Real Analysis Homework 1, due 2007-9-18 in class

Show Your Work to Each Problem

1. (10 points) Let Q be the set of all rationals in the interval $[0,1]$. Let $S = \{I_1, I_2, ..., I_m\}$ be a **finite** collection of closed intervals covering Q. Show that

$$\sum_{k=1}^{m} v(I_k) \geq 1.$$

On the other hand, for any $\varepsilon > 0$, one can find $S = \{I_1, I_2, ..., I_m, \cdots\}$, which is a **countable** collection of closed intervals covering Q, such that

$$\sum_{k=1}^{\infty} v(I_k) < \varepsilon. \tag{0.1}$$

In particular, (0.1) implies that $|Q|_e = 0$. (Now you see the difference between the use of "finite cover" and "countable cover".)

2. (10 points) Find a set $E \subset \mathbf{R}$ with outer measure zero and a function $f : E \to \mathbf{R}$ such that f is continuous on E and $f(E) = [0,1]$. This exercise says that a continuous function can map a set with outer measure zero onto a set with outer measure one.

3. (10 points) Let E_1 and E_2 be two subsets of $\mathbf{R}^n$ such that $E_1 \subset E_2$ and $E_2 - E_1$ is countable. Show that

$$|E_1|_e = |E_2|_e.$$

4. (10 points) Find a continuous function $f(x)$ defined on $[0,1]$ such that $f(x)$ is differentiable on a subset $E \subset [0,1]$ with $|E|_e = 1$ and $f'(x) = 0$ for all $x \in E$, but $f(x)$ is not a constant function.