dx_1\cdots dx_n=\int_{-1}^{1}\left[\int_{E_{x_n}} dx_1\cdots dx_{n-1}\right]dx_n=\int_{-1}^{1}|E_{x_n}|\,dx_n$$

where

$$\begin{aligned}E_{x_n} &= \{(x_1,\cdots,x_{n-1}):(x_1,\cdots,x_{n-1},x_n)\in S\}\\ &= \left\{(x_1,\cdots,x_{n-1}): x_1^2+\cdots+x_{n-1}^2\leq 1-x_n^2\right\}\end{aligned}$$

and so

$$V_n=\int_{-1}^{1} V_{n-1}\left(\sqrt{1-x_n^2}\right)dx_n.$$

¤

Remark 0.2 Similarly we have

$$V_n(r)=\int_{-r}^{r} V_{n-1}\left(\sqrt{r^2-\theta^2}\right)d\theta\quad\text{for any}\quad r>0. \tag{0.1}$$

Lemma 0.3 We have

$$V_n(r)=r^nV_n\quad\text{for any}\quad r>0. \tag{0.2}$$

Proof. We can prove (0.2) using (0.1) and induction. Assume (0.2) holds for all dimensions less than or equal to $n-1$. Then

$$\begin{aligned}V_n(r) &= \int_{-r}^{r} V_{n-1}\left(\sqrt{r^2-\theta^2}\right)d\theta=\int_{-r}^{r} V_{n-1}\left(r\sqrt{1-\left(\frac{\theta}{r}\right)^2}\right)d\theta\\ &= \int_{-1}^{1} V_{n-1}\left(r\sqrt{1-\mu^2}\right)r\,d\mu=\int_{-1}^{1} r^{n-1}V_{n-1}\left(\sqrt{1-\mu^2}\right)r\,d\mu\quad\text{(by induction hypothesis)}\\ &= r^n\int_{-1}^{1} V_{n-1}\left(\sqrt{1-\mu^2}\right)d\mu=r^nV_n.\end{aligned}$$

¤

By the above, we have

$$V_n = \int_{-1}^{1} V_{n-1}\left(\sqrt{1-\theta^2}\right) d\theta = \int_{-1}^{1} \left(1-\theta^2\right)^{\frac{n-1}{2}} V_{n-1} d\theta$$
$$= 2V_{n-1}\int_{0}^{1} \left(1-\theta^2\right)^{\frac{n-1}{2}} d\theta.$$

The proof is done. ¤

4. (10 points) Do Exercise 11 in p. 97.

<u>Solution:</u> We already know that (from calculus)

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

By Tonelli's Theorem we can easily get

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} = \left(\sqrt{\pi}\right)^n.$$

¤