Real Analysis Homework 10, due 2007-11-28 in class

1. (10 points) Given the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}, \quad (x,y) \in I = (0,1) \times (0,1)$$

compute the following iterated integrals (hint: use trigonometric substitution) :

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx.$$

Is $f(x,y) \in L(I)$ or not? Give your reasons.

Solution:

For fixed x we have

$$\int_0^1 f(x,y)\, dy = \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dy = \int_0^{\tan^{-1}\frac{1}{x}} \frac{x^2 - x^2\tan^2\theta}{(x^2 + x^2\tan^2\theta)^2} d(x\tan\theta)$$

$$= \frac{1}{x}\int_0^{\tan^{-1}\frac{1}{x}} \cos 2\theta d\theta = \frac{1}{x} \cdot \sin\left(\tan^{-1}\frac{1}{x}\right)\cos\theta\left(\tan^{-1}\frac{1}{x}\right) = \frac{1}{1+x^2}.$$

Hence

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dy \right) dx = \frac{\pi}{4}.$$

Similarly (by symmetry) we have

$$\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx = \frac{-1}{1+y^2}$$

and so

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx \right) dy = -\frac{\pi}{4}.$$

Since the two iterated integrals are **different**, by Fubini theorem, $f(x,y) \notin L(I)$. □

2. (10 points) Do Exercise 1 in p. 96.

Solution:

(a). We set $E_x = \{y \in \mathbf{R} : (x,y) \in E\}$and $E_y = \{x \in \mathbf{R} : (x,y) \in E\}$. By Tonelli's Theorem, we have

$$\iint_E \chi_E(x,y)\, dxdy = \int_{\mathbf{R}} \left(\int_{E_x} \chi_E(x,y)\, dy \right) dx = \int_{\mathbf{R}} \left(\int_{E_y} \chi_E(x,y)\, dx \right) dy.$$

By assumption we know $\int_{E_x} \chi_E(x,y)\, dy = 0$ a.e. in $x \in \mathbf{R}$ and so $\iint_E \chi_E(x,y)\, dxdy = |E| = 0$. By Tonelli's Theorem again, we have $|E_y| = 0$ a.e. in $y \in \mathbf{R}$.

(b). Let $E = \left\{(x,y) \in \mathbf{R}^2 : f(x,y) = \infty\right\}$. It is a measurable set in $\mathbf{R}^2$. Set $E_x = \{y \in \mathbf{R} : f(x,y) = \infty\}$ and $E_y = \{x \in \mathbf{R} : f(x,y) = \infty\}$. By (a) we have

$$|E| = 0 \text{ in } \mathbf{R}^2 \iff \begin{cases} |E_x| = 0 \text{ in } \mathbf{R} \text{ for a.e. } x \in \mathbf{R} \\ |E_y| = 0 \text{ in } \mathbf{R} \text{ for a.e. } y \in \mathbf{R}. \end{cases}$$

□

3. (10 points) Do Exercise 2 in p. 96.

Solution:

Let $h_1(x,y) = f(x)$. As a function on $\mathbf{R}^{2n}$, it is measurable since $f(x) : \mathbf{R}^n \to \mathbf{R} \bigcup \{\pm\infty\}$ is measurable. More precisely, for any $a \in \mathbf{R}$ we have

$$\left\{(x,y) \in \mathbf{R}^{2n} : f(x,y) > a\right\} = \{x \in \mathbf{R}^n : f(x) > a\} \times \mathbf{R}^n$$

where by repeated application of Lemma 5.2, we know that the RHS is a measurable set in $\mathbf{R}^{2n}$. Similarly, the function $h_2(x,y) = g(y)$ is also a measurable function on $\mathbf{R}^{2n}$. Then by Theorem 4.10, we know that

$$h_1(x,y) \cdot h_2(x,y) = f(x)\,g(y) : \mathbf{R}^{2n} \to \mathbf{R} \bigcup \{\pm\infty\}$$

is also a measurable function on $\mathbf{R}^{2n}$.

Given $E_1 \subset \mathbf{R}^n$, $E_2 \subset \mathbf{R}^n$, both are measurable in $\mathbf{R}^n$. By $\chi_{E_1}(x) \times \chi_{E_2}(y) = \chi_{E_1 \times E_2}(x,y)$, we know that $\chi_{E_1 \times E_2}(x,y) \geq 0$ is a measurable function on $\mathbf{R}^{2n}$. Hence the set $E_1 \times E_2$ is measurable in $\mathbf{R}^{2n}$.

By Tonelli's Theorem

$$\begin{aligned} |E_1 \times E_2| &= \iint_{E_1 \times E_2} \chi_{E_1 \times E_2}(x,y)\,dxdy = \int_{\mathbf{R}^n} \left(\int_{E_2} \chi_{E_1 \times E_2}(x,y)\,dy \right) dx \\ &= \int_{E_1} \left(\int_{E_2} \chi_{E_1 \times E_2}(x,y)\,dy \right) dx = \int_{E_1} \left(\int_{E_2} [\chi_{E_1}(x) \times \chi_{E_2}(y)]\,dy \right) dx \\ &= \int_{E_1} \chi_{E_1}(x)\,dx \cdot \int_{E_2} \chi_{E_2}(y)\,dy = |E_1| \times |E_2|. \end{aligned}$$

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4. (10 points) Do Exercise 3 in p. 96.

Solution:

We first know that $f(x) - f(y)$ is measurable on $(0,1) \times (0,1)$. By Fubini Theorem, if $F(x,y) = f(x) - f(y)$ is integrable on $(0,1) \times (0,1)$, then for a.e. $y \in (0,1)$, $F(x,y) \in L^1(0,1)$ (as a function of x). Hence $f(x) \in L^1(0,1)$. ¤