ODE Lecture Notes, Part III, for Fall Semester 2022

Revised on 2023-1-4

Remark 0.1 ***This is the third part for my 2022 fall semester ODE course. The third part contains material from Chapter 7 (linear system of ODE) of the book.***

Remark 0.2 *This notes is based on the textbook **"Elementary Differential Equations & Boundary Value Problems, 10th Edition" by Boyce & DiPrima.** However, **I will not follow the book exactly**. Lecture notes will be given to you via email whenever necessary.*

Chapter 7.

System of first order linear (homogeneous) equations with constant coefficients.

Any scalar equation can be written as a first order system.

Consider a second order linear equation with constant coefficients (for simplicity, we make the coefficient of $x''(t)$ equal to 1):

$$ax''(t) + bx'(t) + cx(t) = 0, \quad t \in (-\infty, \infty), \tag{1}$$

where $a \neq 0,\ b,\ c$ are constants. We already know how to solve it by looking at the roots of its characteristic polynomial equation. However, there is another way to solve it, which has the advantage of applying the theory of **linear algebra** to study the properties of the solution $x(t)$. Letting $y(t) = x'(t)$, equation (1) can be written as

$$y'(t) = -\frac{c}{a}x(t) - \frac{b}{a}y(t), \quad a \neq 0$$

which is, of course, **not self-contained** for the function $y(t)$. But if we include the equation $x'(t) = y(t)$, then the system of equations

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -\frac{c}{a}x(t) - \frac{b}{a}y(t), \quad t \in (-\infty, \infty) \end{cases} \tag{2}$$

becomes **self-contained** for the **vector-valued function** $v(t) = (x(t), y(t))$ (here we view $v(t)$ as a **column vector**). In terms of vector and matrix notation, we can write (2) as:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ (same as } \frac{dv}{dt} = Av,\ A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\text{)}. \tag{3}$$

We call (3) a 2×2 **system of first order linear (homogeneous) equation with constant coefficients**.

Notation 0.3 *In this chapter we use $M(n)$ to denote the space of all $n \times n$ **real** matrices.*

Lemma 0.4 *Let $A \in M(2)$ be the matrix given in (3). If $x(t)$ satisfies (1) on $t \in (-\infty, \infty)$, then the vector-valued function $v(t) = (x(t), y(t))$, where $y(t) = x'(t)$, satisfies (3) on $t \in (-\infty, \infty)$. On the other hand, if a vector-valued function $v(t) = (x(t), y(t))$ satisfies (3) on $t \in (-\infty, \infty)$, then $y(t) = x'(t)$ and $x(t)$ satisfies (1) on $t \in (-\infty, \infty)$. Moreover, the **eigenvalues** (see Definition 0.8 below) of A are exactly the **two roots** of the **characteristic polynomial equation** of the ODE (1).*

Proof. This is a simple verification. $\square$

Another result similar to Lemma 0.4 is the following:

Lemma 0.5 *If we have a* 2×2 *linear system of the form*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2), \tag{4}$$

then $x(t)$ *will satisfy the equation*

$$x''(t) - (TrA)\, x'(t) + (\det A)\, x(t) = 0, \quad \text{where } \; TrA = a + d, \quad \det A = ad - bc \tag{5}$$

and the same for $y(t)$. *Moreover, the two* ***roots*** r_1, r_2 *of the characteristic polynomial equation of the ODE (5) are the same as the two* ***eigenvalues*** *of the matrix* A.

Remark 0.6 *(**Be careful.**) Assume* r_1 *and* r_2 *are the two roots of the characteristic equation*

$$r^2 - (TrA)\, r + \det A = 0.$$

The above lemma ***does not*** *imply that for* ***any*** *constants* $c_1, c_2, \tilde{c}_1, \tilde{c}_2$, *the function*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ \tilde{c}_1 e^{r_1 t} + \tilde{c}_2 e^{r_2 t} \end{pmatrix}, \quad t \in (-\infty, \infty)$$

is a solution of (4). Since $x(t)$ *and* $y(t)$ *are related by the system of ODE (4), the constants* c_1, c_2 *and the constants* $\tilde{c}_1, \tilde{c}_2$ *must be related. Their relation will involve the* ***eigenvalues and eigenvectors*** *of the matrix* A.

Proof. By (4), we have

$$\begin{cases} x'(t) = ax(t) + by(t) \\ y'(t) = cx(t) + dy(t) \end{cases}$$

and so

$$\begin{aligned} x''(t) &= ax'(t) + by'(t) = ax'(t) + b\left[cx(t) + d\underbrace{y(t)}\right] \\ &= ax'(t) + bcx(t) + d\underbrace{by(t)} = ax'(t) + bcx(t) + d\left(\underbrace{x'(t) - ax(t)}\right), \end{aligned}$$

which gives (5). Similarly, one can check that $y(t)$ also satisfies the equation

$$y''(t) - (a + d)\, y'(t) + (ad - bc)\, y(t) = 0.$$

$\square$

Example 0.7 *Use the idea of Lemma 0.5 to find the* ***general solution*** *of the equation*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}.$$

Solution:

The system is the same as

$$\begin{cases} x'(t) = 2x(t) + 4y(t) \\ y'(t) = x(t) - y(t) \end{cases} \tag{6}$$

and we know that $x(t)$ satisfies the equation

$$x''(t) - x'(t) - 6x(t) = 0$$

and the same for $y(t)$. The characteristic equation for the ODE is $(r-3)(r+2) = 0$, which has two roots $r = 3,\ -2$; hence we have

$$\begin{cases} x(t) = c_1 e^{3t} + c_2 e^{-2t} \\ y(t) = \tilde{c}_1 e^{3t} + \tilde{c}_2 e^{-2t} \end{cases}$$

for some constants $c_1,\ c_2,\ \tilde{c}_1,\ \tilde{c}_2$. However, since $x(t)$ and $y(t)$ are related by the system of equations (6), the constants $\tilde{c}_1,\ \tilde{c}_2$ and the constants $c_1,\ c_2$ must be related. To see this, by the first equation in (6), we have

$$y(t) = \frac{x'(t) - 2x(t)}{4} = \frac{\left[3c_1 e^{3t} - 2c_2 e^{-2t}\right] - 2\left[c_1 e^{3t} + c_2 e^{-2t}\right]}{4} = \frac{c_1}{4} e^{3t} - c_2 e^{-2t},$$

which implies

$$\tilde{c}_1 = \frac{c_1}{4}, \quad \tilde{c}_2 = -c_2.$$

One can verify that, for any constants $c_1,\ c_2$, the function

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

is a solution of (6) defined on $t \in (-\infty, \infty)$. Moreover, we see that the two numbers $3,\ -2$ are the eigenvalues of the matrix A and the two vectors $v_1 = (1, 1/4),\ v_2 = (1, -1)$ are eigenvectors of A corresponding to the eigenvalues $3,\ -2$. □

Eigenvalues and eigenvectors.

The advantage of rewriting the single equation (1) as the system (3) is that we can use **the properties of the matrix** A (like **eigenvalues and eigenvectors**) to study the properties of the solution $x(t)$ and its derivative $x'(t)$ (same as $y(t)$). That is to say, we can use **linear algebra** to help us understand the behavior of $x(t)$ and $x'(t)$ as $t \to \pm\infty$.

We first recall the definition:

Definition 0.8 *Let $A \in M(n)$ ($M(n)$ is the space of all $n \times n$ **real** matrices). We say $\lambda \in \mathbb{R}$ is a **real eigenvalue** of A if there is a **nonzero real vector** $v \neq 0 \in \mathbb{R}^n$ such that $Av = \lambda v$. In such a case, the vector v is called a **real eigenvector** of A corresponding to the real eigenvalue λ. Similarly, we say $\lambda \in \mathbb{C}$ is a **complex eigenvalue** of A (here λ has nonzero imaginary part) if there is a **nonzero complex vector** $v \neq 0 \in \mathbb{C}^n$ such that $Av = \lambda v$. The vector v is called a **complex eigenvector** of A corresponding to the complex eigenvalue λ. Note that **a real matrix can have a complex eigenvalue**.*

Remark 0.9 *Note that if $\lambda \in \mathbb{C}$ is a complex eigenvalue of $A \in M(n)$ with complex eigenvector $v \neq 0 \in \mathbb{C}^n$, then by the identity*

$$A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v} \quad (\textit{i.e. } A\overline{v} = \overline{\lambda}\,\overline{v}),$$

we see that $\overline{\lambda} \in \mathbb{C}$ *is also a complex eigenvalue with corresponding complex eigenvector* $\overline{v}$*. Moreover, if we write* $\lambda = \alpha + i\beta,\ \alpha,\ \beta \neq 0 \in \mathbb{R}$*, and write* $v = u + iw,\ u,\ w \neq 0 \in \mathbb{R}^n$*, then by the identity* $A(u + iw) = (\alpha + i\beta)(u + iw)$ *we have*

$$\begin{cases} Au = \alpha u - \beta w \\ Aw = \beta u + \alpha w. \end{cases} \tag{7}$$

In case $n = 2$*, we can write (7) as the matrix identity*

$$A(u, w) = (Au, Aw) = (\alpha u - \beta w, \beta u + \alpha w) = (u, w) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $(u, w) \in M(2)$ *with* u *as the first column and* w *as the second column.*

From linear algebra, we know the following important fact:

Lemma 0.10 (1). *Let* $A \in M(n)$. λ *is an* ***eigenvalue*** *(real or complex) of* A ***if and only if*** *it satisfies the equation*

$$\det(A - \lambda I) = 0, \tag{8}$$

where I *is the* $n \times n$ *identity matrix.* (2). *If* $\lambda \in \mathbb{R}$ *is a* ***real eigenvalue*** *of* A*, then the set*

$$\ker(A - \lambda I) = \{v \in \mathbb{R}^n : Av = \lambda v \text{ (include } v = 0)\} \subseteq \mathbb{R}^n$$

is a ***vector subspace*** *of* $\mathbb{R}^n$ *with dimension at least* 1 *(call it the* ***eigenspace*** *of* A *corresponding to* $\lambda \in \mathbb{R}$*). Similarly, if* $\lambda \in \mathbb{C}$ *is a* ***complex eigenvalue*** *of* A*, then the set*

$$\ker(A - \lambda I) = \{v \in \mathbb{C}^n : Av = \lambda v \text{ (include } v = 0)\} \subseteq \mathbb{C}^n$$

is a ***vector subspace*** *of* $\mathbb{C}^n$ *with dimension at least* 1 *(call it the* ***eigenspace*** *of* A *corresponding to* $\lambda \in \mathbb{C}$*).*

Remark 0.11 *The equation* $\det(A - \lambda I) = 0$ *for finding eigenvalues* λ *of* A *is also called the* ***characteristic equation*** *of the matrix* A.

0.0.1 System of first order linear equations with constant coefficients.

We define the following:**system of first order linear (homogeneous) equation with constant coefficients**.

Definition 0.12 *Let* $A \in M(n)$ *be an* $n \times n$ *real matrix. The system of equation*

$$\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt}(t) = A\mathbf{x}(t), \quad \mathbf{x} = \mathbf{x}(t) = (x_1(t), \ \dots, x_n(t)) \text{ (column vector)} \in \mathbb{R}^n \tag{9}$$

is called a ***system of first order linear equation with constant coefficients****. A* ***real solution*** *(or just call it a* ***solution****) of (9) is a* ***vector-valued function*** $\mathbf{x}(t) = (x_1(t), \ \dots, x_n(t)) \in \mathbb{R}^n$ *defined at least on some open interval* $t \in I$*. The goal is to find* ***all possible real solutions*** *of (9).*

Remark 0.13 *When we say* $\mathbf{x}(t)$ *is a* ***solution*** *of (9), we always mean that it is a* ***real solution*** *unless otherwise stated.*

Remark 0.14 *To describe the* ***general solution*** $\mathbf{x}(t)$ *of (9), we need* n *arbitrary real constants* $c_1, \dots, c_n$*, which can be viewed as* ***one*** *arbitrary constant* $c = (c_1, \ \dots, c_n) \in \mathbb{R}^n$ *(since (9) is a* ***first order equation in*** $\mathbb{R}^n$*). To determine the constant* $c \in \mathbb{R}^n$ *uniquely, we need an initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$*. See Theorem 0.19 below.*

Remark 0.15 *Any point* $\mathbf{x}_0 \in \mathbb{R}^n$ *satisfying* $A\mathbf{x}_0 = 0$ *is an* ***equilibrium solution*** *of (9), i.e. the function* $\mathbf{x}(t) \equiv \mathbf{x}_0$ *is a solution of* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *defined on* $t \in (-\infty, \infty)$.

Remark 0.16 *(**Important.**) Our goal is to find* ***all possible real solutions (general real solutions)*** *of (9). If there is a* ***complex solution*** $\mathbf{x}(t)$ *of (9), then its* ***real part*** *and* ***imaginary part*** *are both* ***real*** *solutions of (9).*

Lemma 0.17 *If* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are both real solutions to (9) on* $(-\infty, \infty)$, *then their linear combination*

$$\mathbf{z}(t) = c_1\mathbf{x}(t) + c_2\mathbf{y}(t), \quad t \in I$$

is also a solution of (9) on $(-\infty, \infty)$. *Here* c_1, c_2 *are two arbitrary real constants.*

Remark 0.18 *This says that the solution space of (9) has the structure of a* ***vector space.***

Proof. This is obvious. □

We have the following existence and uniqueness property for equation (9):

Theorem 0.19 *(**Existence and uniqueness property.**) Any solution to (9) is defined on* $t \in (-\infty, \infty)$. *Moreover, if (9) has an* ***initial condition*** *of the form*

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n, \tag{10}$$

then there exists a ***unique solution*** $\mathbf{x}(t)$, $t \in (-\infty, \infty)$, *satisfying* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *for all* $t \in (-\infty, \infty)$ *and* $\mathbf{x}(t_0) = \mathbf{x}_0$.

Proof. You can find the proof in some ODE textbook. □

A preliminary result of the system (9) is the following simple but important fact:

Lemma 0.20 *If* λ *is a* ***real*** *eigenvalue of* $A \in M(n)$ *with corresponding* ***real*** *eigenvector* $v \neq 0 \in \mathbb{R}^n$, *then the function*

$$\mathbf{x}(t) = e^{\lambda t}v \in \mathbb{R}^n, \quad t \in (-\infty, \infty) \tag{11}$$

is a ***real*** *solution of* $\mathbf{x}' = A\mathbf{x}$ *defined on* $(-\infty, \infty)$.

Proof. We have $Av = \lambda v$. Hence

$$\frac{d\mathbf{x}}{dt}(t) = \frac{d}{dt}\left(e^{\lambda t}v\right) = \lambda e^{\lambda t}v = A\left(e^{\lambda t}v\right) = A\mathbf{x}(t), \quad t \in (-\infty, \infty).$$

□

Remark 0.21 *(**Important.**) If* $\lambda = \alpha + i\beta \in \mathbb{C}$, $\beta \neq 0$, *is a* ***complex*** *eigenvalue of* $A \in M(n)$ *with corresponding* ***complex*** *eigenvector* $v = u + iw \in \mathbb{C}^n$, *where* u, $w \neq 0 \in \mathbb{R}^n$, *then the function*

$$\mathbf{x}(t) = e^{\lambda t}v = \underbrace{e^{(\alpha+i\beta)t}(u+iw)} \in \mathbb{C}^n, \quad t \in (-\infty, \infty)$$

is a ***complex*** *solution of (9) defined on* $(-\infty, \infty)$. *This is because we have the* ***same*** *identities as in the real case:*

$$\mathbf{x}'(t) = \frac{d}{dt}\left(e^{\lambda t}v\right) = \lambda e^{\lambda t}v = e^{\lambda t}\lambda v = e^{\lambda t}(Av) = A\left(e^{\lambda t}v\right) = A\mathbf{x}(t).$$

We can the ***real part*** $\mathbf{x}_1(t)$ *and* ***imaginary part*** $\mathbf{x}_2(t)$ *of* $\mathbf{x}(t)$ *to get the following* ***two real solutions*** *of* $\mathbf{x}' = A\mathbf{x}$ *:*

$$\begin{cases} \mathbf{x}_1(t) = e^{\alpha t}\left[(\cos\beta t)\,u - (\sin\beta t)\,w\right], & u,\ w \in \mathbb{R}^n, \quad t \in (-\infty,\infty) \\ \mathbf{x}_2(t) = e^{\alpha t}\left[(\sin\beta t)\,u + (\cos\beta t)\,w\right], & u,\ w \in \mathbb{R}^n, \quad t \in (-\infty,\infty). \end{cases} \tag{12}$$

Note that the complex function $\overline{\mathbf{x}}(t) = e^{\overline{\lambda}t}\overline{v}$ *is also a complex solution, but it will produce* ***the same*** *real solutions as in (12). To see this, we note that*

$$\begin{aligned} \overline{\mathbf{x}}(t) &= e^{\overline{\lambda}t}\overline{v} = e^{(\alpha - i\beta)t}(u - iw) = e^{\alpha t}\left[(\cos\beta t) - i(\sin\beta t)\right](u - iw) \\ &= e^{\alpha t}\left[(\cos\beta t)\,u - i(\cos\beta t)\,w - i(\sin\beta t)\,u - (\sin\beta t)\,w\right] \\ &= e^{\alpha t}\left[(\cos\beta t)\,u - (\sin\beta t)\,w\right] + ie^{\alpha t}\left[-(\sin\beta t)\,u - (\cos\beta t)\,w\right], \quad u,\ w \in \mathbb{R}^n, \quad t \in (-\infty,\infty). \end{aligned}$$

To go on, we need some results from linear algebra:

Lemma 0.22 *If* $A \in M(n)$ *has* n ***distinct real eigenvalues*** $\lambda_1,\ \ldots,\ \lambda_n$ *with corresponding nonzero real eigenvectors* $v_1,\ \ldots,\ v_n$*, then* $v_1,\ \ldots,\ v_n$ *are* ***linearly independent*** *in* $\mathbb{R}^n$ *(hence they form a* ***basis*** *of* $\mathbb{R}^n$*).*

Proof. We first claim that v_1 and v_2 are independent. Otherwise, we would have $v_1 = cv_2$ for some constant $c \neq 0$. Hence we get (applying A onto it) $\lambda_1 v_1 = c\lambda_2 v_2$. But we also have $\lambda_1 v_1 = c\lambda_1 v_2$ and so $c\lambda_2 v_2 = c\lambda_1 v_2$. This will force $\lambda_1 = \lambda_2$, impossible. Hence v_1 and v_2 are independent. Similarly if we have $v_3 = \alpha v_1 + \beta v_2$ with $\alpha^2 + \beta^2 \neq 0$, then

$$\begin{cases} \lambda_3 v_3 = \alpha\lambda_1 v_1 + \beta\lambda_2 v_2 \\ \lambda_3 v_3 = \alpha\lambda_3 v_1 + \beta\lambda_3 v_2, \end{cases}$$

which implies

$$\alpha(\lambda_1 - \lambda_3)\,v_1 + \beta(\lambda_2 - \lambda_3)v_2 = 0, \quad \lambda_1 - \lambda_3 \neq 0, \quad \lambda_2 - \lambda_3 \neq 0,$$

and so $\alpha = \beta = 0$, a contradiction. Thus v_3 does not lie on the plane spanned by v_1 and v_2; hence $v_1,\ v_2,\ v_3$ are independent. Keep going and use induction argument to conclude that $v_1,\ \ldots,\ v_n$ are linearly independent. □

Lemma 0.23 *If* $A \in M(n)$ *has* n ***distinct real eigenvalues*** $\lambda_1,\ \ldots,\ \lambda_n$ *with corresponding* ***nonzero real eigenvectors*** $v_1,\ \ldots,\ v_n$*, then*

$$P^{-1}AP = D, \tag{13}$$

where $P = (v_1,\ \ldots,\ v_n)$ *(each* v_i *is a* ***column*** *vector) and* $D = diag(\lambda_1,\ \ldots,\ \lambda_n)$ *is the* ***diagonal matrix*** *with diagonal elements* $\lambda_1,\ \ldots,\ \lambda_n$*. In such a case, we can* ***diagonalize*** *the matrix* A*.*

Proof. Note that we have the identity $AP = PD$ (explain it) and $\det P \neq 0$, hence P^{-1} exists and we obtain $P^{-1}AP = D$. □

Remark 0.24 ***(Important.)*** *Compare the difference between* PD *and* DP*. In general, we* ***do not have*** $PD = DP$ *even if* D *is a diagonal matrix.*

The case when $A \in M(n)$ has n distinct real eigenvalues.

Lemma 0.25 *If $A \in M(n)$ has n* ***distinct real*** *eigenvalues $\lambda_1, \ ..., \ \lambda_n$ with corresponding nonzero real eigenvectors $v_1, \ ..., \ v_n$, then if $\mathbf{x}(t) \in \mathbb{R}^n$ is a solution of (9) on $(-\infty, \infty)$, it can be expressed as*

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n, \quad t \in (-\infty, \infty) \tag{14}$$

for some real constants $c_1, \ ..., \ c_n$. Therefore, the ***general solution*** *of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ in this case (i.e., A has n* ***distinct real*** *eigenvalues) is given by (14). In terms of* ***matrix notation****, one can express (14) as*

$$\mathbf{x}(t) = PD(t) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \text{where } P = (v_1, \ ..., \ v_n) \ (\text{each } v_i \text{ is a } \textbf{column} \text{ vector}).$$

and $D(t)$ is the diagonal matrix diag $\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)$, $t \in (-\infty, \infty)$.

Proof. Since $v_1, \ ..., \ v_n$ is a basis of $\mathbb{R}^n$, at any time $t \in (-\infty, \infty)$ one can decompose $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = a_1(t) v_1 + \cdots + a_n(t) v_n$$

for some coefficient functions $a_1(t), \ ..., \ a_n(t)$. We now have

$$\begin{cases} \mathbf{x}'(t) = a_1'(t) v_1 + \cdots + a_n'(t) v_n \\ A\mathbf{x}(t) = \lambda_1 a_1(t) v_1 + \cdots + \lambda_n a_n(t) v_n. \end{cases}$$

This implies $a_1'(t) = \lambda_1 a_1(t), \ ..., \ a_n'(t) = \lambda_n a_n(t)$. Hence there exist constants $c_1, \ ..., \ c_n$ such that

$$a_1(t) = c_1 e^{\lambda_1 t}, \quad ...., \quad a_n(t) = c_n e^{\lambda_n t}, \quad t \in (-\infty, \infty).$$

The proof is done. □

Remark 0.26 ***(Important.)*** *In case there is a initial condition $\mathbf{x}(0) = \mathbf{x}_0$, then one just solve for $c_1, \ ..., \ c_n$ so that*

$$c_1 v_1 + \cdots + c_n v_n = \mathbf{x}_0.$$

In matrix form we have (note that the matrix P is ***invertible*** *and $D(0) = I$)*

$$\mathbf{x}(0) = PD(0) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = P \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{x}_0 \ (\text{column vector}) \quad \text{or} \quad \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = P^{-1}\mathbf{x}_0.$$

Hence the unique solution of the equation $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, is given by

$$\mathbf{x}(t) = PD(t) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{PD(t) P^{-1}\mathbf{x}_0}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

We summarize the above result as a theorem and prove it using a different "**diagonalization**" method.

Theorem 0.27 *Assume $A \in M(n)$ has n **distinct real** eigenvalues $\lambda_1, \ ..., \ \lambda_n$ with corresponding eigenvectors $v_1, \ ..., \ v_n$ and let $P = (v_1, \ ..., \ v_n)$. Then the **unique solution** to the initial value problem*

$$\mathbf{x}'(t) = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{15}$$

is given by

$$\mathbf{x}(t) = \underbrace{PD(t)P^{-1}}\mathbf{x}_0, \quad t \in (-\infty, \infty) \tag{16}$$

where $D(t)$ is the diagonal matrix $diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)$.

Remark 0.28 *In case the initial condition is $\mathbf{x}(t_0) = \mathbf{x}_0$, where $t_0 \neq 0$, then the unique solution is*

$$\mathbf{x}(t) = \underbrace{PD(t)D^{-1}(t_0)P^{-1}}\mathbf{x}_0, \quad t \in (-\infty, \infty). \tag{17}$$

Proof. (**Diagonalization method; change of variables.**) For convenience, we let D be the diagonal matrix $diag(\lambda_1, \ ..., \ \lambda_n)$. Assume $\mathbf{x}(t) \in \mathbb{R}^n$ is a solution to (15). There is a unique function $\mathbf{y}(t) \in \mathbb{R}^n$ such that

$$\mathbf{x}(t) = P\mathbf{y}(t), \quad t \in (-\infty, \infty), \quad (\text{just let } \mathbf{y}(t) = P^{-1}\mathbf{x}(t)),$$

i.e. in the above we do the linear change of variables $\mathbf{x} = P\mathbf{y}$. The function $\mathbf{y}(t)$ will also satisfy an ODE. We now have (note that $AP = PD$ since $Av_i = \lambda_i v_i$ for each i)

$$\underbrace{\mathbf{x}'(t)} = P\mathbf{y}'(t), \quad \underbrace{A\mathbf{x}(t)} = A(P\mathbf{y}(t)) = (AP)\mathbf{y}(t) = (PD)\mathbf{y}(t),$$

which gives

$$P\mathbf{y}'(t) = (PD)\mathbf{y}(t), \quad \text{i.e.,} \quad \mathbf{y}'(t) = \underbrace{P^{-1}AP}\mathbf{y}(t) = D\mathbf{y}(t). \tag{18}$$

Therefore the function $\mathbf{y}(t)$ satisfies the initial value problem

$$\mathbf{y}'(t) = D\mathbf{y}(t), \quad \mathbf{y}(0) = P^{-1}\mathbf{x}_0, \tag{19}$$

that is, the system for $\mathbf{y}(t)$ is now **decoupled** with (recall that we need to find eigenvectors $v_1, \ ..., \ v_n$ in order to do this change of variables)

$$\frac{dy_1}{dt} = \lambda_1 y_1, \quad \frac{dy_2}{dt} = \lambda_2 y_2, \ ...., \quad \frac{dy_n}{dt} = \lambda_n y_n \tag{20}$$

and it is given by

$$\mathbf{y}(t) = D(t)\underbrace{\mathbf{y}(0)} = D(t)\left(\underbrace{P^{-1}\mathbf{x}_0}\right) = D(t)P^{-1}\mathbf{x}_0.$$

Thus

$$\mathbf{x}(t) = P\mathbf{y}(t) = \underbrace{PD(t)P^{-1}}\mathbf{x}_0$$

and the proof is done. □

Example 0.29 *Consider the 2×2 linear system*

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{21}$$

The coefficients matrix has $\lambda_1 = 2, \ \lambda_2 = -1, \ v_1 = (1,1), \ v_2 = (1,4)$. Thus

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$\mathbf{x}(t) = PD(t)P^{-1}\mathbf{x}_0 = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{x}_0$$
$$= \frac{1}{3}\begin{pmatrix} 4e^{2t} - e^{-t} & -e^{2t} + e^{-t} \\ 4e^{2t} - 4e^{-t} & -e^{2t} + 4e^{-t} \end{pmatrix}\mathbf{x}_0, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is the solution of (21) with initial data $\mathbf{x}_0$. *One can also use the general solution formula*

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

and solve for $c_1,\ c_2$ *satisfying the system*

$$\mathbf{x}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \mathbf{x}_0.$$

2×2 **linear system with constant coefficients.**

In this section we want to find the general solution of the simple 2×2 system with constant coefficients:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2). \tag{22}$$

We already know its general solution if A has **two distinct real eigenvalues**. For other cases, we can use the result of **linear algebra** to help us.

We first recall the following general fact from linear algebra:

Lemma 0.30 *Let* $A,\ P \in M(n)$ *be two real matrices and* P *is* ***invertible****. Then the two matrices* A *and* $P^{-1}AP$ *have* ***the same eigenvalues****. If* λ *is an eigenvalue (*λ *can be* ***real*** *or* ***complex****) of* A *with corresponding eigenvector* $v \neq 0$ *(*v *can be* ***real eigenvector*** *or* ***complex eigenvector****), then* λ *is an eigenvalue of* $P^{-1}AP$ *with corresponding eigenvector* $P^{-1}v$*. Conversely, if* λ *is an eigenvalue (*λ *can be* ***real*** *or* ***complex****) of* $P^{-1}AP$ *with corresponding eigenvector* $v \neq 0$ *(*v *can be* ***real eigenvector*** *or* ***complex eigenvector****), then* λ *is an eigenvalue of* A *with corresponding eigenvector* Pv*.*

Proof. Assume λ is an eigenvalue of A with corresponding eigenvector v. We have

$$Av = \lambda v, \quad v \neq 0.$$

Since $P \in M(n)$ is **invertible**, there is a **unique vector** $w \neq 0$ such that $Pw = v$. Hence the above becomes

$$APw = \lambda Pw = P(\lambda w),$$

which implies

$$\left(P^{-1}AP\right)w = \lambda w, \quad w \neq 0.$$

The above identity says that λ is an eigenvalue of $P^{-1}AP$ with corresponding eigenvector $w = P^{-1}v$.

Conversely, if λ is an eigenvalue of $P^{-1}AP$ with corresponding eigenvector v. We have

$$\left(P^{-1}AP\right)v = \lambda v, \quad v \neq 0,$$

which gives

$$A(Pv) = P(\lambda v) = \lambda(Pv), \quad Pv \neq 0.$$

Hence λ is an eigenvalue of A with corresponding eigenvector Pv. □

Lemma 0.31 *If $A \in M(2)$, then there is an* ***invertible*** *real matrix P such that $P^{-1}AP$ has one of the forms (also known as* ***Jordan canonical form****)*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{23}$$

for some real numbers $\lambda,\ \mu,\ \alpha,\ \beta$ with $\beta \neq 0$.

Proof.

Case 1: $\lambda_1 \neq \lambda_2,\ \lambda_1,\ \lambda_2 \in \mathbb{R}$.

In this case we have the first form in (23) with $P = (v_1, v_2) \in M(2)$.

Case 2: $\lambda_1 = \lambda_2$ (denote it as λ), $\lambda \in \mathbb{R}$.

In this case, there exists a nonzero vector $v_1 \in \mathbb{R}^2$ such that $Av_1 = \lambda v_1$. Let K be the subspace of $\mathbb{R}^2$ given by

$$K = \left\{ v \in \mathbb{R}^2 : Av = \lambda v \right\} = \ker(A - \lambda I), \quad v_1 \in W. \tag{24}$$

If $K = \mathbb{R}^2$, then $A = \lambda I$ and for **any** invertible real matrix P we have

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \tag{25}$$

Hence we assume that $\dim K = 1$ and choose any **nonzero** vector $w \in \mathbb{R}^2$ which is **independent** to v_1. Then we have

$$Aw = \alpha v_1 + \beta w \quad \text{for some number} \quad \alpha \neq 0,\ \beta.$$

Note that if $\alpha = 0$, we will get a contradiction (if $\alpha = 0$, we get $Aw = \beta w$, which implies $\beta = \lambda$ and we have two independent eigenvectors v_1 and w; this gives a contradiction since now $\dim K = 1$). Hence $\alpha \neq 0$. As A has repeated eigenvalue λ, **the number β must be equal to λ**. To see this, note that $\{v_1, w\}$ are **independent** and we have

$$A(v_1, w) = (v_1, w) \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}, \quad \alpha \neq 0,$$

which gives

$$P^{-1}AP = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}, \quad \text{where} \quad P = (v_1, w). \tag{26}$$

The above implies that $P^{-1}AP$ has **eigenvalues** $\lambda,\ \beta$. By Lemma 0.30, A also has eigenvalues $\lambda,\ \beta$. Hence β must be equal to λ, i.e. $\beta = \lambda$. As a conclusion, we have

$$Aw = \alpha v_1 + \lambda w \ \ (\text{same as } \ (A - \lambda I)w = \alpha v_1), \quad \alpha \neq 0. \tag{27}$$

Now if we choose

$$v_2 = \frac{w}{\alpha}, \quad \alpha \neq 0,$$

we will get

$$Av_2 = v_1 + \lambda v_2 \quad (\text{same as } \ (A - \lambda I)v_2 = v_1). \tag{28}$$

This gives the second case if we let

$$P = (v_1, v_2) \in M(2) \quad (v_1,\ v_2 \text{ are column vectors}). \tag{29}$$

Remark 0.32 *(**Quick way to reduce** A **into Jordan canonical form in Case 2.**) Based on the above proof, if* $\dim\ker(A-\lambda I)=1$*, then we can find* $v_2\neq 0$ *satisfying*

$$\begin{cases} (A-\lambda I)\,v_1=0, & v_1\neq 0 \text{ is eigenvector,} \\ (A-\lambda I)\,v_2=v_1. \end{cases} \tag{30}$$

Moreover, $\{v_1,v_2\}$ *is **automatically independent** and is a basis of* $\mathbb{R}^2$*. Using the basis* $\{v_1,v_2\}$*, we have*

$$P^{-1}AP=\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Remark 0.33 *(**Important.**) In the textbook (see p. 433), the vector* v_2 *in (28) is called a **generalized eigenvector** for the map* $A:\mathbb{R}^2\to\mathbb{R}^2$*.Note that the map* $A-\lambda I:\mathbb{R}^2\to\mathbb{R}^2$ *maps the whole* $\mathbb{R}^2$ *onto the line* L *generated by* v_1 *(*L *is the eigenspace corresponding to* λ*) and then maps* L *onto* 0 *(draw a picture here). This is because any vector* $u\in\mathbb{R}^2$ *can be written as* $u=sv_1+tv_2$ *for some* $s,\ t\in\mathbb{R}$*, and then*

$$(A-\lambda I)\,u=(A-\lambda I)\,(sv_1+tv_2)=(A-\lambda I)\,(tv_2)=tv_1$$

and furthermore

$$(A-\lambda I)^2\,u=(A-\lambda I)\,(tv_1)=0,\quad \forall\ u\in\mathbb{R}^2.$$

Case 3: $\lambda_1=\alpha+i\beta,\ \lambda_2=\alpha-i\beta,\ \alpha,\ \beta\in\mathbb{R},\ \beta>0.$

Let $v_1=u+iw,\ v_2=u-iw,\ u,\ w\in\mathbb{R}^2,\ w\neq 0$, be **complex eigenvectors** corresponding to λ_1 and λ_2 respectively. Since **a complex eigenvalue cannot have a real eigenvector,** we must have $w\neq 0$ and

$$\begin{cases} Au=\alpha u-\beta w, \\ Aw=\beta u+\alpha w, \quad w\neq 0, \end{cases} \tag{31}$$

which also implies that $u,\ w$ are **linearly independent** in $\mathbb{R}^2$. To see this, **we first note that** $u\neq 0$. Otherwise, we have $0=-\beta w$, which is impossible since $\beta>0$ and $w\neq 0$. Next, if u is a **multiple** of w, say $u=\tau w$ for some $\tau\neq 0$, we will have

$$A\tau w=\alpha\tau w-\beta w,\qquad \text{i.e. } Aw=\left(\alpha-\frac{\beta}{\tau}\right)w,\quad \tau\neq 0,\quad w\neq 0,\quad \beta>0,$$

which says that A has a **real** eigenvalue $\alpha-\beta/\tau$, a contradiction again. Now choose $P=(u,w)$, which is **invertible**, and (31) implies

$$A\,(u,w)=(u,w)\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\quad P^{-1}AP=\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{32}$$

which gives the third case. □

Remark 0.34 *(**Important.**) Note that the matrix*

$$J:=\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

assume $J^TJ=(\alpha^2+\beta^2)\,I$*. Therefore,* J *is like an **orthogonal matrix** but with a **dilation** by* $\sqrt{\alpha^2+\beta^2}$ *(by definition, an orthogonal matrix* M *satisfies* $M^TM=I$*). One can also write* J *as*

$$J=\sqrt{\alpha^2+\beta^2}\begin{pmatrix} \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} & \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \\ -\frac{\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \end{pmatrix}=\sqrt{\alpha^2+\beta^2}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We see that it is a ***clockwise rotation*** *with angle* θ*, together with a* ***dilation*** *by* $\sqrt{\alpha^2+\beta^2}$ *(explain this). On the other hand, the matrix*

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

corresponds to a ***counterclockwise rotation*** *with angle* θ*, together with a* ***dilation*** *by* $\sqrt{\alpha^2+\beta^2}$ *(explain this).*

Remark 0.35 ***(Important.)*** *In some textbook, the Jordan canonical form of* A *in the case* $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$*, is preferred to have the form*

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{33}$$

From the above proof, if we choose the basis of $\mathbb{R}^2$ *as* $\{w, u\}$ *instead of* $\{u, w\}$ *(recall that* $u + iw$ *is the eigenvector corresponding to* $\alpha + i\beta$*), we will get*

$$A(w,u) = (w,u)\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad P = (w,u), \tag{34}$$

which gives (33). The reason of preferring (33) is that if we look at the matrix multiplication

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta y \\ \beta x + \alpha y \end{pmatrix},$$

then it corresponds to the ***complex number multiplication***

$$(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y).$$

On the other hand, the matrix multiplication

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ -\beta x + \alpha y \end{pmatrix} \tag{35}$$

corresponds to the complex number multiplication

$$(\alpha - i\beta)(x + iy) = (\alpha x + \beta y) + i(-\beta x + \alpha y). \tag{36}$$

Hence we conclude

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \approx \alpha + i\beta, \qquad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \approx \alpha - i\beta. \tag{37}$$

In my teaching, I prefer the Jordan canonical form in (23), which has the advantage that we choose P *as* (u, w) *instead of* (w, u)*.*

By Lemma 0.31, the general solution of (22) when A has repeated eigenvalue λ can be derived. We have:

Theorem 0.36 *Let* $A \in M(2)$*. Consider the* 2×2 *linear system*

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^2, \end{cases} \tag{38}$$

where A *has* ***2 repeated eigenvalues*** λ *(and* $A \neq \lambda I$*) with corresponding eigenvector* v_1 *and generalized eigenvector* v_2 *(i.e.* $Av_1 = \lambda v_1$ *and* $Av_2 = v_1 + \lambda v_2$ *(same as* $(A - \lambda)v_1 = 0$ *and* $(A - \lambda)v_2 = v_1$*). Then the* ***unique solution*** *is given by*

$$\mathbf{x}(t) = P\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}P^{-1}\mathbf{x}_0, \quad t \in (-\infty, \infty), \tag{39}$$

where P is the 2×2 invertible matrix given by

$$P = (v_1, v_2), \qquad P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{40}$$

Note that the matrix identity in (40) is equivalent to the system

$$\begin{cases} Av_1 = \lambda v_1, \\ Av_2 = v_1 + \lambda v_2, \end{cases} \tag{41}$$

and $\mathbf{x}(t)$ can also be written as the vector form

$$\mathbf{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}(t v_1 + v_2), \quad t \in (-\infty, \infty), \tag{42}$$

where $c_1,\ c_2$ solve

$$c_1 v_1 + c_2 v_2 = \mathbf{x}_0 \quad \textit{(this is same as } P^{-1}\mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\textit{)}. \tag{43}$$

Remark 0.37 *(**Important.**) In summary, the general solution of the above system is*

$$\mathbf{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}(t v_1 + v_2), \quad t \in (-\infty, \infty), \tag{44}$$

*where $c_1,\ c_2$ are arbitrary constants and the two nonzero vectors $v_1,\ v_2$ satisfy (41). If you choose different nonzero vectors $v_1,\ v_2$ satisfying (41), you will get the **same** general solution formula as in (44).*

Proof. From (18), we know that if we do the change of variables $\mathbf{x}(t) = P\mathbf{y}(t)$ for some $P \in M(2)$, then $\mathbf{y}(t)$ will satisfy the equation $\mathbf{y}'(t) = \underbrace{P^{-1}AP}\mathbf{y}(t)$ with $P^{-1}\mathbf{x}_0 = \mathbf{y}(0)$. Now we choose $P = (v_1, v_2)$ with

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

then the equation for $\mathbf{y}(t)$ becomes

$$\begin{cases} y_1'(t) = \lambda y_1(t) + y_2(t) \\ y_2'(t) = \lambda y_2(t), \end{cases}, \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}. \tag{45}$$

which is **half-decoupled (good enough)**. We can solve $y_2(t)$ first to get $y_2(t) = e^{\lambda t} y_2(0)$ and obtain

$$y_1'(t) = \lambda y_1(t) + e^{\lambda t} y_2(0),$$

which gives

$$y_1(t) = e^{\lambda t}[y_1(0) + t y_2(0)].$$

We conclude

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \mathbf{y}(0).$$

Therefore,

$$\begin{aligned} \mathbf{x}(t) &= P \underbrace{\begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}} P^{-1}\mathbf{x}_0 = (v_1, v_2) \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \left(e^{\lambda t} v_1, t e^{\lambda t} v_1 + e^{\lambda t} v_2\right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}(t v_1 + v_2). \end{aligned} \tag{46}$$

The proof is done. □

Example 0.38 *(**2 repeated real eigenvalues.**) Find the general solution of the equation*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}, \quad t \in (-\infty, \infty). \tag{47}$$

<u>**Solution**</u>:

We have $\lambda_1 = \lambda_2 = 2$. For eigenvector corresponding to $\lambda = 2$, we solve $(A - \lambda I) v_1 = 0$ to get

$$\begin{cases} x - y = 2x \\ x + 3y = 2y \end{cases} \quad \text{(looking at just one equation is enough !!!)}$$

and get $x+y = 0$, which gives $v_1 = (1, -1)$. For generalized eigenvector v_2, we solve $(A - \lambda I) v_2 = v_1$ (same as $Av_2 = v_1 + \lambda v_2$) to get

$$\begin{cases} x - y = 2x + 1 \\ x + 3y = 2y - 1 \end{cases} \quad \text{(looking at just one equation is enough !!!)}$$

and get $x + y = -1$ (this line is **parallel to** the line $x + y = 0$), which gives (**just pick one solution**) $v_2 = (-1, 0)$. We conclude

$$Av_1 = 2v_1, \quad Av_2 = v_1 + 2v_2.$$

Hence

$$P = (v_1, v_2) \text{ (each is column vector)} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},$$

and

$$P^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ (\textbf{Jordan canonical form})}.$$

Let $\mathbf{x} = P\mathbf{y}$ to get

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which is now **half-decoupled**. The general solution for $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} (t v_1 + v_2) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right). \tag{48}$$

□

Finally, by Lemma 0.31 again, the general solution of (22) when A has complex eigenvalues can be derived. We have:

Theorem 0.39 *Consider the 2×2 linear system*

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^2, \end{cases} \tag{49}$$

*where A has **2 complex conjugate eigenvalues** $\alpha + i\beta,\ \alpha - i\beta,\ \alpha \in \mathbb{R},\ \beta > 0$, with corresponding eigenvectors $u + iw,\ u - iw,\ u,\ w \in \mathbb{R}^2$. Then the unique solution is given by*

$$\mathbf{x}(t) = P \begin{pmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ -e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix} P^{-1}\mathbf{x}_0, \quad t \in (-\infty, \infty), \tag{50}$$

where P is the 2×2 invertible matrix given by

$$P = (u, w), \qquad P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{51}$$

Note that the matrix identity in (51) is equivalent to the system

$$\begin{cases} Au = \alpha u - \beta w, \\ Aw = \beta u + \alpha w, \end{cases} \tag{52}$$

and $\mathbf{x}(t)$ can also be written as the vector form

$$\mathbf{x}(t) = c_1 e^{\alpha t} [(\cos\beta t) u - (\sin\beta t) w] + c_2 e^{\alpha t} [(\sin\beta t) u + (\cos\beta t) w], \quad t \in (-\infty, \infty), \tag{53}$$

where c_1, c_2 solve

$$c_1 u + c_2 w = \mathbf{x}_0 \quad (\text{same as } P^{-1}\mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}). \tag{54}$$

Remark 0.40 *If c_1, c_2 are arbitrary in (53), we get general solution of the ODE $\mathbf{x}'(t) = A\mathbf{x}(t)$.*

Proof. Again, let $\mathbf{x}(t) = P\mathbf{y}(t)$, where $P = (u, w)$. We have

$$\mathbf{y}'(t) = \underbrace{P^{-1}AP}\mathbf{y}(t) = J\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \text{where } J = P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

i.e. we get the system

$$\begin{cases} y_1'(t) = \alpha y_1(t) + \beta y_2(t) \\ y_2'(t) = -\beta y_1(t) + \alpha y_2(t), \end{cases} \tag{55}$$

which is, unfortunately, still **coupled** together. However, for this particular type of 2×2 system (we get this particular form (55) due to the help of linear algebra), we can use **complex exponential function** to help us. We can look at the identity (recall the correspondence between the matrix multiplication by J and the complex number multiplication by $(\alpha - i\beta)$; see (35) and (36))

$$\begin{aligned} &\frac{d}{dt}\left(\underbrace{y_1(t) + iy_2(t)}\right) \\ &= (\alpha y_1(t) + \beta y_2(t)) + i(-\beta y_1(t) + \alpha y_2(t)) = (\alpha - i\beta)\left(\underbrace{y_1(t) + iy_2(t)}\right), \end{aligned} \tag{56}$$

which is a **complex scalar ODE** and it gives the complex solution

$$y_1(t) + iy_2(t) = Ce^{(\alpha - i\beta)t}, \quad C \in \mathbb{C} \tag{57}$$

for some **complex constant** $C = c_1 + ic_2 \in \mathbb{C}$, where c_1, $c_2 \in \mathbb{R}$. By expansion, we have

$$\begin{aligned} y_1(t) + iy_2(t) &= (c_1 + ic_2)\left[\left(e^{\alpha t}\cos\beta t\right) - i\left(e^{\alpha t}\sin\beta t\right)\right] \\ &= \left(c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t\right) + i\left(-c_1 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t\right) \end{aligned}$$

and conclude the general solution $(y_1(t), y_2(t))$ for (55), which is

$$\begin{cases} y_1(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \\ y_2(t) = -c_1 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t, \quad t \in (-\infty, \infty). \end{cases} \tag{58}$$

Hence we conclude

$$\begin{aligned} \mathbf{x}(t) = P\mathbf{y}(t) &= P\begin{pmatrix} c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \\ -c_1 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t \end{pmatrix} \\ &= \left(c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t\right) u + \left(-c_1 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t\right) w \\ &= c_1 e^{\alpha t}[(\cos\beta t) u - (\sin\beta t) w] + c_2 e^{\alpha t}[(\sin\beta t) u + (\cos\beta t) w], \quad t \in (-\infty, \infty). \end{aligned}$$

The proof is done. □

Example 0.41 *Find the general solution of the equation*

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \mathbf{x}, \quad t \in (-\infty, \infty). \tag{59}$$

<u>**Solution**</u>: The matrix has two complex conjugate eigenvalues $\lambda = 2 \pm i = \alpha \pm i\beta,\ \alpha = 2,\ \beta = 1$. Solve

$$\begin{cases} 3x - 2y = (2+i)\,x \\ x + y = (2+i)\,y \quad \text{(this equation looks easier)} \end{cases}$$

to get $x = (1+i)\,y$. Hence a complex eigenvector for $2+i$ is

$$v = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u + iw.$$

According to the proof, if we let

$$P = (u, w) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \end{aligned}$$

which is a canonical form. By (53), the general solution is given by

$$\mathbf{x}(t) = c_1 e^{\alpha t} \left[(\cos \beta t)\, u - (\sin \beta t)\, w \right] + c_2 e^{\alpha t} \left[(\sin \beta t)\, u + (\cos \beta t)\, w \right], \quad t \in (-\infty, \infty), \tag{60}$$

where $u = (1, 1),\ w = (1, 0)$. □

0.0.2 A special 2×2 linear system with variable coefficients of the form $t\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, $t \in (0, \infty)$.

In this section, we look at a special first order system of the form

$$t\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{where} \quad A \in M(2), \quad t \in (0, \infty). \tag{61}$$

To solve (61), similar to the Euler equation, we use the change of variables method. Let $s \in (-\infty, \infty)$ be the new variable given by $s = \log t,\ t \in (0, \infty)$ (this is a one-one onto relation between $t \in (0, \infty)$ and $s \in (-\infty, \infty)$). Then $\mathbf{x}(t)$ becomes $\tilde{\mathbf{x}}(s)$, i.e. $\tilde{\mathbf{x}}(\log t) = \mathbf{x}(t)$. By the chain rule, we have

$$\mathbf{x}'(t) = \tilde{\mathbf{x}}'(s) \frac{ds}{dt} = \frac{1}{t} \tilde{\mathbf{x}}'(s) \quad (\text{same as } \tilde{\mathbf{x}}'(s) = t\mathbf{x}'(t))$$

and so the new equation for $\tilde{\mathbf{x}}(s)$ becomes the standard one:

$$\tilde{\mathbf{x}}'(s) = t\mathbf{x}'(t) = A\mathbf{x} = A\tilde{\mathbf{x}}(s), \quad s \in (-\infty, \infty). \tag{62}$$

Once you know the general solution $\tilde{\mathbf{x}}(s)$ for (62), you can know the general solution $\mathbf{x}(t)$ for (61) by the identity $\mathbf{x}(t) = \tilde{\mathbf{x}}(\log t),\ t \in (0, \infty)$.

0.0.3 Invariant subspace of a general first-order linear system.

Before we look at the 3×3 linear system, we prove the following important result for a general $n \times n$ linear system.

Lemma 0.42 *Let $A \in M(n)$ and let V be a k-dimensional subspace of $\mathbb{R}^n$ which is **invariant** under A (i.e., if $v \in V$, then $Av \in V$), where $1 \leq k \leq n-1$. Let $\mathbf{x}(t)$, $t \in (-\infty, \infty)$, be the solution of the system*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n. \end{cases} \tag{63}$$

Then if $\mathbf{x}_0 \in V$, we will have

$$\mathbf{x}(t) \in V \quad \textit{for all} \quad t \in (-\infty, \infty). \tag{64}$$

Remark 0.43 *If we view $A : \mathbb{R}^n \to \mathbb{R}^n$ as a **vector field** on $\mathbb{R}^n$, then the above result is intuitively clear.*

Solution:

Let $s = n - k$ and pick an **orthonormal basis** $\{v_1, v_2, ..., v_k, w_1, ..., w_s\}$ of $\mathbb{R}^n$ with $\{v_1, v_2, ..., v_k\} \subset V$. One can express $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = a_1(t) v_1 + \cdots + a_k(t) v_k + b_1(t) w_1 + \cdots + b_s(t) w_s, \quad t \in (-\infty, \infty)$$

with

$$b_1(0) = \cdots = b_s(0) = 0 \quad (\text{since } \mathbf{x}(0) \in V).$$

We have for each $1 \leq i \leq s$ the following

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{x}(t), w_i \rangle &= b_i'(t) = \langle A\mathbf{x}(t), w_i \rangle, \\ &= \left\langle A \left(\underbrace{a_1(t) v_1 + \cdots + a_k(t) v_k} + b_1(t) w_1 + \cdots + b_s(t) w_s \right), w_i \right\rangle \\ &= \langle A(b_1(t) w_1 + \cdots + b_s(t) w_s), w_i \rangle = \sum_{j=1}^{s} b_j(t) \langle A w_j, w_i \rangle, \end{aligned}$$

i.e.

$$\begin{cases} b_i'(t) = \sum_{j=1}^{s} b_j(t) \langle A w_j, w_i \rangle = \sum_{j=1}^{s} \langle w_i, A w_j \rangle b_j(t), \quad \forall\, 1 \leq i \leq s \\ b_1(0) = \cdots = b_s(0) = 0. \end{cases} \tag{65}$$

Note that (65) is an $s \times s$ **system of first-order linear equation** for $b_1(t), \ ...\ , b_s(t)$, with **zero** initial condition. By existence and uniqueness theorem, we have

$$b_1(t) \equiv \cdots = b_s(t) \equiv 0, \quad \forall\, t \in (-\infty, \infty), \tag{66}$$

and so

$$\mathbf{x}(t) = a_1(t) v_1 + \cdots + a_k(t) v_k \in V, \quad \forall\, t \in (-\infty, \infty). \tag{67}$$

The proof is done. $\square$

0.0.4 3×3 linear system with constant coefficients.

We now moved to the 3×3 linear system with constant coefficients:

$$\mathbf{x}'(t) = A\mathbf{x}, \quad \text{where } A \in M(3), \quad \mathbf{x}(t) = (x(t), y(t), z(t)), \quad t \in (-\infty, \infty). \tag{68}$$

We want to find its general solution or a particular solution with $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is the initial condition.

In the following, we want to use the "**diagonalization method** (**change of variables method**)" to solve it. Denote the three eigenvalues of A by $\lambda_1,\ \lambda_2$ and λ_3. We have several cases to consider.

The case when $\lambda_1,\ \lambda_2,\ \lambda_3$ are real and distinct.

This is the easiest case. Let $v_1,\ v_2,\ v_3$ be the eigenvectors corresponding to $\lambda_1,\ \lambda_2,\ \lambda_3$. They are independent. The general solution is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3, \tag{69}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants.

The case when $\lambda_1 = \lambda,\ \lambda_2 = \lambda_3 = \sigma,\ \lambda \neq \sigma,\ \lambda,\ \sigma \in \mathbb{R}$.

In this case, we have $\dim\ker(A - \lambda I) = 1$, but the dimension of $\ker(A - \sigma I)$ can be either 1 or 2.

<u>**Case 1**</u>: The eigenspace $\ker(A - \sigma I)$ has dimension 2.

If we can find **two independent** eigenvectors $v_2,\ v_3$ for the repeated eigenvalue σ, then the three eigenvectors $v_1,\ v_2,\ v_3$ are **independent** in $\mathbb{R}^3$ (because if $v_1 = av_2 + bv_3$ for some constants $a,\ b$, we will get a contradiction) and we can **diagonalize** A as

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \quad P = (v_1, v_2, v_3).$$

Then we are in the previous easy case. The **general solution** is given by (one can also use change of variables $\mathbf{x}(t) = P\mathbf{y}(t),\ \mathbf{y}'(t) = (P^{-1}AP)\mathbf{y}(t)$ to see the following)

$$\mathbf{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\sigma t} v_2 + c_3 e^{\sigma t} v_3, \tag{70}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants (to see this, use the change of variables $\mathbf{x}(t) = P\mathbf{y}(t)$ and see that the three equations for $y_1(t),\ y_2(t),\ y_3(t)$ are decoupled).

Example 0.44 *Do Example 4 in p. 403.*

<u>**Solution**</u>:

The matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

has three eigenvalues $\lambda_1 = 2,\ \lambda_2 = \lambda_3 = -1$. For $\lambda_1 = 2$, we solve

$$\begin{cases} y + z = 2x \\ x + z = 2y \\ x + y = 2z \end{cases}$$

and get the relation $x = y = z$ and so we pick $v_1 = (1,1,1)$. For $\lambda = \lambda_2 = \lambda_3 = -1$, we solve

$$\begin{cases} y + z = -x \\ x + z = -y \\ x + y = -z \end{cases}$$

and get the relation $x + y + z = 0$. Hence the eigenspace corresponding to $\lambda = -1$ has dimension 2. One can pick 2 independent eigenvectors $v_2,\ v_3$ for $\lambda = -1$ as

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

By theory, the general solution of the system is given by

$$\mathbf{x}(t) = c_1 e^{2t} v_1 + c_2 e^{-t} v_2 + c_3 e^{-t} v_3,$$

where $\{v_1, v_2, v_3\}$, given above, form a basis in $\mathbb{R}^3$. □

__Case 2__: The eigenspace $\ker(A - \sigma I)$ has dimension 1.

If we can find only **one independent** eigenvector for the repeated eigenvalue σ, then we **cannot** diagonalize the matrix A. Its **Jordan canonical form** is given by:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{71}$$

This is due to the following:

Lemma 0.45 *(**The eigenspace** $\ker(A - \sigma I)$ **has dimension** 1.) Let $A \in M(3)$ with three real eigenvalues $\lambda,\ \sigma,\ \sigma$, where $\lambda \neq \sigma$. Assume $\dim\ker(A - \sigma I) = 1$ (i.e. we can find only **one independent** eigenvector for eigenvalue σ). Then one can find a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ satisfying*

$$(A - \lambda I) v_1 = 0, \qquad (A - \sigma I) v_2 = 0, \qquad (A - \sigma I) v_3 = v_2. \tag{72}$$

As a consequence, we have

$$A(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}, \tag{73}$$

i.e.

$$P^{-1} A P = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}, \quad \text{where} \quad P = (v_1, v_2, v_3) \in M(3). \tag{74}$$

Proof. Clearly we can find two independent eigenvectors $v_1,\ v_2$ such that $Av_1 = \lambda v_1,\ Av_2 = \sigma v_2$, where $\lambda \neq \sigma$. Now we look at the linear transformation

$$A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3. \tag{75}$$

Let $K = \ker(A - \sigma I)$, $R = \operatorname{Im}(A - \sigma I)$. We have $\dim K = 1$ and by the **Rank Theorem** in Linear Algebra (applied to the linear transformation $A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3$), we know that $\dim R = 2$. Hence K is a **line** and R is a **plane** in $\mathbb{R}^3$.

Step 1: Show that $K \subset R$.

We now claim that $K \subset R$. We prove by contradiction and assume $K \not\subseteq R$. Note that the operator $A - \sigma I : R \to R$ is a map from R to R (this is because it maps the whole $\mathbb{R}^3$ onto R, hence it also maps R to R). Now by the identity

$$A = (A - \sigma I) + \sigma I$$

we see that $A : R \to R$ too. Since $A : R \to R$ is a linear map with $\dim R = 2$, on it we have two eigenvalues. Since we assume $K \not\subseteq R$ (which implies that on R one cannot find any nonzero vector $v \neq 0$ with $Av = \sigma v$), both eigenvalues $r_1,\ r_2$ of $A : R \to R$ on R must be **different from** σ and the only possibility is that $r_1 = r_2 = \lambda$. This gives a contradiction (since the three eigenvalues of $A : \mathbb{R}^3 \to \mathbb{R}^3$ are $\lambda,\ \sigma,\ \sigma$). This contradiction implies that $K \subset R$.

Step 2: Show that the equation $(A - \sigma I)\, v_3 = v_2$ has a solution $v_3 \neq 0 \in \mathbb{R}^3$.

As $K \subset R$ (R is the image of $A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3$) and $v_2 \in K \subset R$, there **exists** some vector $v_3 \neq 0 \in \mathbb{R}^3$ (we called v_3 a **generalized eigenvector** of v_2) such that

$$(A - \sigma I)\, v_3 = v_2 \in K \subset R.$$

Step 3: Show that $\{v_1,\ v_2,\ v_3\}$ are independent in $\mathbb{R}^3$.

At this moment, we have obtained three nonzero vectors $v_1,\ v_2,\ v_3$, satisfying

$$(A - \lambda I)\, v_1 = 0, \qquad (A - \sigma I)\, v_2 = 0, \qquad (A - \sigma I)\, v_3 = v_2, \quad \text{where } \lambda \neq \sigma, \tag{76}$$

and we already know that $v_1,\ v_2$ are independent. Now we can infer that $v_1 \in R$ also. This can be easily seen from the identity

$$(A - \sigma I)\, v_1 = Av_1 - \sigma v_1 = (\lambda - \sigma)\, v_1 \in R, \quad \lambda \neq \sigma.$$

Therefore, $\{v_1, v_2\}$ is actually a **basis of the plane** R and if v_3 lies on the plane R, i.e.

$$v_3 = \alpha v_1 + \beta v_2 \quad \text{for some } \alpha,\ \beta,$$

then applying $A - \sigma I$ onto it we can get

$$v_2 = (A - \sigma I)\, v_3 = (A - \sigma I)\,(\alpha v_1 + \beta v_2) = \alpha\,(A - \sigma I)\, v_1 = \alpha\,(\lambda - \sigma)\, v_1,$$

a contradiction. Therefore $\{v_1, v_2, v_3\}$ is **linearly independent in** $\mathbb{R}^3$ and we have (74). The proof is done.

Remark 0.46 *Draw a geometric picture for the above proof.*

Remark 0.47 *(**The picture of the linear map** $A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3$ **acting on the basis** $\{v_1, v_2, v_3\}$.) We have*

$$\begin{cases} v_1 \overset{A-\sigma I}{\to} (\lambda - \sigma)\, v_1 \overset{A-\sigma I}{\to} (\lambda - \sigma)^2\, v_1 \\ v_2 \overset{A-\sigma I}{\to} 0 \overset{A-\sigma I}{\to} 0 \\ v_3 \overset{A-\sigma I}{\to} v_2 \overset{A-\sigma I}{\to} 0. \end{cases}$$

The vectors $\{v_1, v_3\}$ are mapped (by the linear map $A - \sigma I$) into R and the vector v_2 is mapped (by the linear map $A - \sigma I$) into 0.

Remark 0.48 *(**Quick way to reduce** A **into Jordan canonical form in Lemma 0.45.**) In conclusion, we need to solve* $v_1,\ v_2,\ v_3$ *satisfying the system:*

$$(A-\lambda I)\,v_1=0,\qquad (A-\sigma I)\,v_2=0,\qquad (A-\sigma I)\,v_3=v_2. \tag{77}$$

Then from the proof we see that the third equation $(A-\sigma I)\,v_3=v_2$ ***has a solution*** v_3 *and* $\{v_1,v_2,v_3\}$ *is **automatically** linearly independent. Moreover, for fixed chosen* v_2*, the set* $\{v_3\in\mathbb{R}^3:(A-\sigma I)\,v_3=v_2\}$ ***is a line parallel to the line*** $K=\ker(A-\sigma I)$ *(check this simple property by yourself).*

Theorem 0.49 *(**The eigenspace** $\ker(A-\sigma I)$ **has dimension** 1.) Let* $A\in M(3)$ *with three real eigenvalues* $\lambda,\ \sigma,\ \sigma$*, where* $\lambda\neq\sigma$*. Assume* $\dim\ker(A-\sigma I)=1$*. Consider the* 3×3 *linear system*

$$\begin{cases}\mathbf{x}'(t)=A\mathbf{x}\\[2mm] \mathbf{x}(0)=\mathbf{x}_0\in\mathbb{R}^3.\end{cases} \tag{78}$$

Then the unique solution is given by

$$\mathbf{x}(t)=P\begin{pmatrix} e^{\lambda t} & 0 & 0\\ 0 & e^{\sigma t} & te^{\sigma t}\\ 0 & 0 & e^{\sigma t}\end{pmatrix}P^{-1}\mathbf{x}_0, \tag{79}$$

where P *is the* 3×3 *invertible matrix given by*

$$P=(v_1,v_2,v_3)\,,\qquad P^{-1}AP=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \sigma & 1\\ 0 & 0 & \sigma\end{pmatrix}. \tag{80}$$

Here the three vectors $v_1,\ v_2,\ v_3$ *satisfy*

$$(A-\lambda I)\,v_1=0,\qquad (A-\sigma I)\,v_2=0,\qquad (A-\sigma I)\,v_3=v_2. \tag{81}$$

Note that $\mathbf{x}(t)$ *can also be written as the vector form*

$$\mathbf{x}(t)=c_1e^{\lambda t}v_1+c_2e^{\sigma t}v_2+c_3e^{\sigma t}(tv_2+v_3)\,, \tag{82}$$

where $c_1,\ c_2,\ c_3$ *solve*

$$c_1v_1+c_2v_2+c_3v_3=\mathbf{x}_0\quad \text{(this is same as } P^{-1}\mathbf{x}_0=\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}\text{)}. \tag{83}$$

Remark 0.50 *In (81), we call* v_3 *a **generalized eigenvector** for the map* $A:\mathbb{R}^3\to\mathbb{R}^3$*. See Remark 0.33 also.*

Proof. Let $P=(v_1,v_2,v_3)$. In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x}=P\mathbf{y}$) becomes the following "**half-decoupled system**":

$$\mathbf{y}'(t)=\left(P^{-1}AP\right)\mathbf{y}(t)=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \sigma & 1\\ 0 & 0 & \sigma\end{pmatrix}\mathbf{y}(t)\,,\quad \mathbf{y}(t)=\begin{pmatrix}y_1(t)\\ y_2(t)\\ y_3(t)\end{pmatrix}, \tag{84}$$

which gives the **half-decoupled** system

$$\begin{cases} y_1'(t)=\lambda y_1(t)\\[2mm] y_2'(t)=\sigma y_2(t)+y_3(t)\\[2mm] y_3'(t)=\sigma y_3(t)\end{cases}\qquad \mathbf{y}(t)=\begin{pmatrix}y_1(t)\\ y_2(t)\\ y_3(t)\end{pmatrix}.$$

One can solve $y_1(t) = c_1 e^{\lambda t}$ and $y_3(t) = c_3 e^{\sigma t}$ directly and for $y_2(t)$ it satisfies

$$y_2'(t) = \sigma y_2(t) + y_3(t) = \sigma y_2(t) + c_3 e^{\sigma t},$$

which gives

$$y_2(t) = (c_2 + c_3 t) e^{\sigma t},$$

where in the above $c_1,\ c_2,\ c_3$ are integration constants. Hence we get

$$\mathbf{y}(t) = \begin{pmatrix} c_1 e^{\lambda t} \\ (c_2 + c_3 t) e^{\sigma t} \\ c_3 e^{\sigma t} \end{pmatrix}, \quad t \in (-\infty, \infty)$$

and the general solution to the ODE is given by

$$\begin{aligned} \mathbf{x}(t) = P\mathbf{y}(t) = (v_1, v_2, v_3)\,\mathbf{y}(t) &= (v_1, v_2, v_3) \begin{pmatrix} c_1 e^{\lambda t} \\ (c_2 + c_3 t) e^{\sigma t} \\ c_3 e^{\sigma t} \end{pmatrix} \\ &= c_1 e^{\lambda t} v_1 + (c_2 + c_3 t) e^{\sigma t} v_2 + c_3 e^{\sigma t} v_3 \\ &= c_1 e^{\lambda t} v_1 + c_2 e^{\sigma t} v_2 + c_3 e^{\sigma t} (t v_2 + v_3). \end{aligned} \tag{85}$$

The proof is done. □

Example 0.51 *(**The eigenspace** $\ker(A - \sigma I)$ **has dimension** 1.) Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

The polynomial $\det(A - \lambda I) = 0$ is given by $-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$, i.e. $(\lambda - 2)(\lambda - 1)^2 = 0$. The eigenvalues of the coefficient matrix are $\lambda = 2,\ \sigma = 1,\ \sigma = 1$. To find the eigenvector for $\lambda = 2$, we solve

$$\begin{cases} y = 2x \\ z = 2y \\ 2x - 5y + 4z = 2z, \end{cases}$$

which gives the line $y = 2x,\ z = 4x$ and so $v_1 = (1, 2, 4)$. To find the eigenvector for the repeated $\sigma = 1$, we solve

$$\begin{cases} y = x \\ z = y \\ 2x - 5y + 4z = z, \end{cases}$$

to get the relation $x = y = z$ and we pick one eigenvector $v_2 = (1, 1, 1)$. As it is impossible to find another independent eigenvector, we have to find generalized eigenvector. We solve

$$\begin{cases} y = x + 1 \\ z = y + 1 \\ 2x - 5y + 4z = z + 1 \end{cases}$$

to get the relation $y = x + 1,\ z = x + 2$ (this is a line **parallel to** the line $x = y = z$ since any two solutions $u,\ w$ of the equation $(A - \sigma I) v_3 = v_2$ satisfy $(A - \sigma I)(u - w) = 0$ and so $u - w$ is a multiple of v_2) and obtain a **generalized eigenvector** $v_3 = (1, 2, 3)$ (or other possible answers). We see that $v_1,\ v_2,\ v_3$ are linearly independent.

The general solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{t} \left[t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right].$$

□

Example 0.52 *(**The eigenspace** $\ker(A - \sigma I)$ **has dimension** 2.) Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & -3 & 1 \\ 4 & -3 & 2 \\ 6 & -9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

We have

$$\begin{aligned} &\begin{vmatrix} 5-\lambda & -3 & 1 \\ 4 & -3-\lambda & 2 \\ 6 & -9 & 6-\lambda \end{vmatrix} \\ &= (5-\lambda)(-3-\lambda)(6-\lambda) - 36 - 36 + 6(3+\lambda) + 12(6-\lambda) + 18(5-\lambda) \\ &= 6\lambda^2 - 12\lambda - 90 - \lambda^3 + 2\lambda^2 + 15\lambda - 72 + 18 + 6\lambda + 72 - 12\lambda + 90 - 18\lambda \\ &= -\left(\lambda^3 - 8\lambda^2 + 21\lambda - 18\right) = -(\lambda - 2)(\lambda - 3)^2. \end{aligned}$$

Hence we have $\lambda = 2,\ \sigma = 3,\ \sigma = 3$. For $\lambda = 2$, we solve

$$\begin{cases} 5x - 3y + z = 2x \\ 4x - 3y + 2z = 2y \\ 6x - 9y + 6z = 2z, \end{cases}$$

i.e.

$$\begin{cases} 3x - 3y + z = 0 \\ 4x - 5y + 2z = 0 \\ 6x - 9y + 4z = 0 \end{cases}$$

to get the relation $y = 2x,\ z = 3x$, which is a **line**, and obtain $v_1 = (1, 2, 3)$. For $\sigma = 3$, we solve

$$\begin{cases} 5x - 3y + z = 3x \\ 4x - 3y + 2z = 3y \\ 6x - 9y + 6z = 3z, \end{cases}$$

i.e.

$$\begin{cases} 2x - 3y + z = 0 \\ 4x - 6y + 2z = 0 \\ 6x - 9y + 3z = 0 \end{cases}$$

to get the relation $2x - 3y + z = 0$, which is a **plane**, and obtain **two** linearly independent eigenvectors $v_2 = (3, 2, 0)$, $v_3 = (0, 1, 3)$. The general solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

□

The case when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

__Case 1__: The eigenspace $\ker(A - \lambda I)$ has dimension 3.

In such a case, the matrix $A \in M(3)$ must have the form $A = \lambda I$. We can ignore this trivial case.

__Case 2__: The eigenspace $\ker(A - \lambda I)$ has dimension 2.

Assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $\dim\ker(A - \lambda I) = 2$. This means that we can find **two** linearly independent eigenvectors of λ. We claim the following:

Lemma 0.53 *(**The eigenspace** $\ker(A - \lambda I)$ **has dimension** 2.) Let $A \in M(3)$ with three equal real eigenvalues $\lambda,\ \lambda,\ \lambda$. Assume $\dim\ker(A - \lambda I) = 2$. Then we can find a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ satisfying*

$$(A - \lambda I)\, v_1 = 0, \qquad (A - \lambda I)\, v_2 = 0, \qquad (A - \lambda I)\, v_3 = v_2,$$

*where $v_1,\ v_2 \in K$ are **eigenvectors** with $v_1 \notin R,\ v_2 \in R,\ R = \operatorname{Im}(A - \lambda I)$, and v_3 is a **generalized eigenvector** of v_2. As a consequence, we have*

$$A(v_1, v_2, v_3) = (v_1, v_2, v_3)\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{86}$$

i.e.

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \textit{where} \quad P = (v_1, v_2, v_3) \in M(3). \tag{87}$$

Proof. Let $K = \ker(A - \lambda I)$, $\dim K = 2$, $R = \operatorname{Im}(A - \lambda I)$. By the **Rank Theorem**, we know $\dim R = 1$. Hence K is a plane and R is a line.

Step 1: Show that $R \subset K$**.**

To see this, note that the operator $A - \lambda I : R \to R$ (this is because it maps the whole $\mathbb{R}^3$ onto R; hence it also maps R to R). Since $\dim R = 1$, we must have for any $v \neq 0 \in R$ the following

$$(A - \lambda I)\, v = \mu v \quad \text{for some} \quad \mu \in \mathbb{R}.$$

If $\mu \neq 0$, then A has eigenvalue $\lambda + \mu$, a contradiction. Hence $(A - \lambda I)\, v = 0$ and $v \in K$. Thus $R \subset K$.

Step 2: Choose basis $\{v_1, v_2, v_3\}$.

Now we **choose two linearly independent vectors** $v_1,\ v_2$ in $K = \ker(A - \lambda I)$ **with** $v_1 \notin R,\ v_2 \in R$ (**this step is crucial** !!). Then, since $v_2 \in R$, there exists some nonzero vector $v_3 \in \mathbb{R}^3$ such that

$$(A - \lambda I)\, v_3 = v_2 \in R. \tag{88}$$

Such we have $(A - \lambda I)\, v_3 = v_2 \neq 0$, the vector $v_3 \notin K$ and so it is independent to $v_1,\ v_2$ (note that $\{v_1, v_2\}$ is a basis of K). Therefore, $\{v_1, v_2, v_3\}$ are **independent** and we have identity (86).

The proof of Lemma 0.53 is done. □

Remark 0.54 *(**Quick way to reduce** A **into Jordan canonical form in Lemma 0.53.**) In conclusion, we need to solve* $v_1,\ v_2,\ v_3$ *satisfying the system:*

$$(A-\lambda I)\,v_1=0,\qquad (A-\lambda I)\,v_2=0,\qquad (A-\lambda I)\,v_3=v_2,\qquad v_1,\ v_2\in K, \tag{89}$$

and also require $v_1\notin R,\ v_2\in R,\ R=\mathrm{Im}\,(A-\lambda I)$. *The third equation* $(A-\lambda I)\,v_3=v_2$ ***does have a solution*** $v_3\in\mathbb{R}^3$ ***due to*** $v_2\in R$. *Now, automatically, the vectors* $v_1,\ v_2,\ v_3$ *are independent in* $\mathbb{R}^3$. *In solving ODE* $\mathbf{x}'=A\mathbf{x}$, ***you need to find the space*** $R=\mathrm{Im}\,(A-\lambda I)$ ***first*** *in order to find* $v_1\notin R,\ v_2\in R$.

Remark 0.55 *Note that we have*

$$(A-\lambda I)^2\,v=0\quad \textit{for all}\quad v\in\mathbb{R}^3. \tag{90}$$

That is:

$$\mathbb{R}^3\overset{A-\lambda I}{\rightarrow}R\ (R\subset K)\overset{A-\lambda I}{\rightarrow}0. \tag{91}$$

Let $P=(v_1,v_2,v_3)$. In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x}=P\mathbf{y}$) becomes the following "**half-decoupled system**":

$$\mathbf{y}'(t)=\left(P^{-1}AP\right)\mathbf{y}(t)=\begin{pmatrix}\lambda&0&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}\mathbf{y}(t),\quad \mathbf{y}(t)=\begin{pmatrix}y_1(t)\\y_2(t)\\y_3(t)\end{pmatrix} \tag{92}$$

and the general solution to the ODE is given by

$$\begin{aligned}\mathbf{x}(t)&=P\mathbf{y}(t)=(v_1,v_2,v_3)\,\mathbf{y}(t)=(v_1,v_2,v_3)\begin{pmatrix}c_1e^{\lambda t}\\(c_2+c_3t)\,e^{\lambda t}\\c_3e^{\lambda t}\end{pmatrix}\\&=c_1e^{\lambda t}v_1+c_2e^{\lambda t}v_2+c_3e^{\lambda t}\,(tv_2+v_3)\,.\end{aligned} \tag{93}$$

The discussion for this case is done.

We summarize the above in the following:

Theorem 0.56 *(**The eigenspace** $\ker(A-\lambda I)$ **has dimension** 2.) Let* $A\in M(3)$ *with three equal real eigenvalues* $\lambda,\ \lambda,\ \lambda$. *Assume* $\dim\ker(A-\lambda I)=2$. *Consider the* 3×3 *linear system*

$$\begin{cases}\mathbf{x}'(t)=A\mathbf{x}\\[2mm]\mathbf{x}(0)=\mathbf{x}_0\in\mathbb{R}^3.\end{cases} \tag{94}$$

Then the unique solution is given by

$$\mathbf{x}(t)=P\begin{pmatrix}e^{\lambda t}&0&0\\0&e^{\lambda t}&te^{\lambda t}\\0&0&e^{\lambda t}\end{pmatrix}P^{-1}\mathbf{x}_0, \tag{95}$$

where P *is the* 3×3 *invertible matrix given by*

$$P=(v_1,v_2,v_3)\,,\qquad P^{-1}AP=\begin{pmatrix}\lambda&0&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix} \tag{96}$$

and $v_1,\ v_2$ *are* ***eigenvectors*** *with* $v_1\notin R,\ v_2\in R,\ R=\mathrm{Im}\,(A-\lambda I)$, *and* v_3 *is a* ***generalized eigenvector*** *of* v_2, *i.e. we have*

$$(A-\lambda I)\,v_1=0,\qquad (A-\lambda I)\,v_2=0,\qquad (A-\lambda I)\,v_3=v_2. \tag{97}$$

Note that $\mathbf{x}(t)$ *can also be written as the vector form*

$$\mathbf{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2 + c_3 e^{\lambda t} (t v_2 + v_3), \tag{98}$$

where $c_1,\ c_2,\ c_3$ *solve*

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{x}_0 \quad \textit{(this is same as } P^{-1}\mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}\textit{).} \tag{99}$$

Example 0.57 *Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

The characteristic polynomial of the coefficient matrix is

$$\begin{aligned} &\begin{vmatrix} 5-\lambda & -3 & -2 \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} \\ &= (5-\lambda)(-5-\lambda)(3-\lambda) - 48 - 48 + 8(5+\lambda) + 12(5-\lambda) + 24(3-\lambda) \\ &= 3\lambda^2 - \lambda^3 + 1 - 3\lambda = -(\lambda-1)^3. \end{aligned}$$

Hence we have $\lambda_1 = \lambda_2 = \lambda_3 = 1$. To find the eigenvector for $\lambda = 1$, we solve

$$\begin{cases} 5x - 3y - 2z = x \\ 8x - 5y - 4z = y \\ -4x + 3y + 3z = z \end{cases}$$

and obtain $4x - 3y - 2z = 0$. Thus one can find **two** linearly independent eigenvectors $v_1,\ v_2$. The space $K = \ker(A - I)$ is given by the plane $4x - 3y - 2z = 0$.

To go further, we need to find the **image space** $R = \operatorname{Im}(A - I)$ first. The **image** of the matrix (which is the space of all possible linear combinations of the three column vectors)

$$A - \lambda I = A - I = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

is a line R given by $\{t(1,2,-1) : t \in (-\infty,\infty)\}$. We note that $R \subset K$ since the vector $(1,2,-1)$ lies on the plane $4x - 3y - 2z = 0$.

According to the proof, we must choose two linearly independent vectors $v_1,\ v_2$ **in** K **with** $v_1 \notin R,\ v_2 \in R$. Thus we choose $v_1 = (3,4,0),\ v_2 = (1,2,-1)$. Finally, we solve $(A - I)v_3 = v_2$ (same as $Av_3 = v_3 + v_2$) to get

$$\begin{cases} 5x - 3y - 2z = x + 1 \\ 8x - 5y - 4z = y + 2 \\ -4x + 3y + 3z = z - 1 \end{cases}$$

and get $4x-3y-2z=1$ (this is a plane **parallel to** $4x-3y-2z=0$, since any two solutions $u,\ w$ of the equation $(A-I)\,v_3=v_2$ satisfy $(A-I)\,(u-w)=0$ and so $u-w$ lies on the plane $4x-3y-2z=0$). So we choose $v_3=(0,1,-2)$. We see that $v_1,\ v_2,\ v_3$ are linearly independent.

The general solution is given by

$$\mathbf{x}\,(t)=c_1e^t\begin{pmatrix}3\\4\\0\end{pmatrix}+c_2e^t\begin{pmatrix}1\\2\\-1\end{pmatrix}+c_3e^t\left(t\begin{pmatrix}1\\2\\-1\end{pmatrix}+\begin{pmatrix}0\\1\\-2\end{pmatrix}\right). \tag{100}$$

□

Remark 0.58 *(**Be careful !!**) If we do not choose $v_2\in R$, then the system $(A-I)\,v_3=v_2$ **may not have** a solution. For example, choose $v_2=(3,4,0)\in K,\ v_2\notin R$. Then we solve*

$$\begin{cases}5x-3y-2z=x+3\\ 8x-5y-4z=y+4\\ -4x+3y+3z=z+0\end{cases}$$

*and see that there is **no solution** for v_3 at all.*

<u>**Case 3**</u>: The eigenspace $\ker(A-\lambda I)$ has dimension 1.

Assume $\lambda_1=\lambda_2=\lambda_3=\lambda$ and $\dim\ker(A-\lambda I)=1$. This means that we can find **only one** independent eigenvector of λ. In this case Lemma 0.53 becomes:

Lemma 0.59 *(**The eigenspace** $\ker(A-\lambda I)$ **has dimension** 1.) Let $A\in M(3)$ with three equal real eigenvalues $\lambda,\ \lambda,\ \lambda$. Assume $\dim\ker(A-\lambda I)=1$. Then we can find a basis $\{v_1,v_2,v_3\}$ of $\mathbb{R}^3$ satisfying*

$$(A-\lambda I)\,v_1=0,\qquad (A-\lambda I)\,v_2=v_1,\qquad (A-\lambda I)\,v_3=v_2. \tag{101}$$

*where v_1 is **eigenvector** and $v_2,\ v_3$ are **generalized eigenvector** of $v_1,\ v_2$ respectively. As a consequence, we have*

$$A\,(v_1,v_2,v_3)=(v_1,v_2,v_3)\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}, \tag{102}$$

i.e.

$$P^{-1}AP=\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix},\quad \textit{where}\quad P=(v_1,v_2,v_3)\in M(3). \tag{103}$$

Proof. Let $K=\ker(A-\lambda I)$, $R=\operatorname{Im}(A-\lambda I)$, $\dim K=1$. By the **Rank Theorem**, we know $\dim R=2$. Hence K is a **line** and R is a **plane** in $\mathbb{R}^3$. Let $v_1\neq 0\in K$ be an eigenvector of A for λ.

Step 1: **Show that** $K\subset R$.

Assume $K\not\subset R$. We will derive a contradiction. Note that on the vector space R the linear map $A-\lambda I:R\to R$ has two eigenvalues $\sigma_1,\ \sigma_2$, **both are nonzero (since we assume** $K\not\subset R$**)**, which will imply that $A:\mathbb{R}^3\to\mathbb{R}^3$ has eigenvalues $\lambda+\sigma_1$ and $\lambda+\sigma_2$, both are different from λ, a contradiction. This contradiction implies that $K\subset R$ (and we also have $\sigma_1=\sigma_2=0$; however, $A-\lambda I:R\to R$ is not a zero map, otherwise we will have $R\subset K$, a contradiction).

Step 2: Show that $(A-\lambda I)R=K$.

Next we claim that $(A-\lambda I)R=K$. To see this, note that $A-\lambda I:R\to R$ and $(A-\lambda I)R$ is **one-dimensional** due to $K\subset R$ (here we apply the Rank Theorem to the linear map $A-\lambda I:R\to R$, where $\dim R=2$). Therefore, $(A-\lambda I)R$ is a line L in the plane R spanned by some nonzero vector $v\neq 0$, where $v\in L\subset R$. But then we have

$$(A-\lambda I)v=t_0 v \quad \text{for some} \quad t_0\in\mathbb{R}$$

and if $t_0\neq 0$, the map $A:\mathbb{R}^3\to\mathbb{R}^3$ has new eigenvalue $\lambda+t_0$, a contradiction. Hence we must have $t_0=0$ and then $(A-\lambda I)v=0$, meaning that $v\in K$. As a result, we have $L=K$ and so $(A-\lambda I)R=K$.

Step 3: Choose basis $\{v_1,v_2,v_3\}$.

Now let $v_1\in K$ be an eigenvector of A. By $(A-\lambda I)R=K$, there exists some nonzero vector $v_2\in R,\ v_2\notin K$, such that (see Remark 0.63 below too)

$$(A-\lambda I)v_2=v_1,\quad v_1\in K. \tag{104}$$

Finally, since $v_2\in R$ and $R=\operatorname{Im}(A-\lambda I)$, there exists some nonzero vector $v_3\in\mathbb{R}^3$ such that

$$(A-\lambda I)v_3=v_2\quad (\text{note that now } (A-\lambda I)^2v_3=v_1). \tag{105}$$

Step 4: Show that $\{v_1,\ v_2,\ v_3\}$ **are automatically independent in** $\mathbb{R}^3$.

We then claim that $v_1,\ v_2,\ v_3$ are linearly independent. If not, then (we already know that $v_1,\ v_2$ are independent)

$$v_3=\alpha v_1+\beta v_2\in R\quad\text{for some } \alpha,\ \beta.$$

Applying $A-\lambda I$ onto it to get

$$v_2=(A-\lambda I)v_3=(A-\lambda I)(\alpha v_1+\beta v_2)=\beta(A-\lambda I)v_2=\beta v_1\in K,$$

a contradiction since $v_2\notin K$. Therefore we have $v_3\in\mathbb{R}^3,\ v_3\notin R$, and (102) cab be achieved. □

Remark 0.60 *Draw a diagram for the above proof.*

Remark 0.61 *We have the picture for the above proof:*

$$\mathbb{R}^3\overset{A-\lambda I}{\to}R\ (K\subset R)\overset{A-\lambda I}{\to}K\overset{A-\lambda I}{\to}0. \tag{106}$$

Remark 0.62 *In conclusion, we need to solve* $v_1,\ v_2,\ v_3$ *satisfying the system:*

$$(A-\lambda I)v_1=0,\qquad (A-\lambda I)v_2=v_1,\qquad (A-\lambda I)v_3=v_2. \tag{107}$$

Again, same as before, we say $v_2,\ v_3$ *are* ***generalized eigenvector*** *of* $v_1,\ v_2$ *respectively.*

Remark 0.63 ***(Important.)*** *We claim that when we solve the equation* $(A-\lambda I)v_2=v_1$, *we* ***automatically*** *have* $v_2\in R,\ v_2\notin K$ *(clearly* $v_2\neq 0$*). Clearly we first have* $v_2\notin K$, *otherwise we will have* $(A-\lambda I)v_2=0$ *and it implies* $v_1=0$. *Now we claim* $v_2\in R$. *To see this, assume we have* $v_2\in\mathbb{R}^3,\ v_2\notin R$, *such that*

$$(A-\lambda I)v_2=v_1,\quad \textit{where}\quad v_1\in K,\quad v_1\neq 0.$$

Then for **any** $v \in \mathbb{R}^3$ *there exists some vector* $w \in R$ *such that*

$$v = v_2 + w, \quad v_2 \notin R, \quad w \in R, \quad \dim R = 2.$$

This implies

$(A - \lambda I)\, v = (A - \lambda I)\,(v_2 + w) = v_1 + (A - \lambda I)\, w \in K$ *(here we use the property* $(A - \lambda I)\, R = K$*)*

and so $\dim \operatorname{Im}(A - \lambda I) = 1$, *a contradiction (since we already have* $\dim \operatorname{Im}(A - \lambda I) = 2$*). In view of this, if we solve the equation*

$$(A - \lambda I)\, v_2 = v_1, \quad \text{where} \quad v_1 \in K, \quad v_1 \neq 0.$$

then ***we automatically have*** $v_2 \in R,\ v_2 \notin K$. *Then one can go directly to find* $v_3 \in \mathbb{R}^3$ *such that*

$$(A - \lambda I)\, v_3 = v_2 \in R.$$

Clearly, we have $v_3 \neq 0$. *If* $v_3 \in R$, *we would have* $v_3 = \alpha v_1 + \beta v_2$ *for some* $\alpha,\ \beta \in \mathbb{R}$, *which gives*

$$v_2 = (A - \lambda I)\, v_3 = (A - \lambda I)\,(\alpha v_1 + \beta v_2) = (A - \lambda I)\,(\beta v_2) = \beta v_1,$$

a contradiction.

Remark 0.64 *In summary, we have the following: Assume* A *is a* 3×3 *real matrix with* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, *then if* $\ker(A - \lambda I)$ *has three independent eigenvectors, then*

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(in this case, } A = \lambda I\text{)}$$

and if $\ker(A - \lambda I)$ *has two independent eigenvectors, then*

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

and if $\ker(A - \lambda I)$ *has only one independent eigenvector, then*

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Let $P = (v_1, v_2, v_3)$. In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x} = P\mathbf{y}$) becomes the following **"half-decoupled system"**:

$$\mathbf{y}'(t) = \left(P^{-1}AP\right)\mathbf{y}(t) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \mathbf{y}(t), \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

and so

$$\begin{cases} y_1' = \lambda y_1 + y_2 \\ y_2' = \lambda y_2 + y_3 \\ y_3' = \lambda y_3. \end{cases}$$

We get $y_3(t) = c_3 e^{\lambda t}$ and

$$y_2(t) = (c_2 + c_3 t)\, e^{\lambda t}$$

and

$$y_1(t) = \left(c_1 + c_2 t + \frac{c_3}{2} t^2\right) e^{\lambda t}.$$

We conclude the general solution to the ODE

$$\begin{aligned}
\mathbf{x}(t) = P\mathbf{y}(t) = (v_1, v_2, v_3)\,\mathbf{y}(t) &= (v_1, v_2, v_3) \begin{pmatrix} \left(c_1 + c_2 t + \frac{c_3}{2} t^2\right) e^{\lambda t} \\ (c_2 + c_3 t)\, e^{\lambda t} \\ c_3 e^{\lambda t} \end{pmatrix} \\
&= \left(c_1 + c_2 t + \frac{c_3}{2} t^2\right) e^{\lambda t} v_1 + (c_2 + c_3 t)\, e^{\lambda t} v_2 + c_3 e^{\lambda t} v_3 \\
&= c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} (t v_1 + v_2) + c_3 e^{\lambda t} \left(\frac{t^2}{2} v_1 + t v_2 + v_3\right). \qquad (108)
\end{aligned}$$

Note that, in the above, v_1 is **eigenvector** and $v_2,\ v_3$ are **generalized eigenvectors**. The discussion for this case is done. □

Same as before, we can summarize the above in the following:

Theorem 0.65 *(**The eigenspace** $\ker(A - \lambda I)$ **has dimension** 1.) Let $A \in M(3)$ with three equal real eigenvalues $\lambda,\ \lambda,\ \lambda$. Assume $\dim\ker(A - \lambda I) = 1$. Consider the 3×3 linear system*

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^3. \end{cases} \qquad (109)$$

Then the unique solution is given by

$$\mathbf{x}(t) = P \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} & \frac{t^2}{2} e^{\lambda t} \\ 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix} P^{-1} \mathbf{x}_0, \qquad (110)$$

where P is the 3×3 invertible matrix given by

$$P = (v_1, v_2, v_3), \qquad P^{-1} A P = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}. \qquad (111)$$

Here the three vectors $v_1,\ v_2,\ v_3$ satisfy

$$(A - \lambda I)\, v_1 = 0, \qquad (A - \lambda I)\, v_2 = v_1, \qquad (A - \lambda I)\, v_3 = v_2. \qquad (112)$$

Note that $\mathbf{x}(t)$ can also be written as the vector form

$$\mathbf{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} (t v_1 + v_2) + c_3 e^{\lambda t} \left(\frac{t^2}{2} v_1 + t v_2 + v_3\right) \qquad (113)$$

where $c_1,\ c_2,\ c_3$ solve

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{x}_0 \quad \left(\textit{this is same as } P^{-1}\mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}\right). \qquad (114)$$

Example 0.66 *Find the general solution of the system*

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

The characteristic polynomial of the coefficient matrix is

$$\begin{aligned}
&\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 (4-\lambda) + 4 + 3 + 3(1-\lambda) + 2(1-\lambda) - 2(4-\lambda) \\
&= \left(\lambda^2 - 2\lambda + 1\right)(4-\lambda) + 4 - 3\lambda \\
&= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda-2)^3 .
\end{aligned}$$

Hence we have $\lambda_1 = \lambda_2 = \lambda_3 = 2$. To find the eigenvector for $\lambda = 2$, we solve

$$\begin{cases} x + y + z = 2x \\ 2x + y - z = 2y \\ -3x + 2y + 4z = 2z \end{cases}$$

and we obtain $x = 0,\ y + z = 0$. Thus we can find **only one** independent eigenvector $v_1 = (0, 1, -1)$. The image of the matrix

$$A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

is the plane R given by $x - y - z = 0$. We can see that the line $K = \{t(0,1,-1) : t \in \mathbb{R}\}$ lies in the plane R.

Then we solve $(A-2)\,v_2 = v_1$ to get

$$\begin{cases} x + y + z = 2x \\ 2x + y - z = 2y + 1 \\ -3x + 2y + 4z = 2z - 1 \end{cases}$$

and obtain $x = 1,\ y + z = 1$. We can pick $v_2 = (1, 1, 0)$ (note that here we have $v_2 \in R,\ v_2 \notin K$; which is always the case due to Remark 0.63). Finally, we solve $(A-2)\,v_3 = v_2$ to get

$$\begin{cases} x + y + z = 2x + 1 \\ 2x + y - z = 2y + 1 \\ -3x + 2y + 4z = 2z \end{cases}$$

and obtain $x = 2,\ y + z = 3$. We can pick $v_3 = (2, 3, 0)$. Hence the general solution is given by

$$\mathbf{x}(t) = c_1 e^{2t} v_1 + c_2 e^{2t} (t v_1 + v_2) + c_3 e^{2t} \left(\frac{t^2}{2} v_1 + t v_2 + v_3\right) = \cdots .$$

□

The case when $\lambda_1 = \lambda,\ \lambda_2 = \alpha + i\beta,\ \lambda_3 = \alpha - i\beta$.

Assume we have three eigenvalues $\lambda \in \mathbb{R}$ and $\alpha + i\beta,\ \alpha - i\beta$, where $\alpha,\ \beta \in \mathbb{R},\ \beta > 0$. Let $v_1 \in \mathbb{R}^3,\ v_1 \neq 0$, be an eigenvector of λ and let $v_2 + i v_3,\ v_2,\ v_3 \in \mathbb{R}^3,\ v_3 \neq 0$, be an eigenvector of

$\alpha + i\beta$. Then we know that $v_2 - iv_3$ is an eigenvector of $\alpha - i\beta$. Now v_1, v_2, v_3 satisfy the following identities:

$$Av_1 = \lambda v_1, \quad Av_2 = \alpha v_2 - \beta v_3, \quad Av_3 = \beta v_2 + \alpha v_3. \tag{115}$$

From it we immediately see that $v_2 \neq 0$ (otherwise we get $\beta v_3 = 0$, impossible).

Now we check that the three vectors $v_1,\ v_2,\ v_3$ satisfying (115) must be **independent** in $\mathbb{R}^3$. To begin with, they are all **nonzero** vectors and if $v_2 = \tau v_3$ for some $\tau \neq 0,\ \tau \in \mathbb{R}$, then the second and third equations in (115) give

$$\begin{cases} A\tau v_3 = \alpha\tau v_3 - \beta v_3 \text{ (same as } Av_3 = \left(\alpha - \frac{\beta}{\tau}\right) v_3, & \tau \neq 0, \quad \beta > 0 \\ Av_3 = \beta\tau v_3 + \alpha v_3 \text{ (same as } Av_3 = (\alpha + \beta\tau)\, v_3, & \tau \neq 0, \quad \beta > 0, \end{cases}$$

which says that A has **two different real eigenvalues** $\alpha - \beta/\tau,\ \alpha + \beta\tau$ (they are equal if and only if $\tau = \pm i$, impossible). Hence $v_2,\ v_3$ are independent (note that here the proof is slightly different from the 2×2 case).

Finally, if $v_1 = sv_2 + tv_3$ for some $s,\ t \in \mathbb{R}$ (at least one of them is not zero), then we will have

$$\underbrace{\lambda\,(sv_2 + tv_3)} = \lambda v_1 = Av_1 = A\,(sv_2 + tv_3) = \underbrace{s\,(\alpha v_2 - \beta v_3) + t\,(\beta v_2 + \alpha v_3)},$$

which gives

$$s\,(\alpha v_2 - \beta v_3) + t\,(\beta v_2 + \alpha v_3) - \lambda\,(sv_2 + tv_3) = 0,$$

i.e.

$$[s\,(\alpha - \lambda) + t\beta]\, v_2 + [t\,(\alpha - \lambda) - s\beta]\, v_3 = 0,$$

same as

$$\begin{cases} (\alpha - \lambda)\, s + \beta t = 0 \\ -\beta s + (\alpha - \lambda)\, t = 0. \end{cases} \tag{116}$$

Since the above 2×2 system of equations has a nonzero solution (s, t), we must have

$$\begin{vmatrix} \alpha - \lambda & \beta \\ -\beta & \alpha - \lambda \end{vmatrix} = 0 \quad \text{(same as } (\alpha - \lambda)^2 + \beta^2 = 0),$$

a contradiction. Therefore, $\{v_1, v_2, v_3\}$ is a **basis** of $\mathbb{R}^3$ and we have

$$AP = P \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}, \quad P = (v_1, v_2, v_3)\,, \quad P^{-1} \text{ exists.}$$

Let $\mathbf{x}\,(t) = P\mathbf{y}\,(t)$. The solution for the system $\mathbf{y}'\,(t) = (P^{-1}AP)\,\mathbf{y}\,(t)$ is

$$\begin{cases} y_1' = \lambda y_1 \\ y_2' = \alpha y_2 + \beta y_3 \\ y_3' = -\beta y_2 + \alpha y_3 \end{cases} \tag{117}$$

and the general solution is given by (see the proof of Theorem 0.39 for the general solutions of $y_2\,(t)$ and $y_3\,(t)$)

$$\begin{cases} y_1\,(t) = c_1 e^{\lambda t} \\ y_2\,(t) = c_2 e^{\alpha t} \cos \beta t + c_3 e^{\alpha t} \sin \beta t \\ y_3\,(t) = -c_2 e^{\alpha t} \sin \beta t + c_3 e^{\alpha t} \cos \beta t. \end{cases} \tag{118}$$

Hence the general solution $\mathbf{x}(t)$, $t \in (-\infty, \infty)$, to the ODE $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = P\mathbf{y}(t) = (v_1, v_2, v_3) \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\alpha t} \cos \beta t + c_3 e^{\alpha t} \sin \beta t \\ -c_2 e^{\alpha t} \sin \beta t + c_3 e^{\alpha t} \cos \beta t \end{pmatrix}$$
$$= c_1 e^{\lambda t} v_1 + c_2 e^{\alpha t} [(\cos \beta t) v_2 - (\sin \beta t) v_3] + c_3 e^{\alpha t} [(\sin \beta t) v_2 + (\cos \beta t) v_3]. \tag{119}$$

The discussion for this case is done. □

Example 0.67 *Find the general solution of the system*

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

The characteristic polynomial of the matrix is of the coefficient matrix are

$$\begin{vmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} = -\lambda^3 - 4\lambda^2 - 7\lambda - 6 = -(\lambda + 2)\left(\lambda^2 + 2\lambda + 3\right).$$

Hence the eigenvalues are $\lambda_1 = -2,\ \lambda_2 = -1 + \sqrt{2}i,\ \lambda_3 = -1 - \sqrt{2}i$. To find the eigenvector for $\lambda = -2$, we solve

$$\begin{cases} -3x + 2z = -2x \\ x - y = -2y \\ -2x - y = -2z \end{cases}$$

and we obtain $x = 2z,\ y = -2z$. Thus $v_1 = (2, -2, 1)$. To find the eigenvector for $\lambda = -1 + \sqrt{2}i$, we solve

$$\begin{cases} -3x + 2z = \left(-1 + \sqrt{2}i\right) x \\ x - y = \left(-1 + \sqrt{2}i\right) y \text{ (same as } x = \left(\sqrt{2}i\right) y) \\ -2x - y = \left(-1 + \sqrt{2}i\right) z. \end{cases}$$

We first note that $y \neq 0$. Otherwise, we will get zero eigenvector. By **scaling**, we can choose $y = 1$ and by the equation $x = \left(\sqrt{2}i\right) y$, we can get $x = \sqrt{2}i$ and plug it into the first equation to get

$$-3\sqrt{2}i + 2z = \left(-1 + \sqrt{2}i\right)\sqrt{2}i \text{ (same as } z = -1 + \sqrt{2}i).$$

Hence we get the complex eigenvector

$$v = \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 + \sqrt{2}i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} = v_2 + iv_3.$$

So we get $v_2 = (0, 1, -1)$ and $v_3 = \left(\sqrt{2}, 0, \sqrt{2}\right)$. The general solution is given by

$$\mathbf{x}(t) = c_1 e^{-2t} v_1 + c_2 e^{-t} \left[\left(\cos \sqrt{2}t\right) v_2 - \left(\sin \sqrt{2}t\right) v_3\right] + c_3 e^{-t} \left[\left(\sin \sqrt{2}t\right) v_2 + \left(\cos \sqrt{2}t\right) v_3\right],$$

where $v_1,\ v_2,\ v_3$ are given in the above. □

Summary of solving $\mathbf{x}' = A\mathbf{x}$ for $A \in M(3)$ with repeated real eigenvalues (read this section by yourself).

We only focus on the following three more difficult cases.

(1). Assume A has three real eigenvalues $\lambda,\ \sigma,\ \sigma$, where $\lambda \neq \sigma$; and $\dim\ker(A - \sigma I) = 1$.

First solve $(A - \lambda I)v_1 = 0,\ v_1 \neq 0$, and $(A - \sigma I)v_2 = 0,\ v_2 \neq 0$. In this case, we have

$$K = \ker(A - \sigma I) \subset R = \operatorname{Im}(A - \sigma I), \quad \dim K = 1, \quad \dim R = 2 \tag{120}$$

one can find $v_3 \in \mathbb{R}^3$ satisfying

$$(A - \sigma I)v_3 = v_2, \quad \text{where} \quad v_2 \in K \subset R.$$

These three vectors $v_1,\ v_2,\ v_3$ will **automatically form a basis in $\mathbb{R}^3$** and the general solution formula is given by

$$\mathbf{x}(t) = c_1 e^{\lambda t}v_1 + c_2 e^{\sigma t}v_2 + c_3 e^{\sigma t}(tv_2 + v_3), \tag{121}$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants.

Remark 0.68 *Conclusion:* $\dim K = 1,\ \dim R = 2,\ K \subset R$, *and*

$$\begin{cases} (A - \lambda I)v_1 = 0, \\ (A - \sigma I)v_2 = 0, \quad v_2 \in K \subset R \\ (A - \sigma I)v_3 = v_2, \quad v_3 \in \mathbb{R}^3. \end{cases} \tag{122}$$

The Jordan canonical form is

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{123}$$

(2). Assume A has three real eigenvalues $\lambda,\ \lambda,\ \lambda$; and $\dim\ker(A - \lambda I) = 2$.

In this case, we have

$$R = \operatorname{Im}(A - \lambda I) \subset K = \ker(A - \lambda I), \quad \dim R = 1, \quad \dim K = 2. \tag{124}$$

We then find nonzero vectors $v_1,\ v_2$ in $K = \ker(A - \lambda I)$ with $v_1 \notin R \subset K,\ v_2 \in R \subset K$ (**this step is crucial** !!) satisfying $(A - \lambda I)v_1 = 0$ and $(A - \lambda I)v_2 = 0$. Since $v_2 \in R$, one can find $v_3 \in \mathbb{R}^3$ satisfying

$$(A - \lambda I)v_3 = v_2 \in R \subset K. \tag{125}$$

These three vectors $v_1,\ v_2,\ v_3$ **will automatically form a basis in $\mathbb{R}^3$** and the general solution formula is given by

$$\mathbf{x}(t) = c_1 e^{\lambda t}v_1 + c_2 e^{\lambda t}v_2 + c_3 e^{\lambda t}(tv_2 + v_3), \tag{126}$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants.

Remark 0.69 *Conclusion:* $\dim R = 1,\ \dim K = 2,\ R \subset K$, *and*

$$\begin{cases} (A - \lambda I)v_1 = 0, \quad v_1 \in K, \quad v_1 \notin R \\ (A - \lambda I)v_2 = 0, \quad v_2 \in K, \quad v_2 \in R \\ (A - \lambda I)v_3 = v_2, \quad v_3 \in \mathbb{R}^3. \end{cases} \tag{127}$$

The Jordan canonical form is

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{128}$$

(3) . Assume A has three real eigenvalues $\lambda,\ \lambda,\ \lambda$; and $\dim\ker(A-\lambda I)=1$.

In this case, we have

$$K=\ker(A-\lambda I)\subset R=\operatorname{Im}(A-\lambda I),\quad \dim K=1,\quad \dim R=2,\quad (A-\lambda I)R=K. \tag{129}$$

We first solve $(A-\lambda I)v_1=0,\ v_1\in K$. Then by $(A-\lambda I)R=K$, we can solve

$$(A-\lambda I)v_2=v_1,\quad v_1\in K.$$

The solution v_2 will **automatically satisfy** $v_2\in R,\ v_2\notin K$. Since $v_2\in R$, one can find $v_3\in\mathbb{R}^3$ satisfying

$$(A-\lambda I)v_3=v_2\in R,\quad v_2\notin K.$$

These three vectors $v_1,\ v_2,\ v_3$ **will automatically form a basis** in $\mathbb{R}^3$ and the general solution formula is given by

$$\mathbf{x}(t)=c_1e^{\lambda t}v_1+c_2e^{\lambda t}(tv_1+v_2)+c_3e^{\lambda t}\left(\frac{t^2}{2}v_1+tv_2+v_3\right), \tag{130}$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants.

Remark 0.70 *Conclusion:* $\dim K=1,\ \dim R=2,\ K\subset R$, *and*

$$\begin{cases}(A-\lambda I)v_1=0, & v_1\in R,\quad v_1\in K\\ (A-\lambda I)v_2=v_1, & v_2\in R,\quad v_2\notin K\\ (A-\lambda I)v_3=v_2, & v_3\in\mathbb{R}^3.\end{cases} \tag{131}$$

The Jordan canonical form is

$$\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}. \tag{132}$$

Some facts from linear algebra and its applications to ODE.

In this section, we review some useful facts from linear algebra. They play important roles in the ODE $\mathbf{x}'=A\mathbf{x}$, where now $A\in M(n)$ is an $n\times n$ real matrix.

Definition 0.71 *A matrix $A\in M(n)$ is called **symmetric** if it satisfies $A^T=A$. If it satisfies $A^T=-A$, we say it is **anti-symmetric**.*

The following fact is interesting in Linear Algebra:

Lemma 0.72 ***Any** matrix $A\in M(n)$ can be decomposed as $A=B+C$, where $B\in M(n)$ is a symmetric matrix and $C\in M(n)$ is an anti-symmetric matrix.*

Remark 0.73 *Unfortunately, Lemma 0.72 does not help us too much in solving the equation $\mathbf{x}'(t)=A\mathbf{x}(t)$ for general $A\in M(n)$. However, if A is symmetric or anti-symmetric, the system $\mathbf{x}'(t)=A\mathbf{x}(t)$ has special interesting properties.*

Proof. We have

$$A=\frac{A+A^T}{2}+\frac{A-A^T}{2}:=B+C.$$

Then B is symmetric and C is anti-symmetric. □

A very important property for symmetric matrices is the following:

Lemma 0.74 *If $A \in M(n)$ is a* ***symmetric*** *matrix (i.e. $A^T = A$), then*

1. All eigenvalues of are **real**, i.e. it has n real eigenvalues $\lambda_1, \ldots \lambda_n$ (some may be repeated).
2. One can find n **orthonormal** eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ for these eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
3. Let $P = (v_1, \ldots, v_n)$. Then P is an **orthogonal** matrix with $P^T P = I$ (i.e. $P^T = P^{-1}$) and we have
$$P^{-1}AP = P^T AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{133}$$
The above says that an $n \times n$ **symmetric** matrix can always be **diagonalized**.

Proof. We will omit it (the proof is actually quite straightforward). See any linear algebra textbook. □

A consequence of the above lemma is:

Corollary 0.75 *Assume that $A \in M(n)$ is* ***symmetric****. Then any solution $\mathbf{x}(t)$ to the equation $\mathbf{x}' = A\mathbf{x}$ has the form*
$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n, \quad t \in (-\infty, \infty), \tag{134}$$
where $c_1, \ldots, c_n \in \mathbb{R}$ are arbitrary constants and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are the n eigenvalues of A, and $v_1, \ldots, v_n$ are their corresponding eigenvectors, which can be chosen to form an ***orthonormal basis*** *of $\mathbb{R}^n$. In particular, we have*
$$|\mathbf{x}(t)| = \sqrt{\left(c_1 e^{\lambda_1 t}\right)^2 + \cdots + \left(c_n e^{\lambda_n t}\right)^2}, \quad t \in (-\infty, \infty). \tag{135}$$

Another important result is the following:

Lemma 0.76 *If $A \in M(n)$ is an $n \times n$* ***anti-symmetric*** *matrix (i.e. $A^T = -A$), then*

1. *All diagonal elements of A are 0. In particular, $TrA = 0$.*
2. *All eigenvalues of A are either 0 or* ***pure imaginary****.*

Remark 0.77 *If $A \in M(n)$ is anti-symmetric, then A^2 is symmetric.*

Remark 0.78 *Assume $A \in M(n)$ is anti-symmetric and $Av = 0$ for some $v \neq 0 \in \mathbb{R}^n$ (i.e. v is eigenvector with eigenvalue $\lambda = 0$). Then for all $w \in \mathbb{R}^n$, we have $\langle Aw, v\rangle = 0$. This is due to*
$$\langle Aw, v\rangle = \langle w, A^T v\rangle = \langle w, -Av\rangle = \langle w, 0\rangle = 0. \tag{136}$$

Remark 0.79 *In particular, if $A \in M(2)$ is a nonzero matrix, then its two eigenvalues have the form $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$ for some $\beta > 0$. This is because A has the form*
$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \quad \beta \neq 0.$$

Proof. (1) is clear. For (2), let $\lambda \in \mathbb{R}$ be a ***real*** eigenvalue. Then there exists some nonzero $v \in \mathbb{R}^n$ such that $Av = \lambda v$. Hence
$$\lambda |v|^2 = \langle \lambda v, v\rangle_{\mathbb{R}^n} = \langle Av, v\rangle_{\mathbb{R}^n} = \langle v, A^T v\rangle_{\mathbb{R}^n} = \langle v, -Av\rangle_{\mathbb{R}^n} = \langle v, -\lambda v\rangle_{\mathbb{R}^n} = -\langle v, \lambda v\rangle_{\mathbb{R}^n} = -\lambda |v|^2,$$
which implies that $\lambda = 0$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

On the other hand, if λ is a **complex** eigenvalue, then there exists some nonzero **complex vector** $v \in \mathbb{C}^n$ such that $Av = \lambda v$. Using **complex inner product** $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ (see Remark 0.81 below) we have

$$\langle Av, v \rangle_{\mathbb{C}} = \left\langle v, \overline{A^T} v \right\rangle_{\mathbb{C}}$$

and so (note that A is a real matrix and so $\overline{A^T} = A^T = -A$)

$$\lambda |v|^2 = \langle \lambda v, v \rangle_{\mathbb{C}} = \langle Av, v \rangle_{\mathbb{C}} = \left\langle v, \overline{A^T} v \right\rangle_{\mathbb{C}} = \left\langle v, A^T v \right\rangle_{\mathbb{C}} = \langle v, -Av \rangle_{\mathbb{C}} = -\langle v, \lambda v \rangle_{\mathbb{C}} = -\bar{\lambda} |v|^2$$

and so $\lambda = -\bar{\lambda}$. Thus λ is **pure imaginary**. The proof is done. □

Remark 0.80 *(**Linear algebra fact.**) Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$ and we have $A = (a_{ij})_{1 \le i,j \le n} \in M(n)$. Then for any two vectors $u,\ w \in \mathbb{R}^n$, we have the identity*

$$\langle Au, w \rangle = \left\langle u, A^T w \right\rangle. \tag{137}$$

*To see this, since $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a **bilinear map**, it suffices to verify the identity*

$$\langle Ae_i, e_j \rangle = \left\langle e_i, A^T e_j \right\rangle, \quad 1 \le i,\ j \le n, \tag{138}$$

*where $\{e_1, \ldots, e_n\}$ is the **standard basis** of $\mathbb{R}^n$. However, (138) is clear since we have*

$$\langle Ae_i, e_j \rangle = \left\langle e_i, A^T e_j \right\rangle = a_{ij}, \quad 1 \le i,\ j \le n. \tag{139}$$

Remark 0.81 *The **(standard) complex inner product** $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ between two **complex vectors** $v = (v_1, \ldots, v_n),\ w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, is defined as*

$$\langle v, w \rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} v_i \bar{w}_i \in \mathbb{C}, \quad \bar{w}_i \text{ is the complex conjugate of } w_i. \tag{140}$$

Note that under this definition, we have

$$\langle w, v \rangle_{\mathbb{C}^n} = \overline{\langle v, w \rangle_{\mathbb{C}^n}}, \quad \langle v, v \rangle_{\mathbb{C}^n} = |v|^2 \ge 0, \quad \forall\ v \in \mathbb{C}^n \tag{141}$$

and

$$\langle \lambda v, v \rangle_{\mathbb{C}^n} = \lambda |v|^2, \quad \langle v, \lambda v \rangle_{\mathbb{C}^n} = \bar{\lambda} |v|^2, \quad \lambda \in \mathbb{C}. \tag{142}$$

*Let $C(n)$ denote **the space of all** $n \times n$ **complex matrices**. Using the above definition, one can check the identity*

$$\langle Av, w \rangle_{\mathbb{C}^n} = \left\langle v, \overline{A^T} w \right\rangle_{\mathbb{C}^n} \quad \text{for any} \quad A \in C(n), \quad v,\ w \in \mathbb{C}^n.$$

For simplicity, you can look at the case $n = 2$ and use a 2×2 complex matrix A to verify it.

Application of lemma 0.76 to ODE is the following:

Lemma 0.82 *(**Preserving inner product.**) Assume that A is an $n \times n$ **anti-symmetric** real matrix. Then for any two solutions $\mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t) \in \mathbb{R}^n$ to the linear system of equations*

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad A \in M(n), \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad t \in (-\infty, \infty), \tag{143}$$

*their inner product $\left\langle \mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t) \right\rangle$ is **independent** of time. In particular, if $\mathbf{x}(t)$ is a solution of (143), its length $|\mathbf{x}(t)|$ is a **constant** for all $t \in (-\infty, \infty)$.*

Proof. By

$$\frac{d}{dt}\left\langle \mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\right\rangle$$
$$= \left\langle A\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\right\rangle + \left\langle \mathbf{x}^{(1)}(t), A\mathbf{x}^{(2)}(t)\right\rangle = \left\langle \mathbf{x}^{(1)}(t), \left(A^T + A\right)\mathbf{x}^{(2)}(t)\right\rangle = 0,$$

the conclusion is proved. □

Example 0.83 ***(Important.)*** *Assume that A is a 3×3* ***anti-symmetric*** *nonzero real matrix, given by*

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in M(3), \quad \det A = 0. \tag{144}$$

We know its eigenvalues are 0, $\pm\beta i$, where $\beta = \sqrt{a^2+b^2+c^2} > 0$. Denote the eigenvector corresponding to 0 as v and denote the complex eigenvector corresponding to βi as $u + iw$, $u \neq 0$, $w \neq 0 \in \mathbb{R}^3$. We have

$$Av = 0, \quad Au = -\beta w, \quad Aw = \beta u \tag{145}$$

and the Jordan canonical form of A is

$$P^{-1}AP = J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}, \quad P = (v, u, w) \in M(3).$$

The general solution $\mathbf{x}(t)$ of the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ is given by (denote the eigenvector corresponding to 0 as v)

$$\mathbf{x}(t) = c_1 v + c_2\left[(\cos\beta t)\, u - (\sin\beta t)\, w\right] + c_3\left[(\sin\beta t)\, u + (\cos\beta t)\, w\right], \tag{146}$$

where c_1, c_2, c_3 are arbitrary constants. In particular, we get the following three real solutions

$$\begin{cases} \mathbf{x}^{(1)}(t) = v, \\ \mathbf{x}^{(2)}(t) = (\cos\beta t)\, u - (\sin\beta t)\, w, \\ \mathbf{x}^{(3)}(t) = (\sin\beta t)\, u + (\cos\beta t)\, w, \quad t \in (-\infty, \infty). \end{cases}$$

Since the inner product

$$\left\langle \mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\right\rangle = (\cos\beta t)\langle v, u\rangle - (\sin\beta t)\langle v, w\rangle, \quad \beta > 0$$

is ***independent*** *of time, we must have*

$$\langle v, u\rangle = \langle v, w\rangle = 0. \tag{147}$$

Also, since the inner product

$$\left\langle \mathbf{x}^{(2)}(t), \mathbf{x}^{(3)}(t)\right\rangle = \langle(\cos\beta t)\, u - (\sin\beta t)\, w, (\sin\beta t)\, u + (\cos\beta t)\, w\rangle$$
$$= (\cos\beta t)(\sin\beta t)\left[\langle u, u\rangle - \langle w, w\rangle\right] + \left[(\cos\beta t)^2 - (\sin\beta t)^2\right]\langle u, w\rangle$$

is ***independent*** *of time, we must have*

$$\langle u, u\rangle = \langle w, w\rangle, \quad \langle u, w\rangle = 0. \tag{148}$$

Therefore, if we choose v, u, w to be ***unit vectors****, then the matrix $P = (v, u, w)$ will have the property $P^T P = I$, i.e. it is an* ***orthogonal matrix*** *and we have $P^T AP = J$. Note: use (145) to describe the geometric meaning of the linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$.*

Lemma 0.84 *Let $A,\ B$ be two $n\times n$ real matrices with $B=(\mathbf{b}_1,\ ...,\ \mathbf{b}_n)$ where each $\mathbf{b}_i$ is a column vector. Then*

$$\det(A\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\det(\mathbf{b}_1,\ A\mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\cdots+\det(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ A\mathbf{b}_n)=TrA\cdot\det B, \quad (149)$$

where TrA denotes the trace of A.

Proof. Define the map $F:\mathbb{R}^n\times\cdots\times\mathbb{R}^n\to\mathbb{R}$ by

$$\begin{aligned}&F(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)\\ &:=\det(A\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\det(\mathbf{b}_1,\ A\mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\cdots+\det(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ A\mathbf{b}_n).\end{aligned}$$

One can check that F is an **alternating multilinear map** with $F(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)=0$ if $\mathbf{b}_i=\mathbf{b}_j$ for some $i\neq j$ (explain this). Similarly, the map

$$G(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n):=TrA\cdot\det B=TrA\cdot\det(\mathbf{b}_1,\ ...,\ \mathbf{b}_n)$$

is also an **alternating multilinear map** with $G(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)=0$ if $\mathbf{b}_i=\mathbf{b}_j$ for some $i\neq j$. Hence it suffices to check that the identity (149) holds for the case $B=(\mathbf{e}_1,\ ...,\ \mathbf{e}_n)$, where $\{\mathbf{e}_1,\ ...,\ \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. In such a case, we have (use cancellation to explain why we have $F(\mathbf{e}_1,\ ...,\ \mathbf{e}_n)=TrA$)

$$F(\mathbf{e}_1,\ ...,\ \mathbf{e}_n)=G(\mathbf{e}_1,\ ...,\ \mathbf{e}_n)=TrA.$$

The proof is done. □

Example 0.85 *We look at the simple case when $n=2$. Let $A\in M(2)$ be a fixed matrix. Define $F:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ by the following:*

$$F(\mathbf{b}_1,\mathbf{b}_2)=\det(A\mathbf{b}_1,\mathbf{b}_2)+\det(\mathbf{b}_1,A\mathbf{b}_2),\quad \mathbf{b}_1\in\mathbb{R}^2,\quad \mathbf{b}_2\in\mathbb{R}^2.$$

It is a bilinear map, satisfying

$$\begin{cases}F(\mathbf{b}_2,\mathbf{b}_1)=\det(A\mathbf{b}_2,\mathbf{b}_1)+\det(\mathbf{b}_2,A\mathbf{b}_1)=-F(\mathbf{b}_1,\mathbf{b}_2)\ \textit{(alternating)},\quad \forall\ \mathbf{b}_1\in\mathbb{R}^2,\ \mathbf{b}_2\in\mathbb{R}^2\\ F(\mathbf{b},\mathbf{b})=\det(A\mathbf{b},\mathbf{b})+\det(\mathbf{b},A\mathbf{b})=0,\quad \forall\ \mathbf{b}\in\mathbb{R}^2\\ F(\mathbf{e}_1,\mathbf{e}_2)=\det(A\mathbf{e}_1,\mathbf{e}_2)+\det(\mathbf{e}_1,A\mathbf{e}_2)=TrA.\end{cases}$$

Lemma 0.86 *Let $t\in I$ (some interval) and let $A(t)=(\mathbf{a}_1(t),\mathbf{a}_2(t),\ ...,\mathbf{a}_n(t))$ be an **invertible** time-dependent $n\times n$ real matrix, where each $\mathbf{a}_i(t)$ is a column vector. Then we have the identity*

$$\begin{aligned}&\frac{d}{dt}\det A(t)\\ &=Tr\left((A(t))^{-1}\frac{dA}{dt}(t)\right)\det A(t)=Tr\left(\left(\frac{dA}{dt}(t)\right)(A(t))^{-1}\right)\det A(t),\quad t\in I,\end{aligned}\quad (150)$$

where $(dA/dt)(t)$ is defined as

$$\frac{dA}{dt}(t)=(\mathbf{a}_1'(t),\mathbf{a}_2'(t),\ ...,\mathbf{a}_n'(t)),\quad t\in I.\quad (151)$$

Remark 0.87 *Note that for any two matrices $M,\ N\in M(n)$, we have $Tr(MN)=Tr(NM)$. However, be careful that although we have*

$$\det(MN)=\det M\cdot\det N,$$

*we **do not have***

$$Tr(MN)=TrM\cdot TrN$$

in general.

Proof. This is a consequence of the previous lemma. Write $A(t) = (\mathbf{a}_1(t), \mathbf{a}_2(t), \ ..., \mathbf{a}_n(t))$, where $\mathbf{a}_i(t)$ are column vectors. Then

$$\begin{aligned}&\frac{d}{dt}\det A(t)\\&= \det(\mathbf{a}_1'(t), \mathbf{a}_2(t), \ ..., \mathbf{a}_n(t)) + \det(\mathbf{a}_1(t), \mathbf{a}_2'(t), \ ..., \mathbf{a}_n(t)) + \cdots + \det(\mathbf{a}_1(t), \mathbf{a}_2(t), \ ..., \mathbf{a}_n'(t))\end{aligned}$$

and we note that

$$A'(t) = \frac{dA}{dt}(t) = (\mathbf{a}_1'(t), \mathbf{a}_2'(t), \ ..., \mathbf{a}_n'(t))$$

and if we let $P(t) = A'(t)(A(t))^{-1}$ ($P(t)$ is the multiplication of two matrices), then (in the following we denote the standard basis of $\mathbb{R}^n$ as $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$)

$$P(t)\mathbf{a}_1(t) = \left(A'(t)(A(t))^{-1}\right)\mathbf{a}_1(t) = A'(t)\left((A(t))^{-1}\mathbf{a}_1(t)\right) = A'(t)\mathbf{e}_1 = \mathbf{a}_1'(t)$$

and similarly

$$P(t)\mathbf{a}_2(t) = \left(A'(t)(A(t))^{-1}\right)\mathbf{a}_2(t) = A'(t)\left((A(t))^{-1}\mathbf{a}_2(t)\right) = A'(t)\mathbf{e}_2 = \mathbf{a}_2'(t), \quad etc.$$

Hence, by Lemma 0.84, we have

$$\begin{aligned}&\frac{d}{dt}\det A(t)\\&= \begin{cases}\det(P(t)\mathbf{a}_1(t), \mathbf{a}_2(t), \ ..., \mathbf{a}_n(t)) + \det(\mathbf{a}_1(t), P(t)\mathbf{a}_2(t), \ ..., \mathbf{a}_n(t))\\ + \cdots + \det(\mathbf{a}_1(t), \mathbf{a}_2(t), \ ..., P(t)\mathbf{a}_n(t))\end{cases}\\&= Tr(P(t))\det A(t) = Tr\left(A'(t)(A(t))^{-1}\right)\det A(t) = Tr\left((A(t))^{-1}A'(t)\right)\det A(t). \qquad (152)\end{aligned}$$

The proof is done. □

Remark 0.88 *In the last identity of (152), we have used the identity that $Tr(AB) = Tr(BA)$ for any two matrices $A, \ B \in M(n)$.*

Application of Lemma 0.84 to ODE is:

Lemma 0.89 *Consider the ODE*

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad A \in M(n), \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad t \in (-\infty, \infty) \qquad (153)$$

with initial conditions $\mathbf{x}(0) = v_1, \ v_2, \ ..., \ v_n \in \mathbb{R}^n$ respectively. Denote the corresponding solutions as

$$\mathbf{x}^{(1)}(t), \ \mathbf{x}^{(2)}(t), \ ..., \ \mathbf{x}^{(n)}(t), \quad t \in (-\infty, \infty)$$

respectively and let $B(t)$ be the $n \times n$ matrix given by (each $\mathbf{x}^{(i)}(t)$ below is a column vector)

$$B(t) = \left(\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ ..., \mathbf{x}^{(n)}(t)\right) \in M(n), \quad t \in (-\infty, \infty). \qquad (154)$$

Then we have

$$\begin{cases}\dfrac{d}{dt}B(t) = AB(t), \quad B(t) \in M(n),\\ \dfrac{d}{dt}\det B(t) = (TrA)\det B(t), \quad \det B(t) \in \mathbb{R}, \quad t \in (-\infty, \infty).\end{cases} \qquad (155)$$

As a consequence we have

$$\det B(t) = e^{(TrA)t}\det B(0), \quad B(0) = (v_1, v_2, \ ..., v_n) \in M(n). \qquad (156)$$

In particular, if $\{v_1, v_2, ..., v_n\}$ *is a* ***basis*** *of* $\mathbb{R}^n$*, then we have* $\det B(0) \neq 0$ *and*

$$\left\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), ..., \mathbf{x}^{(n)}(t)\right\} \tag{157}$$

is also a ***basis*** *of* $\mathbb{R}^n$ *for all* $t \in (-\infty, \infty)$*. Moreover, for each* $t \in (-\infty, \infty)$*, the* ***volume*** *of the parallelepiped spanned by* $\left\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), ..., \mathbf{x}^{(n)}(t)\right\}$ *is the same as the* ***volume*** *of the parallelepiped spanned by* $\{v_1, v_2, ..., v_n\}$ *if and only if* $TrA = 0$*. In particular, if* $A \in M(n)$ *is* ***anti-symmetric****, then it is* ***volume-preserving.***

Proof. We know that $\left\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), ..., \mathbf{x}^{(n)}(t)\right\}$ is a **basis** of $\mathbb{R}^n$ if and only if $\det B(t) \neq 0$, where

$$B(t) = \left(\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), ..., \mathbf{x}^{(n)}(t)\right) \in M(n), \quad t \in (-\infty, \infty).$$

We clearly have the equation $B'(t) = AB(t)$, which, together with Lemma 0.84, implies

$$\begin{aligned}
&\frac{d}{dt}\det B(t) \\
&= \det\left(\dot{\mathbf{x}}^{(1)}(t), \mathbf{x}^{(2)}(t), \cdots\right) + \det\left(\mathbf{x}^{(1)}(t), \dot{\mathbf{x}}^{(2)}(t), \cdots\right) + \cdots \\
&= \begin{cases} \det\left(A\mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \cdots\right) + \det\left(\mathbf{x}^{(1)}(t),\ A\mathbf{x}^{(2)}(t),\ \cdots\right) \\ + \cdots + \det\left(\mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ ...,\ A\mathbf{x}^{(n)}(t)\right) \end{cases} \\
&= (TrA)\det B(t), \quad t \in (-\infty, \infty),
\end{aligned} \tag{158}$$

which gives

$$\det B(t) = e^{(TrA)t}\det B(0), \quad \forall\, t \in (-\infty, \infty).$$

Hence the proof of (155) and (156) is done. Moreover, if $\det B(0) \neq 0$, then $\det B(t) \neq 0$ for all $t \in (-\infty, \infty)$.

Finally, we note that the **volume** of the parallelepiped spanned by $\left\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), ..., \mathbf{x}^{(n)}(t)\right\}$ is given by $|\det B(t)|$, where

$$|\det B(t)| = \left|e^{(TrA)t}\det B(0)\right| = e^{(TrA)t}|\det B(0)|, \quad \forall\, t \in (-\infty, \infty).$$

Hence the volume is unchanged if and only if $TrA = 0$. □

Remark 0.90 *In case we assume that* $B(t)$ *is* ***invertible*** *for all time, then we can apply formula (150) to get*

$$\begin{aligned}
\frac{d}{dt}\det B(t) &= Tr\left(\underbrace{\frac{dB}{dt}(t)}\cdot(B(t))^{-1}\right)\det B(t) = Tr\left(\underbrace{AB(t)}\cdot(B(t))^{-1}\right)\det B(t) \\
&= (TrA)\det B(t), \quad t \in (-\infty, \infty),
\end{aligned}$$

which is (155).

As a consequence of Lemma 0.89, we have the following important geometric result (we will omit its proof):

Theorem 0.91 *Consider the ODE*

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad A \in M(n), \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad t \in (-\infty, \infty) \tag{159}$$

and view $A : \mathbb{R}^n \to \mathbb{R}^n$ *as a vector field on* $\mathbb{R}^n$*. Let* $\Omega(0)$ *be a domain in* $\mathbb{R}^n$ *with* n*-dimensional volume* $V(0) = vol(\Omega(0))$*. At time* $t_0 \in (-\infty, \infty)$*, the domain* $\Omega(0)$ *will be carried by the vector field* A *(along solution curves) and becomes a domain* $\Omega(t_0)$*, given by*

$$\Omega(t_0) = \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} = \mathbf{x}(t_0),\ \textit{where } \mathbf{x}(t)\ \textit{ satisfies (159) with } \mathbf{x}(0) \in \Omega(0)\}. \tag{160}$$

Then we have

$$V(t_0) = vol(\Omega(t_0)) = e^{(TrA)t_0}vol(\Omega(0)) = e^{(TrA)t_0}V(0), \quad \forall\, t_0 \in (-\infty, \infty). \tag{161}$$

Thus we can conclude the following result:

Theorem 0.92 *Consider the $n \times n$ linear system $\mathbf{x}' = A\mathbf{x}$, where $A \in M(n)$ and $\mathbf{x}(t) \in \mathbb{R}^n$. We can view $A : \mathbb{R}^n \to \mathbb{R}^n$ as a **vector field** on $\mathbb{R}^n$ and it will generate a **flow** on $\mathbb{R}^n$ (every point $\mathbf{x}(0) \in \mathbb{R}^n$ is moved along its trajectory $\mathbf{x}(t)$ for $t \in (-\infty, \infty)$).*

1. *If $TrA > 0$, the flow is **volume-increasing** (as time is increasing).*
2. *If $TrA = 0$, the flow is **volume-preserving** (for example, when A is **anti-symmetric**).*
3. *If $TrA < 0$, the flow is **volume-decreasing** (as time is increasing).*

Remark 0.93 *(**Be careful.**) Assume $n = 2$. we note that area-preserving **does not** necessarily imply **length-preserving,** which is equivalent to **preserving the inner product**. Conversely, if the flow is length-preserving, then it must be area-preserving. Therefore, if $A \in M(2)$ is anti-symmetric, it is **length-preserving** and so **area-preserving**. But an **area-preserving** 2×2 linear system is not necessarily anti-symmetric.*

Example 0.94 *Look at the simple system*

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}, \quad TrA = 0.$$

*Its general solution is given by $\mathbf{x}(t) = (c_1 e^{3t}, c_2 e^{-3t})$. It is area-preserving **but not** length-preserving. The unit vector $\mathbf{x}(0) = (1, 0)$ under the flow becomes $\mathbf{x}(t) = (e^{3t}, 0)$, no longer an unit vector.*

Stability and Phase Portrait.

Remark 0.95 *Throughout this section, we assume that $\det A \neq 0$, where $A \in M(n)$, unless otherwise stated. Under the assumption $\det A \neq 0$, **the only equilibrium solution** of the ODE $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) \equiv 0$. If $\det A = 0$, the ODE $\mathbf{x}' = A\mathbf{x}$ will have **infinitely many** equilibrium solutions (any $\mathbf{x}_0 \in \ker A$ will be an equilibrium solution).*

Consider the ODE $\mathbf{x}' = A\mathbf{x}$, where $A \in M(n)$. In this section, we assume that $\det A \neq 0$ unless otherwise stated. One can view the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ as a **vector field** $\mathbf{x} \to A\mathbf{x}$ on $\mathbb{R}^n$. Since $\det A \neq 0$, the only point $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = 0$ is the origin $\mathbf{x} = 0 = (0, 0, ..., 0)$. Hence **the only equilibrium solution** of the ODE $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) \equiv 0$, $t \in (-\infty, \infty)$. In some textbook, the origin $\mathbf{x} = 0 = (0, 0, ..., 0)$ (or equilibrium solution $\mathbf{x}(t) \equiv 0$) is also called a **singularity** of the vector field $A : \mathbb{R}^n \to \mathbb{R}^n$ or **singularity** of the ODE $\mathbf{x}' = A\mathbf{x}$.

The collection of **all trajectories** (**with the direction of motion attached on each trajectory as time is increasing**) of solution curves $\mathbf{x}(t) \in \mathbb{R}^n$, $t \in (-\infty, \infty)$, of $\mathbf{x}' = A\mathbf{x}$ is called the **phase portrait** of the ODE. The background space $\mathbb{R}^n$ is called the **phase space**.

A solution curve $\mathbf{x}(t) \in \mathbb{R}^n$ can be viewed as the **motion** (in the direction of increasing t) of the particle $\mathbf{x}(0)$ in $\mathbb{R}^n$ influenced by the **vector field** $A : \mathbb{R}^n \to \mathbb{R}^n$. By uniqueness of the solution to ODE with initial condition, **different trajectories will not intersect at all**. The **phase portrait** of $\mathbf{x}' = A\mathbf{x}$ describes the **dynamics** (or **flow**) of the vector field A on $\mathbb{R}^n$.

As we shall see, the most important result is:

Theorem 0.96 *(**Not precise ...**) The "**dynamics**" of the ODE $\mathbf{x}' = A\mathbf{x}$ is determined essentially by the **sign** of the eigenvalues of the matrix A.*

The definition of saddle, sink, source for general $A \in M(n)$, $\det A \neq 0$, $n \in \mathbb{N}$.

Definition 0.97 *Assume* $A \in M(n)$, $\det A \neq 0$, *has* n ***distinct real*** *eigenvalues such that some of them are* ***positive*** *and some of them are* ***negative***, *then we say the system* $\mathbf{x}' = A\mathbf{x}$ *has a* ***saddle*** *at the origin* $\mathbf{x} = 0$, *which is* ***the only equilibrium point*** *of the system.*

Lemma 0.98 *(**Properties of a saddle for** $A \in M(n)$.) A **saddle** equilibrium point is **unstable,** which means that for **any** small* $\varepsilon > 0$, *there is a point* $\mathbf{x}_0 \in \mathbb{R}^n$ *with* $|\mathbf{x}_0| < \varepsilon$ *such that the trajectory* $\mathbf{x}(t)$ *of the ODE* $\mathbf{x}' = A\mathbf{x}$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *will take* $\mathbf{x}_0$ *away from the origin* $\mathbf{x} = 0$ *as* $t \to \infty$. *Moreover, we have* $\lim_{t\to\infty} |\mathbf{x}(t)| = +\infty$.

Proof. For any small $\varepsilon > 0$, there is an eigenvector $v \neq 0$ with $|v| < \varepsilon$ such that $Av = \lambda v$, $\lambda > 0$ and the solution $\mathbf{x}(t) = e^{\lambda t} v$, $t \in (-\infty, \infty)$, will take $\mathbf{x}(0) = v$ away from the origin $\mathbf{x} = 0$ as $t \to \infty$. Moreover, we have $\lim_{t\to\infty} |\mathbf{x}(t)| = +\infty$. The proof is done. □

Example 0.99 *Consider the linear system*

$$\mathbf{x}'(t) = \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix} \mathbf{x}(t), \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Show that the origin is a ***saddle*** *and* ***draw its phase portrait.***

<u>**Solution**</u>:

To plot the phase portrait, it is easier to plot it on the $\mathbf{y}$-plane and then come back to $\mathbf{x}$-plane. We know the relation between $\mathbf{x}(t) = (x_1(t), x_2(t))$ and $\mathbf{y}(t) = (y_1(t), y_2(t))$ is $\mathbf{x}(t) = P\mathbf{y}(t)$. Here $P \in M(2)$ is a matrix making $P^{-1}AP = J$ (Jordan canonical form).

The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 2$ with corresponding eigenvectors $v_1 = (1, 0)$, $v_2 = (-1, 1)$. If we let $\mathbf{x} = P\mathbf{y}$, $P = (v_1, v_2)$, then in terms of $\mathbf{y}(t)$ the system becomes

$$\begin{cases} y_1'(t) = -y_1(t) \\ y_2'(t) = 2y_2(t) \end{cases}$$

and the phase portrait in the $\mathbf{y} = (y_1, y_2)$ space looks like the following (the general solution for $\mathbf{y}(t)$ is $\mathbf{y} = (c_1 e^{-t}, c_2 e^{2t}) = c_1 e^{-t}\mathbf{e}_1 + c_2 e^{2t}\mathbf{e}_2$, $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$):

Picture Here.

By the relation $\mathbf{x} = P\mathbf{y}$, the phase portrait in the $\mathbf{x} = (x_1, x_2)$ space looks like the following (the two perpendicular vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are mapped onto v_1, v_2 respectively):

Picture Here.

The general solution $\mathbf{x}(t)$ to the system is given by

$$\mathbf{x}(t) = P\mathbf{y}(t) = c_1 e^{-t} v_1 + c_2 e^{2t} v_2, \quad \mathbf{x}(0) = c_1 v_1 + c_2 v_2, \quad v_1 = (1, 0), \quad v_2 = (-1, 1)$$

$$c_2 e^{2t} v_2$$

and we see that if $c_2 = 0$, we have $\mathbf{x}(t) \to 0$ as $t \to \infty$ ($\mathbf{x}(t)$ is now lying on the v_1-axis) and if $c_2 \neq 0$, we have $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. Moreover, for $c_2 \neq 0$, **the asymptotic direction**, as $t \to \infty$, of the vector $\mathbf{x}(t)$ is

$$\lim_{t\to\infty} \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} = \begin{cases} \frac{v_2}{|v_2|} & \text{if} \quad c_2 > 0 \\ -\frac{v_2}{|v_2|} & \text{if} \quad c_2 < 0. \end{cases} \tag{162}$$

Similarly, if $c_1 = 0$, we have $\mathbf{x}(t) \to 0$ as $t \to -\infty$ ($\mathbf{x}(t)$ is now lying on the v_2-axis) and if $c_1 \neq 0$, we have $|\mathbf{x}(t)| \to \infty$ as $t \to -\infty$. Moreover, for $c_1 \neq 0$, **the asymptotic direction**, as $t \to -\infty$, of the vector $\mathbf{x}(t)$ is

$$\lim_{t\to-\infty} \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} = \begin{cases} \frac{v_1}{|v_1|} & \text{if} \quad c_1 > 0 \\ -\frac{v_1}{|v_1|} & \text{if} \quad c_1 < 0. \end{cases} \tag{163}$$

The equilibrium point 0 is a **saddle point**, which is **unstable**. □

Remark 0.100 *In the above example, all trajectories, as $t \to \infty$, are tangent to either v_1-axis ($c_2 = 0$) or v_2-axis ($c_2 \neq 0$). We note that both v_1 and v_2 are eigenvectors.*

Motivated by the above example, we have the following important result for a linear system:

Lemma 0.101 *(**Asymptotic directions are eigenvector directions.**) Consider the ODE $\mathbf{x}' = A\mathbf{x}$ where $A \in M(n)$ ($\det A = 0$ is allowed). Assume $\mathbf{x}(t)$ is a **nonzero solution** and it satisfies*

$$\lim_{t\to\infty} \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} = v \quad \textit{for some } \textbf{\textit{unit}} \textit{ vector } v \in \mathbb{R}^n. \tag{164}$$

*Then $v \neq 0$ must be an **eigenvector** of A. Similarly, if we have*

$$\lim_{t\to-\infty} \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} = w \quad \textit{for some } \textbf{\textit{unit}} \textit{ vector } w \in \mathbb{R}^n, \tag{165}$$

*then $w \neq 0$ must be an **eigenvector** of A.*

Remark 0.102 *Since $\mathbf{x} = 0$ is an equilibrium solution of $\mathbf{x}' = A\mathbf{x}$, we have $\mathbf{x}(t_0) \neq 0$ at some $t_0 \in (-\infty, \infty)$ if and only if $\mathbf{x}(t) \neq 0$ for all $t \in (-\infty, \infty)$.*

Remark 0.103 *Another related result is: Assume $\mathbf{x}(t)$ is a **nonzero solution** of $\mathbf{x}' = A\mathbf{x}$ ($\det A = 0$ is allowed) and it satisfies*

$$\lim_{t\to\infty} \mathbf{x}(t) = p \quad \textit{for some} \quad p \in \mathbb{R}^n. \tag{166}$$

*Then we must have $Ap = 0$. That is, the point $p \in \mathbb{R}^n$ must be an **equilibrium point** of the vector field $A : \mathbb{R}^n \to \mathbb{R}^n$. The proof is to use the identity*

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t A\mathbf{x}(s)\,ds = \int_0^T A\mathbf{x}(s)\,ds + \int_T^t A\mathbf{x}(s)\,ds, \quad 0 < T < t$$

and let $t \to \infty$ to obtain a contradiction if $Ap \neq 0$. In particular, if $\det A \neq 0$, then we must have $p = 0$.

To prove Lemma 0.101, we first need a simple **calculus result**:

Lemma 0.104 *Let $\mathbf{x}(t) : (-\infty, \infty) \to \mathbb{R}^n$ be a continuous function satisfying*

$$\begin{cases} (1).\ \int_0^\infty \mathbf{x}(s)\,ds \textit{ converges (i.e., each component of the integral converges)}, \\ (2).\ \lim_{s\to\infty} \mathbf{x}(s) = p \in \mathbb{R}^n. \end{cases} \tag{167}$$

Then we must have $p = 0$.

Proof. This is a simple exercise. □

Proof of Lemma 0.101:

Let
$$\tilde{\mathbf{x}}(t) := \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}, \quad t \in (-\infty, \infty), \quad \lim_{t\to\infty} \tilde{\mathbf{x}}(t) = v.$$
We can compute
$$\tilde{\mathbf{x}}'(t) = \frac{|\mathbf{x}(t)|\mathbf{x}'(t) - \mathbf{x}(t)\frac{d}{dt}|\mathbf{x}(t)|}{|\mathbf{x}(t)|^2} = \frac{|\mathbf{x}(t)|\mathbf{x}'(t) - \mathbf{x}(t)\frac{\langle \mathbf{x}(t),\ \mathbf{x}'(t)\rangle}{|\mathbf{x}(t)|}}{|\mathbf{x}(t)|^2}$$
$$= \frac{|\mathbf{x}(t)|A\mathbf{x}(t) - \mathbf{x}(t)\frac{\langle \mathbf{x}(t),\ A\mathbf{x}(t)\rangle}{|\mathbf{x}(t)|}}{|\mathbf{x}(t)|^2} = A\tilde{\mathbf{x}}(t) - \langle \tilde{\mathbf{x}}(t),\ A\tilde{\mathbf{x}}(t)\rangle\tilde{\mathbf{x}}(t), \quad t \in (-\infty, \infty)$$
and get
$$\tilde{\mathbf{x}}(t) - \tilde{\mathbf{x}}(0) = \int_0^t \tilde{\mathbf{x}}'(s)\,ds = \int_0^t \left[A\tilde{\mathbf{x}}(s) - \langle \tilde{\mathbf{x}}(s),\ A\tilde{\mathbf{x}}(s)\rangle\tilde{\mathbf{x}}(s)\right]ds. \tag{168}$$
We note that the integral in (168) converges due to
$$\lim_{t\to\infty}\int_0^t \left[A\tilde{\mathbf{x}}(s) - \langle \tilde{\mathbf{x}}(s),\ A\tilde{\mathbf{x}}(s)\rangle\tilde{\mathbf{x}}(s)\right]ds = \lim_{t\to\infty}\left(\tilde{\mathbf{x}}(t) - \tilde{\mathbf{x}}(0)\right) = v - \tilde{\mathbf{x}}(0).$$
Moreover, the integrand in (168) also converges due to the following
$$\lim_{s\to\infty}\left[A\tilde{\mathbf{x}}(s) - \langle \tilde{\mathbf{x}}(s),\ A\tilde{\mathbf{x}}(s)\rangle\tilde{\mathbf{x}}(s)\right] = Av - \langle v, Av\rangle v.$$
Hence Lemma 0.104 can be applied and we conclude
$$Av - \langle v, Av\rangle v = 0 \quad (\text{same as } Av = \langle v, Av\rangle v), \quad \text{where } v \neq 0.$$
Therefore the unit vector $v \neq 0$ must be an **eigenvector** of A with corresponding eigenvalue $\langle v, Av\rangle$.

The proof for the case (165) is similar. □

Definition 0.105 *Assume $A \in M(n)$, $\det A \neq 0$, and all eigenvalues* ***(may be real or complex)*** *of A have* ***negative*** *real parts, then $\mathbf{x} = 0$ is called a* ***sink.*** *An equilibrium point which is a* ***sink*** *is* ***"asymptotically stable",*** *which means that* ***all*** *trajectories $\mathbf{x}(t)$ tend to 0 as $t \to \infty$***.***

Lemma 0.106 ***(Properties of a sink for** $A \in M(n)$.**)*** *Let $A \in M(n)$ with $\det A \neq 0$. If $\mathbf{x} = 0$ is a* ***sink*** *of the $n \times n$ system $\mathbf{x}' = A\mathbf{x}$, then any solution $\mathbf{x}(t)$ of the ODE $\mathbf{x}' = A\mathbf{x}$ satisfies*
$$\lim_{t\to\infty}\mathbf{x}(t) = 0. \tag{169}$$
Also for any solution $\mathbf{x}(t)$ to the equation with $\mathbf{x}(0) \neq 0$ we have
$$\lim_{t\to-\infty}|\mathbf{x}(t)| = \infty. \tag{170}$$

Proof. We prove Lemma 0.106 for the case $n = 2$ only (the proof for the case $n = 3$ is similar; however, for the proof of general $n \in \mathbb{N}$, we need to know the **Jordan canonical form** of $A \in M(n)$, which is beyond our scope).

By the discussion in Section 0.0.1 (see (16), (39), (50)), we have
$$\mathbf{x}(t) = PJ(t)P^{-1}\mathbf{x}_0, \quad t \in (-\infty, \infty), \tag{171}$$

where $J(t)$ has one of the forms:

$$J(t) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ -e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{pmatrix}$$

with $\lambda < 0,\ \mu < 0$ ($\lambda = \mu$ is possible), $\alpha < 0$. We see that $\lim_{t\to\infty} J(t) = 0$ (zero matrix). The proof of the first limit is done.

For the second limit, use the identity $P^{-1}\mathbf{x}(t) = J(t)P^{-1}\mathbf{x}(0)$ and note that

$$\lim_{t\to-\infty} \left| \underbrace{J(t)P^{-1}\mathbf{x}(0)} \right| = \infty \quad \text{for any} \quad \mathbf{x}(0) \neq 0,$$

which implies

$$\lim_{t\to-\infty} |\mathbf{x}(t)| = \lim_{t\to-\infty} \left| P\left(\underbrace{J(t)P^{-1}\mathbf{x}(0)} \right) \right| = \infty.$$

The proof is done. □

Definition 0.107 *Assume $A \in M(n)$, $\det A \neq 0$, and all eigenvalues (**may be real or complex**) of A have **positive** real parts, then $\mathbf{x} = 0$ is called a **source.** An equilibrium point which is a **source** is **unstable** (see Lemma 0.98 for its meaning).*

Lemma 0.108 *(**Properties of a source for** $A \in M(n)$**.**) Let $A \in M(n)$ with $\det A \neq 0$. If $\mathbf{x} = 0$ is a **source** of the $n \times n$ system $\mathbf{x}' = A\mathbf{x}$, then any solution $\mathbf{x}(t)$ of the ODE $\mathbf{x}' = A\mathbf{x}$ satisfies*

$$\lim_{t\to-\infty} \mathbf{x}(t) = 0. \tag{172}$$

Also for any solution $\mathbf{x}(t)$ to the equation with $\mathbf{x}(0) \neq 0$ we have

$$\lim_{t\to\infty} |\mathbf{x}(t)| = \infty. \tag{173}$$

Proof. The proof is similar to the previous lemma, we omit it. □

The definition of center for $A \in M(2)$, $\det A \neq 0$.

From now on, we focus on the case $A \in M(2)$ with $\det A \neq 0$.

Definition 0.109 *Assume $A \in M(2)$, $\det A \neq 0$, and all eigenvalues of A are **pure imaginary,** then $\mathbf{x} = 0$ is called a **center.** An equilibrium point which is a **center** is **stable** (in the sense that solutions **will not drift away** from the origin $\mathbf{x} = 0$ as $t \to \infty$). However, it is **not** asymptotically stable.*

Lemma 0.110 *(**Properties of a center for** $A \in M(2)$**.**) Let $A \in M(2)$ with $\det A \neq 0$. If $\mathbf{x} = 0$ is a **center** of the 2×2 system $\mathbf{x}' = A\mathbf{x}$, then all solutions $\mathbf{x}(t)$ with $\mathbf{x}(0) \neq 0$ are **periodic with period** $2\pi/\beta$, where the eigenvalues of A are $\pm i\beta$, $\beta > 0$. Each trajectory*

$$\left\{\mathbf{x}(t) \in \mathbb{R}^2 : \mathbf{x}(0) \neq 0 \in \mathbb{R}^2,\ t \in (-\infty, \infty)\right\}$$

*is a periodic **ellipse** (with center at the origin 0) in the plane with period $2\pi/\beta$.*

Proof. By (50), the solution $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = P \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} P^{-1}\mathbf{x}_0, \quad \beta > 0 \tag{174}$$

for some invertible matrix $P \in M(2)$. If we let $P^{-1}\mathbf{x}_0 = \mathbf{y}_0$, then the trajectory

$$\mathbf{y}(t) = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} \mathbf{y}_0, \quad t \in (-\infty, \infty)$$

is a **circle** (with radius $|\mathbf{y}_0|$) in the $\mathbf{y}$-plane. Note that $\mathbf{y}(t)$ is a **clockwise rotation** of $\mathbf{y}_0$ by angle βt. Therefore, the trajectory of $\mathbf{y}(t)$ is moving in the **clockwise direction**, with period equal to $2\pi/\beta$. The curve $\mathbf{x}(t) = P\mathbf{y}(t)$ is now an **ellipse** (with center at the origin 0) in the $\mathbf{x}$-plane with period $2\pi/\beta$. We have the following phase portraits in the $\mathbf{x} = (x_1, x_2)$ space and the $\mathbf{y} = (y_1, y_2)$ space:

Picture Here.

If $\det P > 0$ (P preserves orientation), $\mathbf{x}(t)$ is moving in the **clockwise direction** and if $\det P < 0$ (P reverses orientation), $\mathbf{x}(t)$ is moving in the **counterclockwise direction.** □

Remark 0.111 *There is a fact in plane geometry saying that a* ***nonsingular*** *linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$* ***maps a circle onto an ellipse****. You can look up this fact by google search ...*

0.0.5 More on the phase portrait of a sink (or a source) for $A \in M(2)$.

Assume $A \in M(2)$, $\det A \neq 0$. The phase portrait of a **sink** (all eigenvalues, **real or complex,** of A have **negative** real parts) for a 2×2 linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ can be divided into **four subcases.** They are known as **focus** (trivial case), **node, improper node** and **spiral sink** (different textbooks may have different names for these).

Similarly, the phase portrait of a **source** (all eigenvalues, **real or complex,** of A have **positive** real parts) for a 2×2 linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ can be divided into **four subcases.** They are also known as **focus** (trivial case), **node, improper node** and **spiral source**.

For simplicity, in the following, we will discuss the case of a **sink** only. The discussion for a **source** is similar (**just reverse the direction of each trajectory $\mathbf{x}(t)$**).

We look at one example for each case (except the **focus** case, which is rather easy). For easier drawing of the picture, we assume the matrix A **is already in the canonical form**. If not, then you have to apply a **linear transformation** P to the phase portrait in canonical form (remember that $\mathbf{x} = P\mathbf{y}$, where in the $\mathbf{y}$-plane the matrix is in canonical form).

Example 0.112 *(A is* ***diagonalizable****; the equilibrium point $\mathbf{x} = 0$ is called a* ***node.****) Assume $A \in M(2)$ has the canonical form (****diagonalizable****)*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{where } \lambda_1 < \lambda_2 < 0. \tag{175}$$

Sketch the phase portrait of the equation $\mathbf{x}' = A\mathbf{x}$.

Remark 0.113 *In case A is diagonalizable with $\lambda_1 = \lambda_2 = \lambda < 0$, we must have $A = \lambda I$ and each trajectory is given by*

$$\mathbf{x}(t) = \left(c_1 e^{\lambda t}, c_2 e^{\lambda t}\right) = e^{\lambda t}(c_1, c_2), \quad t \in (-\infty, \infty).$$

It is a ray approaching the origin $(0,0)$ as $t \to \infty$. In this case, we call the origin a ***focus.***

<u>**Solution**</u>:

We write the solution as

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}, \quad t \in (-\infty, \infty)$$

for arbitrary c_1 and c_2, where $\mathbf{x}(0) = (c_1, c_2)$. The first important observation is that, for $c_2 \neq 0$, $x(t) = c_1 e^{\lambda_1 t}$ tends to 0 faster than $y(t) = c_2 e^{\lambda_2 t}$ as $t \to \infty$. Therefore, as $t \to \infty$, **the y-coordinate is dominant** (more visible). We will see that, for $c_2 \neq 0$, $(x(t), y(t))$ is asymptotically **tangent to the y-axis** at the origin $(0,0)$ as $t \to \infty$.

For convenience, we focus on **the phase portrait on the first quadrant** and assume $c_1 > 0$, $c_2 > 0$. The trajectory $\mathbf{x}(t) = \left(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}\right) \to (0,0)$, (∞, ∞) as $t \to \infty$, $-\infty$ respectively, and it lies on the **graph** of the following function for all $t \in (-\infty, \infty)$:

$$y = kx^{\alpha}, \quad \text{where} \quad k = \frac{c_2}{c_1^{\alpha}} > 0, \quad \alpha = \frac{\lambda_2}{\lambda_1} \in (0,1), \tag{176}$$

where we have $dy/dx = \infty$ at $x = 0$. Therefore, $(x(t), y(t))$ is asymptotically **tangent to the y-axis** at the origin $(0,0)$ as $t \to \infty$ ($x(t)$ tends to 0 faster than $y(t)$ as $t \to \infty$). The tangent vector $(x'(t), y'(t)) = \left(c_1\lambda_1 e^{\lambda_1 t}, c_2\lambda_2 e^{\lambda_2 t}\right)$ is pointing in the $(-,-)$ direction and it has positive slope

$$m(t) = \frac{c_2\lambda_2 e^{\lambda_2 t}}{c_1\lambda_1 e^{\lambda_1 t}} > 0, \quad \forall\, t \in (-\infty, \infty).$$

We see that $m(t) \to 0^+$ as $t \to -\infty$ and $m(t) \to \infty$ as $t \to \infty$. As c_1, c_2 run over all possible positive numbers, the constant k ranges between 0 and ∞. **That means, by varying the constant $k \in (0,\infty)$ in (176), we can find all trajectories (in graph form) of the ODE in the first quadrant.** For each ray $y = mx$, $m > 0$, in the first quadrant, the trajectory $\mathbf{x}(t) = (x(t), y(t)) = \left(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}\right)$, $\mathbf{x}(0) = (c_1, c_2)$, intersects the ray at a **unique** time t_0, given by

$$t_0 = \frac{1}{\lambda_2 - \lambda_1} \log\left(\frac{mc_1}{c_2}\right) \in (-\infty, \infty).$$

The phase portrait of the ODE in the other three quadrants are similar to that in the first quadrant. That is, $(x(t), y(t))$ is asymptotically **tangent to the y-axis** at the origin $(0,0)$ as $t \to \infty$. The equilibrium point $\mathbf{x} = (0,0)$ is **globally asymptotically stable** due to the fact that **ALL** trajectories tend to $(0,0)$ as $t \to \infty$. We have the following phase portrait:

Picture Here.

The remaining cases are $c_1 = 0$, $c_2 \neq 0$ and $c_1 \neq 0$, $c_2 = 0$. For these two cases, the trajectory $\mathbf{x}(t)$ either lies on the y-axis or on the x-axis. The only two trajectories $\mathbf{x}(t)$ in $\mathbb{R}^2$ which **will not tangent to the y-axis as** $t \to \infty$ is the case when $c_2 = 0$ (but with $c_1 > 0$ or $c_1 < 0$). □

Remark 0.114 *The matrix A in (175) has two independent eigenvectors $v_1 = (1,0)$ and $v_2 = (0,1)$. **All trajectories** $\mathbf{x}(t)$ in $\mathbb{R}^2$ tend to the origin $(0,0)$ and **tangent to** either the x-**axis** (v_1-**axis**) or the y-**axis** (v_2-**axis**) as $t \to \infty$.*

Remark 0.115 *For the general case, if A has two eigenvalues $\lambda_1 < \lambda_2 < 0$ with corresponding eigenvectors v_1, v_2, then since it is diagonalizable, the general solution is given by*

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2, \quad \lambda_1 < \lambda_2 < 0,$$

*where $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. Due to $\lambda_1 < \lambda_2 < 0$, for $c_2 \neq 0$, $c_1 e^{\lambda_1 t} v_1$ will tend to zero faster than $c_2 e^{\lambda_2 t} v_2$. Hence, for $c_2 \neq 0$, $\mathbf{x}(t)$ is **asymptotically tangent to the** v_2-**axis** at the origin $(0,0)$ as $t \to \infty$. For $c_2 = 0$, $\mathbf{x}(t)$ lies on the v_1-**axis** and tends to $(0,0)$ as $t \to \infty$.*

Example 0.116 *(A is **not diagonalizable**; the equilibrium point $\mathbf{x} = 0$ is called an **improper node.**) Assume $A \in M(2)$ has the canonical form*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda_1 = \lambda_2 = \lambda < 0. \tag{177}$$

Sketch the phase portrait of the equation $\mathbf{x}' = A\mathbf{x}$.

Solution:

There is **only one** independent eigenvector $v_1 = (1, 0)$ for the matrix A. The general solution is given by

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (c_1 + c_2 t) e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}, \quad \lambda < 0, \quad t \in (-\infty, \infty)$$

with $\mathbf{x}(t) \to (0,0)$ as $t \to \infty$ for any initial data $\mathbf{x}_0 = (c_1, c_2) \in \mathbb{R}^2$. We see that, **regardless of the values of** c_2, $y(t) = c_2 e^{\lambda t}$ tends to 0 faster than $x(t) = (c_1 + c_2 t) e^{\lambda t}$ as $t \to \infty$. Therefore, as $t \to \infty$, the x-coordinate is dominant and $(x(t), y(t))$ is asymptotically **tangent to the x-axis** at the origin $(0,0)$ as $t \to \infty$, i.e., tangent to the **eigenvector axis** as $t \to \infty$**.**

Case 1: $c_2 > 0$ (regardless of the sign of c_1).

We now have

$$y(t) = c_2 e^{\lambda t} > 0, \quad \forall\, t \in (-\infty, \infty), \quad \lambda < 0$$

and see that $y(t)$ is always decreasing. As for $x(t)$, it will become positive eventually with

$$x(t) = (c_1 + c_2 t) e^{\lambda t} = \begin{cases} \text{negative if } t \in \left(-\infty, -\frac{c_1}{c_2}\right) \\ \text{positive if } t \in \left(-\frac{c_1}{c_2}, \infty\right). \end{cases}$$

For each ray $y = mx$, $m > 0$, in the first quadrant, the trajectory $(x(t), y(t)) = \left((c_1 + c_2 t) e^{\lambda t}, c_2 e^{\lambda t}\right)$ intersects it at a **unique** time t_0, given by

$$t_0 = \frac{1}{m} - \frac{c_1}{c_2} > -\frac{c_1}{c_2}, \quad m > 0, \quad x(t_0) > 0.$$

By the above discussion, we can draw the phase portrait of the system for $c_2 > 0$, given by the following (**all trajectories are tangent to the x-axis as $t \to \infty$ through the first quadrant**)

Picture Here.

Motivated by (176), for the entire trajectory (i.e. for all $t \in (-\infty, \infty)$), one can **express x as a function of** $y = c_2 e^{\lambda t}$ as

$$x = (c_1 + c_2 t) \cdot e^{\lambda t} = \left[c_1 + c_2 \left(\frac{1}{\lambda} \log \frac{y}{c_2}\right)\right] \cdot \frac{y}{c_2}$$
$$= \frac{c_1}{c_2} y + \frac{y}{\lambda} \log\left(\frac{y}{c_2}\right), \quad t = \frac{1}{\lambda} \log \frac{y}{c_2},$$

and get

$$\left.\frac{dx}{dy}\right|_{y=0} = \left.\left(\frac{c_1}{c_2} + \frac{1}{\lambda} + \frac{1}{\lambda} \log\left(\frac{y}{c_2}\right)\right)\right|_{y=0} = \infty.$$

By this, we see that the trajectory is tangent to the x-axis as $t \to \infty$. On the other hand, for the entire trajectory (i.e. for all $t \in (-\infty, \infty)$), y cannot be expressed as a function of x.

Case 2: $c_2 < 0$ (regardless of the sign of c_1).

The discussion is similar to that for Case 1. **All trajectories are tangent to the x-axis as $t \to \infty$ through the third quadrant.** We omit it. The phase portrait of the system for $c_2 < 0$ is given by the following

Picture Here.

Case 3: $c_2 = 0$ (but $c_1 \neq 0$).

In this case, we have

$$\mathbf{x}(t) = \left(c_1 e^{\lambda t}, 0\right), \quad t \in (-\infty, \infty), \quad \lambda < 0.$$

The trajectory lies on the x-axis and tends to $(0,0)$ as $t \to \infty$. The phase portrait of the system for $c_2 = 0$ is given by the following

Picture Here.

In conclusion, the phase portrait of the system **on the whole plane** is given by:

Picture Here.

□

Remark 0.117 *The matrix A in (177) has* ***only one*** *independent eigenvector $v_1 = (1,0)$; hence* ***all trajectories*** *in $\mathbb{R}^2$ tend to the origin $(0,0)$ and* ***tangent to*** *the x-**axis** (v_1-**axis**) as $t \to \infty$. Compare with Remark 0.114.*

Remark 0.118 *For the general case (i.e. A has two repeated eigenvalues $\lambda < 0$ and is not diagonalizable), the solution is given by*

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}(t v_1 + v_2) \\ &= e^{\lambda t}\left[c_1 v_1 + c_2(t v_1 + v_2)\right], \quad t \in (-\infty, \infty), \quad \lambda < 0,\end{aligned}$$

where $Av_1 = \lambda v_1$, $Av_2 = \lambda v_2 + v_1$. From it one can see that ***all trajectories*** *tend to the origin $(0,0)$ and* ***tangent to*** *the v_1-**axis** as $t \to \infty$ (can you see this ?).*

Example 0.119 *(The equilibrium point $\mathbf{x} = 0$ is called a* ***spiral sink.****) Assume $A \in M(2)$ has the canonical form (**with complex eigenvalues** $\alpha \pm i\beta$, $\alpha < 0$, $\beta > 0$)*

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \alpha < 0, \quad \beta > 0.$$

Sketch the phase portrait of the equation $\mathbf{x}' = A\mathbf{x}$.

<u>**Solution**</u>:

The general solution is given by

$$\begin{aligned}\mathbf{x}(t) &= \begin{pmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ -e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix} \mathbf{x}_0 \\ &= e^{\alpha t}\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} \mathbf{x}_0, \quad \alpha < 0, \quad \beta > 0\end{aligned}$$

and so $\mathbf{x}(t) \to 0$ as $t \to \infty$ for any initial data $\mathbf{x}_0 = (c_1, c_2) \in \mathbb{R}^2$. Note that the effect of the matrix

$$\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}, \quad \beta > 0$$

is rotation in the **clockwise** direction (as time is increasing). The phase portrait is given by

Picture Here.

□

Using trace and determinant to determine the phase portrait of $\mathbf{x}' = A\mathbf{x}$, where $A \in M(2)$, $\det A \neq 0$.

Let $A \in M(2)$ with $\det A \neq 0$. The only equilibrium point of the system $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = 0$. The characteristic equation for the eigenvalues λ of A is given by

$$\lambda^2 - (TrA)\lambda + \det A = 0, \quad \lambda = \frac{TrA \pm \sqrt{(TrA)^2 - 4\det A}}{2}. \tag{178}$$

Hence one can use the values of $TrA = \lambda_1 + \lambda_2$ and $\det A = \lambda_1\lambda_2$ to determine the phase portrait of the system. We first note that we have

$$(TrA)^2 - 4\det A = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2. \tag{179}$$

Note that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\begin{cases} TrA = \lambda_1 + \lambda_2 = a + d \\ \det A = \lambda_1\lambda_2 = ad - bc \\ (TrA)^2 - 4\det A = (\lambda_1 - \lambda_2)^2 = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc. \end{cases} \tag{180}$$

Remark 0.120 *By the third identity in (180), if $bc \geq 0$, then* ***it is impossible to have complex conjugate eigenvalues****.*

If we want to use TrA and $\det A$ to determine the phase portrait of $\mathbf{x}' = A\mathbf{x}$, where $A \in M(2)$, $\det A \neq 0$, the most useful one is the following:

Lemma 0.121 *(**The case when** $\det A < 0$.) Let $A \in M(2)$ with $\det A \neq 0$. The equation $\mathbf{x}' = A\mathbf{x}$ has $\det A < 0$ if and only if we have a **saddle** at $\mathbf{x} = 0$.*

Remark 0.122 *There is no definite relation between a **saddle** at $\mathbf{x} = 0$ and the value of TrA.*

Proof. This is clear due to the identity $\det A = \lambda_1\lambda_2$. If $\mathbf{x} = 0$ is a saddle, by definition, we must have $\lambda_1 > 0$ and $\lambda_2 < 0$ and so $\det A < 0$. Conversely, if $\det A = \lambda_1\lambda_2 < 0$, then λ_1 and λ_2 cannot be complex conjugate and they must be real numbers with different signs. Hence $\mathbf{x} = 0$ is a saddle. □

Next, we note the following:

Lemma 0.123 *Let $A \in M(2)$ with $\det A \neq 0$. The two eigenvalues λ_1, λ_2 of A are:*

1. *Real and distinct if $(TrA)^2 - 4\det A > 0$.*
2. *Real and repeated if $(TrA)^2 - 4\det A = 0$.*
3. *Complex conjugate if $(TrA)^2 - 4\det A < 0$.*

Proof. This is obvious. □

Lemma 0.124 *(**The case when** $(TrA)^2 - 4\det A > 0$.) Let $A \in M(2)$ with $\det A \neq 0$. If $(TrA)^2 - 4\det A > 0$, then $\lambda_1 \neq \lambda_2$ are **real** and **distinct** and we have the following three cases for the equilibrium point $\mathbf{x} = 0$ of the ODE $\mathbf{x}' = A\mathbf{x}$:*

1. If $\det A = \lambda_1\lambda_2 > 0$ and $TrA > 0$, then $\lambda_1 > 0,\ \lambda_2 > 0$, and we have a **source** (which is a **node**) **(unstable)**.

2. If $\det A = \lambda_1\lambda_2 > 0$ and $TrA < 0$, then $\lambda_1 < 0,\ \lambda_2 < 0$, and we have a **sink** (which is a **node**) **(asymptotically stable)**.

3. If $\det A = \lambda_1\lambda_2 < 0$ (regardless of the sign of TrA), then one eigenvalue is positive, one eigenvalue is negative, and we have a **saddle (unstable)**.

Remark 0.125 *In Case 3 of the above, it is possible to have $TrA = 0$.*

Proof. This is obvious. □

Lemma 0.126 *(**The case when** $(TrA)^2 - 4\det A = 0$.) Let $A \in M(2)$ with $\det A \neq 0$. If $(TrA)^2 - 4\det A = 0$, we have $\lambda_1 = \lambda_2 = (TrA)/2$ and we have the following two cases for the equilibrium point $\mathbf{x} = 0$ of the ODE $\mathbf{x}' = A\mathbf{x}$:*

1. *If $TrA > 0$, we have a **source** (which is either a **focus**, i.e. $A = \lambda I$, or an **improper node**) **(unstable)**.*

2. *If $TrA < 0$, we have a **sink** (which is either a **focus**, i.e. $A = \lambda I$, or an **improper node**) **(asymptotically stable)**.*

Remark 0.127 *(**Be careful.**) Note that one cannot use the conditions $(TrA)^2 - 4\det A = 0$ and $TrA < 0$ to distinguish between **focus** and **improper node.** The following two matrices*

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda < 0$$

*have **the same** trace and determinant, but one is a **focus** and the other is an **improper node.***

Proof. This is obvious. □

Lemma 0.128 *(**The case when** $(TrA)^2 - 4\det A < 0$.) Let $A \in M(2)$ with $\det A \neq 0$. If $(TrA)^2 - 4\det A < 0$, we have the following three cases for the equilibrium point $\mathbf{x} = 0$ of the ODE $\mathbf{x}' = A\mathbf{x}$:*

1. *If $TrA > 0$, we have a **spiral source (unstable)**.*

2. *If $TrA < 0$, we have a **spiral sink (asymptotically stable)**.*

3. *If $TrA = 0$, we have a **center (stable)**.*

Proof. This is obvious. □

Example 0.129 *Let $A \in M(2)$. Determine the nature of the equilibrium point $\mathbf{x} = 0$ of the ODE $\mathbf{x}' = A\mathbf{x}$ for each of the following cases:*

$$\begin{cases} (1).\ \det A < 0, \\ (2).\ \det A > 0, \quad TrA > 0, \\ (3).\ \det A > 0, \quad TrA < 0, \\ (4).\ \det A > 0, \quad TrA = 0. \end{cases}$$

Solution:

For (1), $\mathbf{x}=0$ is a **saddle**.

For (2), by the Jordan canonical form of A, given by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \beta > 0, \tag{181}$$

we must have $\lambda > 0,\ \mu > 0$ and $\alpha > 0$. Hence $\mathbf{x}=0$ is a **source** (can be any of the four subcases).

For (3), we must have $\lambda < 0,\ \mu < 0$ and $\alpha < 0$. Hence $\mathbf{x}=0$ is a **sink** (can be any of the four subcases).

For (4), the first two canonical forms in (181) cannot happen and we must have $\alpha = 0$ in the third canonical form. Hence $\mathbf{x}=0$ is a **center**. □

0.0.6 Conclusion:

To determine the nature of the equilibrium point $\mathbf{x}=0$ of the system $\mathbf{x}'=A\mathbf{x}$, where $A \in M(2)$, $\det A \neq 0$, we look at the sign of

$$(TrA)^2 - 4\det A$$

first, then look at the signs of TrA and $\det A$.

If we let $x = TrA$ and let $y = \det A$, then on the xy-plane, we have the following picture (also see the book "Differential Equations, Dynamical Systems, and an Introduction to Chaos" by Hirsch, Smale, Devaney, p. 63):

Picture Here.

End of the Third Part of the Course, 2023-1-5