Solutions to Homework 1

1. (10 points) Let Q be the set of all rationals in the interval $[0,1]$. Let $S = \{I_1, I_2, ..., I_m\}$ be a **finite** collection of closed intervals covering Q. Show that
$$\sum_{k=1}^{m} v(I_k) \geq 1. \tag{0.1}$$
On the other hand, for any $\varepsilon > 0$, one can find $S = \{I_1, I_2, ..., I_m, \cdots\}$, which is a **countable** collection of closed intervals covering Q, such that
$$\sum_{k=1}^{\infty} v(I_k) < \varepsilon. \tag{0.2}$$
In particular, (0.2) implies that $|Q|_e = 0$. (Now you see the difference between the use of "finite cover" and "countable cover".)

<u>Solution:</u> For (0.1), we first assume that the intervals $I_1, I_2, \ldots, I_m$ are **nonoverlapping**. In such case we clearly have (0.1).

For arbitrary intervals $I_1, I_2, \ldots, I_m$ with overlapping, one can throw away the overlapping part and the remaining nonoverlapping part, which we denote it as $J_1, J_2, \ldots, J_n$, satisfies $\sum_{k=1}^{n} v(J_k) \geq 1$. Therefore we have (0.1).

For (0.2), it has been done in class. ¤

2. (10 points) Find a set $E \subset \mathbf{R}$ with outer measure zero and a function $f : E \to \mathbf{R}$ such that f is continuous on E and $f(E) = [0,1]$. This exercise says that a continuous function can map a set with outer measure zero onto a set with outer measure one.

<u>Solution:</u> Choose $E = C$ to be the **Cantor set** contained in $[0,1]$ and let $f(x)$ be the continuous Cantor Lebesgue function defined on $[0,1]$ (see book p. 35). We know that when restricted to C, $f(x)$ is still a **continuous** function on C. Moreover, one can easily see that $f(C) = f([0,1]) = [0,1]$ (for example, we have $f\left(\frac{1}{3}, \frac{2}{3}\right) = f\left\{\frac{1}{3}, \frac{2}{3}\right\}$, etc.). ¤

3. (10 points) Let E_1 and E_2 be two subsets of $\mathbf{R}^n$ such that $E_1 \subset E_2$ and $E_2 - E_1$ is countable. Show that
$$|E_1|_e = |E_2|_e.$$

<u>Solution:</u> Clearly we have $|E_1|_e \leq |E_2|_e$. Also
$$|E_2|_e \leq |E_1|_e + |E_2 - E_1|_e = |E_1|_e$$
which implies $|E_1|_e = |E_2|_e$. ¤

4. (10 points) Find a continuous function $f(x)$ defined on $[0,1]$ such that $f(x)$ is differentiable on a subset $E \subset [0,1]$ with $|E|_e = 1$ and $f'(x) = 0$ for all $x \in E$, but $f(x)$ is not a constant function.

Solution: Let $f(x)$ be the Cantor Lebesgue function defined on $[0,1]$ as given in p. 35 of the book. We know $f(x)$ is differentiable on the open set $O = O_1 \cup O_2 \cup O_3 \cup \cdots$, where

$$O_1 = \left(\frac{1}{3}, \frac{2}{3}\right), \quad O_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right), \quad O_3 = \left(\frac{1}{27}, \frac{2}{27}\right) \cup \cdots, \quad O_4 = \cdots$$

The total length of these open intervals is given by

$$\frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right] = 1.$$

□

The example below shows you how to obtain a set which is measurable, but **not** Borel measurable.

Let $f(x) : [0,1] \to [0,1]$ be the Cantor-Lebesgue function and let $g(x) = x + f(x)$. It is easy to see that $g(x) : [0,1] \to [0,2]$ is a strictly increasing continuous function. Hence $g(x)$ is a homeomorphism of $[0,1]$ onto $[0,2]$. On each interval $I_1,\ I_2,\ I_3,\ ...$, removed in the construction of the Cantor set, say the interval $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, the function $g(x)$ becomes $g(x) = x + \frac{1}{2}$. Hence $g(x)$ sends I_1 onto an open interval **with the same length**. Using this observation one can see that

$$\left|g\left(\cup_{k=1}^{\infty} I_k\right)\right| = \left|\cup_{k=1}^{\infty} g(I_k)\right| = \sum_{k=1}^{\infty} |g(I_k)| = \sum_{k=1}^{\infty} |I_k| = 1$$

which implies $|g(C)| = 2 - 1 = 1$, where C is the Cantor set.

Since $g(C)$ has positive measure, by Corollary 3.39 in the book, there exists a non-measurable set $B \subset g(C)$. Now consider the set $A = g^{-1}(B) \subset C$. It has measure zero, hence it is measurable. However it can not be Borel measurable. If A were Borel measurable, then since $g(x)$ is a homeomorphism, it would imply that $B = g(A)$ is also Borel measurable. But this is impossible since B is a non-measurable set.