Real Analysis Final Exam, 2008-1-8

Name: ID #:

Show detailed argument to each problem.

1. (10 points) Let $E \subset \mathbf{R}^n$ be a measurable set and $q(x) \geq 0$ be a nonnegative measurable function on E. Show that for any constant $\alpha > 0$ we have the inequality

$$\int_E q\,dx \cdot \int_E q^{\alpha}dx \leq |E| \cdot \int_E q^{1+\alpha}dx.$$

solution:

By Hölder inequality

$$\int_E q\,dx \leq \left(\int_E q^{1+\alpha}dx\right)^{\frac{1}{1+\alpha}} \left(\int_E dx\right)^{\frac{\alpha}{1+\alpha}}, \quad \int_E q^{\alpha}dx \leq \left(\int_E (q^{\alpha})^{\frac{1+\alpha}{\alpha}}\,dx\right)^{\frac{\alpha}{1+\alpha}} \left(\int_E dx\right)^{\frac{1}{1+\alpha}}$$

the proof is done. ¤

2. (10 points) Assume $|E| < \infty$ and u is a measurable function on E, which is everywhere positive. For $0 < p < \infty$, set

$$\Phi_p(u) := \left(\frac{1}{|E|}\int_E |u(x)|^p\,dx\right)^{1/p} = \left(\frac{1}{|E|}\int_E u^p(x)\,dx\right)^{1/p}. \tag{0.1}$$

Assume that

$$\begin{cases} (1).\ u^p(x) \to 1 \text{ uniformly on } E \text{ as } p \to 0^+. \\ (2).\ \frac{d}{dp}\left(\int_E u^p(x)\,dx\right) = \int_E \frac{d}{dp}\left(u^p(x)\right)dx. \\ (3).\ |\log u(x)| \leq C \text{ for all } x \in E, \text{ for some constant } C > 0. \end{cases}$$

Evaluate the limit $\lim_{p\to 0^+}\Phi_p(u)$ in terms of an integral involving u. (Compare with the limit $\lim_{p\to\infty}\Phi_p(u) = \|u\|_{\infty}$.)

solution:

As $p \to 0^+$, $\Phi_p(u)$ has the form 1^{∞}. Hence we can use Lopital rule. Note that

$$\log \Phi_p(u) = \frac{\log\left(\frac{1}{|E|}\int_E u^p(x)\,dx\right)}{p} \qquad \left(\frac{0}{0}\text{ form}\right)$$

and so

$$\lim_{p\to 0^+}\log\Phi_p(u) = \lim_{p\to 0^+}\frac{\frac{d}{dp}\log\left(\frac{1}{|E|}\int_E u^p(x)\,dx\right)}{1} = \lim_{p\to 0^+}\frac{\frac{1}{|E|}\frac{d}{dp}\int_E u^p(x)\,dx}{\frac{1}{|E|}\int_E u^p(x)\,dx}$$

$$= \lim_{p\to 0^+}\frac{\frac{1}{|E|}\int_E \frac{d}{dp}\left(u^p(x)\right)dx}{\frac{1}{|E|}\int_E u^p(x)\,dx} = \lim_{p\to 0^+}\frac{\frac{1}{|E|}\int_E u^p(x)\log u(x)\,dx}{\frac{1}{|E|}\int_E u^p(x)\,dx} = \frac{1}{|E|}\int_E \log u(x)\,dx.$$

Hence we have the formula

$$\lim_{p\to 0^+}\Phi_p(u) = \exp\left(\frac{1}{|E|}\int_E \log u(x)\,dx\right).$$

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3. (10 points) If $f \in L^p(\mathbf{R}^n)$, where $0 < p < \infty$ is a constant, show that

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p \, dy = 0$$

for almost everywhere $x \in \mathbf{R}^n$. (This is a previous homework problem.)

<u>solution:</u>

Let $\{r_k\}_{k=1}^{\infty}$ be the set of all rational numbers. For any k and any $0 < p < \infty$, the function $|f(y) - r_k|^p$ is clearly locally integrable on $\mathbf{R}^n$. By Theorem 7.11 of the book, for each k we have

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|^p \, dy = |f(x) - r_k|^p \tag{0.2}$$

for a.e. $x \in \mathbf{R}^n$. Let Z_k be the set such that (0.2) is not valid, $|Z_k| = 0$, and set $Z = \bigcup_{k=1}^{\infty} Z_k$, $|Z| = 0$. If $x \notin Z$, then by the inequality (in below, Q is centered at x and k is arbitrary)

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |f(y) - f(x)|^p \, dy \le \frac{1}{|Q|} \int_Q \{|f(y) - r_k| + |r_k - f(x)|\}^p \, dy \\ &\le \frac{C(p)}{|Q|} \int_Q |f(y) - r_k|^p \, dy + \frac{C(p)}{|Q|} \int_Q |r_k - f(x)|^p \, dy \\ &= \frac{C(p)}{|Q|} \int_Q |f(y) - r_k|^p \, dy + C(p) |r_k - f(x)|^p . \end{aligned}$$

Hence

$$\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|^p \, dy \le 2C(p) |r_k - f(x)|^p \quad \text{for all} \quad x \notin Z$$

By choosing r_k approximating $f(x)$ (note that $f(x)$ is finite almost everywhere in $\mathbf{R}^n$; without loss of generality, we can assume it is finite everywhere), we obtain

$$\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p \, dy = 0 \quad \text{for all} \quad x \notin Z$$

for any $0 < p < \infty$. Hence (0.2) holds for a.e. $x \in \mathbf{R}^n$. ¤

4. (10 points) Give an example of a bounded continuous function f on $(0, \infty)$ such that $\lim_{x \to \infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$. Give your reasons. (This is exercise 12 in p. 86.)

<u>solution:</u>

For example, one can take

$$f(x) = \begin{cases} \frac{1}{\log 2}, & x \in (0, 2) \\ \frac{1}{\log x}, & x \in [2, \infty). \end{cases}$$

Then $f(x)$ is bounded continuous on $(0, \infty)$ with $\lim_{x \to \infty} f(x) = 0$, but for any $p > 0$ we have

$$\int_2^{\infty} f^p(x) \, dx = \int_2^{\infty} \left(\frac{1}{\log x} \right)^p dx = \int_{\log 2}^{\infty} \frac{e^y}{y^p} dy = \infty.$$

Hence $f \notin L^p(0, \infty)$ for any $p > 0$. ¤

5. (10 points) Give an example of a set $E \subset \mathbf{R}$ such that the number $x = 0$ is a point of density of E, but it is not a Lebesgue point of the function $\chi_E(x)$. Give your reasons.

solution:

Let $E = (-1, 1) \setminus \{0\}$. We have

$$\lim_{Q \searrow 0} \frac{|E \bigcap Q|}{|Q|} = 1.$$

Hence $x = 0$ is a point of density of E. Since $0 \notin E$, we cannot have

$$\lim_{Q \searrow 0} \frac{1}{|Q|} \int_Q |\chi_E(y) - \chi_E(0)| \, dy = 0.$$

Hence $x = 0$ is **not** a Lebesgue point of the function $\chi_E(x)$. ¤

6. (15 points)

   (a) (10 points) Let $0 < p < \infty$. Assume $f_k,\ f \in L^p(E)$, $f_k \to f$ a.e. on E, and $\|f_k\|_p \to \|f\|_p$. Show that $\|f_k - f\|_p \to 0$ as $k \to \infty$. (This is a slight modification of exercise 12 in p. 144.)

   (b) (5 points) If we replace $0 < p < \infty$ in above by $p = \infty$, do we have the same conclusion or not? Give your reasons.

solution:

For (a): For any $0 < p < \infty$, we have the inequality

$$|f - f_k|^p \le 2^p |f|^p + 2^p |f_k|^p$$

we have (by Fatou's Lemma)

$$\int_E \liminf_k \left(2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right) \le \liminf_k \int_E \left(2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right)$$

where

$$\int_E \liminf_k \left(2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right) = \int_E 2^{p+1} |f|^p$$

and

$$\liminf_k \int_E \left(2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p\right) = \int_E 2^{p+1} |f|^p - \limsup_k \int_E |f - f_k|^p.$$

Since we assume $f \in L^p$, the integral $\int_E 2^{p+1} |f|^p$ is finite. Hence we conclude

$$\limsup_k \int_E |f - f_k|^p \le 0.$$

This implies $\|f_k - f\|_p \to 0$ as $k \to \infty$.

For (b): For $p = \infty$, the conclusion fails. Take $E = \mathbf{R}$ and $f_k = \chi_{(-k,k)},\ f = 1$. Then $f_k,\ f \in L^\infty(\mathbf{R}),\ f_k \to f$ a.e. on $\mathbf{R}$, and $\|f_k\|_\infty = \|f\|_\infty = 1$. But we do not have $\|f_k - f\|_\infty \to 0$ as $k \to \infty$.
Another example is $E = (0, 1),\ f_k = \chi_{(1/k, 1)},\ f = 1,\ k \in \mathbf{N}$. Then we have the same conclusion. ¤

7. (15 points) Let $E = (0, \infty) \times (0, 1) \subset \mathbf{R}^2$ and $f(x, y) = y e^{-xy} \sin x,\ (x, y) \in E$.

   (a) (5 points) Show that $f(x, y)$ is integrable on E, i.e., $f \in L(E)$.

   (b) (10 points) Evaluate the integral

$$\iint_E f(x, y) \, dx dy.$$

solution:

For (a), note that

$$\iint_E |f(x,y)|\, dxdy \le \iint_E ye^{-xy} dxdy$$

and by Tonelli Theorem we have

$$\iint_E ye^{-xy} dxdy = \int_0^1 \left[\int_0^\infty ye^{-xy} dx\right] dy = \int_0^1 dy = 1$$

which implies that $f \in L(E)$.

For (b), we apply Fubini Theorem and get

$$\iint_E f(x,y)\, dxdy = \int_0^1 y\left(\int_0^\infty e^{-xy}\sin x dx\right) dy.$$

Recall the elementary formula

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C, \qquad a,\ b \text{ are constants}$$

we see that for each $y > 0$

$$\int_0^\infty e^{-xy}\sin x dx = \left.\frac{e^{-xy}(-y\sin x - \cos x)}{y^2+1}\right|_{x=0}^{x=\infty} = \frac{1}{y^2+1}$$

and so

$$\int_0^1 y\left(\int_0^\infty e^{-xy}\sin x dx\right) dy = \int_0^1 \frac{y}{y^2+1} dy = \frac{1}{2}\log 2.$$

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8. (20 points) Answer Yes or No to each statement. Just provide the answers.

   (a) If x is a point of density of a measurable set E, then $x \in E$. ANS: __________

   (b) Let $0 < p < \infty$. If $\|f_k - f\|_{p,E} \to 0$ as $k \to \infty$, then there exists a subsequence $f_{k_j} \to f$ almost everywhere in E as $j \to \infty$. ANS: __________

   (c) Let f and g be two nonnegative measurable functions on measurable set E satisfying

   $$|\{x \in E : f(x) > y\}| = |\{x \in E : g(x) > y\}| \quad \text{for all} \quad y > 0.$$

   Then we must have $\int_E f = \int_E g$. ANS: __________

   (d) $\|f_k - f\|_{\infty,E} \to 0$ as $k \to \infty$ **if and only if** $f_k \to f$ uniformly on E except on a set $Z \subset E$ of measure zero. ANS: __________

solution:

The answers to the five problems are: NO, YES, YES, YES. ¤