ODE Lecture Notes, Part I, for Fall Semester 2020

Revised on 2020-11-19

1 Introduction.

Remark 1.1 *This notes is based on the textbook* ***"Elementary Differential Equations & Boundary Value Problems, 10th Edition" by Boyce & DiPrima.*** *However,* ***I will not follow the book exactly****. Lecture notes will be given to you via email whenever necessary.*

1.1 Notation and terminology.

Here are some basics you need to know:

- What is an ordinary differential equation (ODE)?

Let $x(t)$ be a function depending on time. Roughly speaking, an **equation** relating $x(t)$ and its derivatives $x'(t)$, $x''(t)$, ..., and perhaps some known functions of t, is called an **ODE**. For example, the following are all ODEs for x :

$$x' + e^x = \sin t, \quad x''' + 5x'' - 7x = 0, \quad \sin\left(x'' + x^2\right) = \cos\left(x + t^3\right), \quad \left(x'\right)^2 + x^2 = 1 + t^2 \qquad (1)$$

where $x = x(t)$, $x' = x'(t)$, etc.

- To **"solve"** an ODE, say $x' + e^x = \sin t$, means to find a family of **explicit** functions $x = x(t)$ (with **integration constant** C as a parameter), where each one of them is **differentiable** on **some open interval** $I = (a, b)$, so that we have

$$x'(t) + e^{x(t)} = \sin t, \qquad \text{for all} \quad t \in (a, b). \qquad (2)$$

  In such a case we say the function $x(t)$ is a **solution** of the ODE on (a, b). Note that although the variable t in the above ODE (2) can be allowed to be $t \in (-\infty, \infty)$, its solution $x(t)$ may, in general, not be defined on all $t \in (-\infty, \infty)$. You will know the domain of $x(t)$ **only after** you have solved the equation. The size of the domain interval (a, b) of solution $x(t)$ also depends on the integration constant C. In case there is difficulty finding explicit solutions $x(t)$, we will try to derive "**solution formula**" (involving indefinite integrals) or derive "**method of solutions**" for the ODE.

- There are only a few types of ODEs that can be (easily) solved **explicitly**. Note that one can rewrite the above ODE (2) in the **general explicit form**

$$x'(t) = \frac{dx}{dt} = f(t, x(t)) \quad \text{(or, for simplicity, just write it as just } x' = f(t, x)\text{)} \qquad (3)$$

  for some 2-variable continuous function $f(t, x)$, where $f(t, x) = \sin t - e^x$ is defined and continuous on its domain $D := (-\infty, \infty) \times (-\infty, \infty)$.

- (**Existence and Uniqueness Theorem, EUT**.) If an ODE has the form **(3)** where $f(t, x)$ is a **continuous** function on a domain $D \subset \mathbb{R}^2$, then for each point $(t_0, x_0) \in D$, there **is** a solution (**may not be unique**) $x(t)$ defined on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$ satisfying $(t, x(t)) \in D$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and

$$\begin{cases} x'(t) = f(t, x(t)), & \forall\, t \in (t_0 - \varepsilon, t_0 + \varepsilon) \\ x(t_0) = x_0. \end{cases} \qquad (4)$$

In this course, we always assume $f(t,x)$ is at least a **continuous function** on its domain $D \subset \mathbb{R}^2$ unless otherwise stated. The condition $x(t_0) = x_0$ in (4) is called an **initial condition** of the ODE $x' = f(t,x)$. **Its purpose is to determine the integration constant C uniquely.**

- Moreover, if $f(t,x)$ is "**better than continuous**" (will explain this later on) near $(t_0, x_0) \in D$, then the solution $x(t)$ of (4) is **unique** near $t = t_0$, i.e. there exists some small $\delta > 0$ such that there is **only one solution** satisfying

$$\begin{cases} x'(t) = f(t, x(t)), & \forall\, t \in (t_0 - \delta, t_0 + \delta) \\ x(t_0) = x_0. \end{cases}$$

- If the ODE $x' = f(t,x)$ (continuous function) has **no initial condition**, then it has **infinitely many** solutions (**with integration constant C as a parameter**). For ODE **not** of the form (3), it may have have **no solution at all (no matter what the initial condition is**). For example, the ODE

$$e^{x'(t)} = -1$$

has no solution at all. Note that one **cannot** rewrite the above ODE in the form (3). That also explains why it has no solution at all. Another example with **no solution** is the following ODE with initial condition

$$x(t)\,x'(t) = 1, \quad x(0) = 0, \tag{5}$$

i.e. one cannot find a differentiable function $x(t)$ define on some $(-\varepsilon, \varepsilon)$ satisfying (5) (impossible to have $x(0)\,x'(0) = 0x'(0) = 1$). Note that the initial condition happens to be at a **bad place** if we write the equation as

$$x'(t) = \frac{1}{x(t)} = f(t, x(t)), \quad x(0) = 0.$$

One can see that it has no solution at all since $f(t,x) = 1/x$ is **undefined** at $(0,0)$. On the other hand, if we replace $x(0) = 0$ by $x(0) = \lambda \neq 0$, then it has a solution defined near $t = 0$.

- The **order** of an ODE is "the order of the highest derivative" involved in the equation. For example, the order in the equations of (1) is 1, 3, 2, 1 respectively. A **first-order** ODE for an unknown function $x = x(t)$ has the **general explicit form**

$$x' = f(t,x). \tag{6}$$

In some rare situation, we also look at **first-order** ODE with the **general implicit form** $F(t, x, x') = 0$. For example, $\sin(x' + x) = e^x$.

- We say a **differentiable** function $x = x(t)$ is a **solution** to the first-order equation $x' = f(t,x)$ if there is some open interval $(a,b) \subset \mathbb{R}$ (domain of x) such that $(t, x(t)) \in D$ (domain of $f(t,x)$) for all $t \in (a,b)$ and

$$\frac{dx}{dt} = f(t, x(t)), \quad \forall\, t \in (a,b). \tag{7}$$

The **graph** of the curve $(t, x(t))$, $t \in (a,b)$, in the tx-plane is also called an **integral (solution) curve** of the equation. Geometrically, the slope of the integral curve is equal to the "**slope field**" (in the (t,x) plane) determined by the function $f(t,x)$ everywhere.

- The problem of solving a differential equation (7) subject to an initial condition $x(t_0) = x_0$, that is

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{8}$$

is called an **initial value problem** (ivp). Here, t_0 is a fixed value of time and x_0 is a fixed value of x and we assume that $(t_0, x_0) \in D$ (the domain of $f(t,x)$). Geometrically, the ivp (8) requires an integral curve $x = x(t)$ plotted in the tx-plane to pass through the fixed point (t_0, x_0). The curve is everywhere **tangent** to the slope field on the region D given by $f(t, x)$.

Example 1.2 *Consider the ODE with initial condition*

$$x(t)\, x'(t) = 1, \quad x(0) = \varepsilon > 0, \tag{9}$$

where $\varepsilon > 0$ is a constant. Show that the initial value problem has a unique solution $x(t)$ and find it ***explicitly****. What is the domain (maximal domain) of $x(t)$?*

Solution:

Any differentiable function $x(t)$ satisfying $x(t)\, x'(t) = 1$ near $t = 0$ must satisfy

$$\frac{d}{dt}\left(\frac{x^2(t)}{2}\right) = 1,$$

which gives

$$\frac{x^2(t)}{2} = t + C \quad \text{for some integration constant } C$$

and the initial condition implies

$$\frac{x^2(t)}{2} = t + \frac{\varepsilon^2}{2} \ \text{(i.e. } x(t) = \sqrt{2t + \varepsilon^2}\text{; the function } x(t) = -\sqrt{2t+\varepsilon^2} \text{the is not a solution).}$$

The solution of the ivp is defined on $t \in (-\varepsilon^2/2, \infty)$ and is unique. □

Remark 1.3 *If the initial condition is $x(0) = 0$, we get $x(t) = \sqrt{2t}$, which is not differentiable at $t = 0$ (or you may say $x'(0) = +\infty$). This matches with our previous discussion in (5).*

Example 1.4 *Find domain interval for solutions $x(t)$ of the ODE with initial condition:*

$$x' = f(t, x) = tx^2, \qquad x(0) = 1, \tag{10}$$

where t is allowed to lie on the interval $(-\infty, \infty)$ in the equation. What is the answer if we change the condition as $x(0) = -1$.

Solution:

We shall see that the ODE is **separable** shortly and the solution (unique in this example) is given by

$$x(t) = \frac{1}{1 - t^2/2}, \quad t \in \left(-\sqrt{2}, \sqrt{2}\right), \quad x(0) = 1. \tag{11}$$

It is defined on the maximal time interval $\left(-\sqrt{2}, \sqrt{2}\right)$ only (even if $f(t,x) = tx^2$ is defined on $(-\infty, \infty) \times (-\infty, \infty)$). The maximal time interval is finite is due to the term x^2, not the term t. The solution blows up to $+\infty$ as $t \to \pm\sqrt{2}$.

For $x(0) = -1$, the answer is

$$x(t) = \frac{-1}{1 + t^2/2}, \quad t \in (-\infty, \infty), \quad x(0) = -1. \tag{12}$$

Now it is defined on **the maximal time interval** $(-\infty, \infty)$. □

Remark 1.5 *Plot integral curves for (11) and (12) roughly.*

1.2 Simple example: a falling object.

Example 1.6 *(This is Examples 1, 2 in p. 2-4.). A* ***falling object*** *equation in physics has the form (based on Newtonian mechanics)*

$$m\frac{dv}{dt} = F \text{ (force)} = mg \text{ (gravitational force)} - \gamma v \text{ (air resistance)},$$

where $v = v(t)$ is the velocity of the falling object (with mass m), g is the gravitational constant, and γ is also a constant related to ***air resistance****. To find the solutions of the equation (we take $m = 10kg,\ g = 9.8m/s^2,\ \gamma = 2kg/s$)*

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v = v(t), \quad t \in [0, T] \text{ (some interval, at } t = T \text{ the object hit the ground)}, \tag{13}$$

we can rewrite it as $\frac{dv}{dt} + \frac{v}{5} = 9.8$ and multiply the equation by $e^{\frac{1}{5}t}$ (this is a ***trick*** *!!!, the function $e^{\frac{1}{5}t}$ is called an* ***integrating factor*** *of the ODE) to get*

$$e^{\frac{1}{5}t}\left(\frac{dv}{dt} + \frac{v}{5}\right) = 9.8e^{\frac{1}{5}t},$$

which can be written as

$$\frac{d}{dt}\left(e^{\frac{1}{5}t}v(t)\right) = 9.8e^{\frac{1}{5}t}, \quad t \in [0, T]. \tag{14}$$

*Now there are two ways to solve it (**I prefer the first way**). The first way is to integrate (**indefinite integral**) both sides of (14) and obtain the identity*

$$e^{\frac{1}{5}t}v(t) = 49e^{\frac{1}{5}t} + C, \tag{15}$$

for some integration constant C. The constant C is determined by the ***initial velocity*** *(given initial condition) $v(0)$ of the object (we take it positive in the downward direction, negative in the upward direction). Letting $t = 0$ in (15) gives $C = v(0) - 49$ and we conclude*

$$v(t) = (v(0) - 49)e^{-\frac{1}{5}t} + 49, \quad t \in [0, T]. \tag{16}$$

*The second way is to integrate (**definite integral**) both sides of (14) over the interval $[0, t]$, $t > 0$, and obtain*

$$e^{\frac{1}{5}t}v(t) - e^0 v(0) = \int_0^t 9.8e^{\frac{1}{5}s}ds = 49e^{\frac{1}{5}t} - 49, \quad t \in [0, T]$$

and conclude the same result as in (16). Note that, for any $v(0)$, if we forget the existence of the ground, the solution (16) can be defined on the time interval $(-\infty, \infty)$ satisfying the asymptotic behavior $\lim_{t\to\infty} v(t) = 49$ (no matter what the initial velocity is), which is called the ***asymptotic stable solution*** *of the equation. It is also an* ***equilibrium solution*** *(a solution which is* ***independent of time****) of the equation (i.e. the function $v(t) \equiv 49$ is also a solution of the equation). Note that if $v(0) = 49$, then we will have $v(t) \equiv 49$ for all time.*

Remark 1.7 *Draw a picture of the "**slope field**" and "**equilibrium solution**".*

Example 1.8 *(Example 3 in p. 5-6). Read it yourself.*

1.3 Existence of a solution.

Theorem 1.9 ***(Existence of a solution.)*** *Let $a,\ b>0$ be two constants. Let $R:|t-t_0|\le a,\ |x-x_0|\le b$ be a rectangle contained in D and $f(t,x): D\subseteq\mathbb{R}^2\to\mathbb{R}$ is* ***continuous*** *with*

$$|f(t,x)|\le M \quad \text{for all} \quad (t,x)\in R\subset D, \tag{17}$$

for some constant $M>0$. Then there exists a solution $x(t)$ to the ODE (ivp)

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{18}$$

defined for $|t-t_0|\le h$, where $h=\min(a,b/M)$. Moreover, we also have $(t,x(t))\in R$ for all $|t-t_0|\le h$, i.e.

$$|x(t)-x_0| = |x(t)-x(t_0)|\le b, \quad \forall \quad |t-t_0|\le h.$$

Note that the solution to (18) ***may not be unique.***

Proof. We will not prove the theorem. It is enough to know the result. □

Remark 1.10 *We will discuss the issue concerning the* ***uniqueness of a solution*** *later on. At this moment, it is important to know that for the ivp*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \quad (x_0,t_0)\in D \end{cases} \tag{19}$$

may have more than one *(or* ***infinitely many****) solutions if we* ***only assume*** *that $f(t,x): D\subseteq\mathbb{R}^2\to\mathbb{R}$ is* ***continuous.*** *Look at the example*

$$\begin{cases} x' = x^{\frac{1}{3}} \\ x(0) = 0, \quad (0,0)\in\mathbb{R}^2, \end{cases} \tag{20}$$

where now $f(t,x)=x^{\frac{1}{3}}:\mathbb{R}^2\to\mathbb{R}$ is a continuous function on $\mathbb{R}^2$. Note that the function $f(t,x)=x^{\frac{1}{3}}$ is only continuous near the point $(0,0)$, not differential at $(0,0)$ (or more precisely, ***not Lipschitz continuous with respect to*** *x* ***near*** *$(0,0)$). One can see that the following two functions are both solutions to the ivp (20) defined on $(-\infty,\infty)$:*

$$(1).\ x(t)\equiv 0, \quad \forall\ t\in(-\infty,\infty),$$

$$(2).\ x(t)=\begin{cases} 0, & t\in(-\infty,0) \\ \sqrt{\frac{8}{27}}t^{\frac{3}{2}}, & t\in[0,\infty). \end{cases}$$

Note that both functions are continuously differentiable satisfying (20) on $(-\infty,\infty)$.

1.4 Gronwall inequality and uniqueness of a solution.

To prove uniqueness of solution to a given ODE with initial condition, we need the following famous inequality:

Lemma 1.11 ***(Gronwall inequality, simple form.)*** *Let $a<b$. Assume $u(t):[a,b]\to\mathbb{R}$ is continuous and satisfies the inequality*

$$u(t)\le C+K\int_a^t u(s)\,ds \quad \text{for all} \quad t\in[a,b], \tag{21}$$

where $C\in(-\infty,\infty)$ and $K\ge 0$ are constants. Then we have

$$u(t)\le Ce^{K(t-a)} \quad \text{for all} \quad t\in[a,b]. \tag{22}$$

Remark 1.12 *In particular, if $u(t) \geq 0$ and $C = 0$, then we have*

$$0 \leq u(t) \leq Ce^{K(t-a)} = 0$$

and so $u(t) \equiv 0$ on $[a, b]$. This is what we need later on.

Remark 1.13 *Note that the function $v(t) = Ce^{K(t-a)}$, $t \in [a, b]$, satisfies*

$$v(t) = C + K \int_a^t v(s)\, ds, \quad \forall\, t \in [a, b].$$

Therefore, the estimate (22) is ***optimal.***

Remark 1.14 *(**Gronwall inequality, another version.**) If we replace (21) by*

$$u(t) \leq C + K \int_t^b u(s)\, ds \quad \text{for all} \quad t \in [a, b], \tag{23}$$

then we have

$$u(t) \leq Ce^{K(b-t)} \quad \text{for all} \quad t \in [a, b]. \tag{24}$$

Proof. Let

$$U(t) = C + K \int_a^t u(s)\, ds, \quad U(a) = C, \quad t \in [a, b].$$

By the assumption, we have $u(t) \leq U(t)$ on $[a, b]$. Also $U'(t) = Ku(t) \leq KU(t)$ on $[a, b]$ (here we use $K \geq 0$). Hence we have

$$U'(t) \leq KU(t) \quad \text{on} \quad [a, b], \qquad U(a) = C,$$

which gives

$$\frac{d}{dt}\left(U(t)\, e^{-K(t-a)}\right) = \left[U'(t) - KU(t)\right] e^{-K(t-a)} \leq 0$$

for all $t \in [a, b]$, and so $U(t)\, e^{-K(t-a)} \leq U(a)\, e^{-K(a-a)} = C$ for all $t \in [a, b]$. Therefore $U(t) \leq Ce^{K(t-a)}$ for all $t \in [a, b]$ and so

$$u(t) \leq U(t) \leq Ce^{K(t-a)}, \quad \forall\, t \in [a, b].$$

The proof is done. □

We can use **Gronwall inequality** to prove the uniqueness of a solution to the ivp $x' = f(t, x)$, $x(t_0) = x_0$.

Definition 1.15 *Let $D \subseteq \mathbb{R}^2$ be a domain (open connected set) and let $f(t, x) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. If there exists a constant $k > 0$ such that*

$$|f(t, x_1) - f(t, x_2)| \leq k\, |x_1 - x_2|, \quad \forall\ (t, x_1),\ (t, x_2) \in D,$$

*then we say $f(t, x)$ is **Lipschitz continuous** on D with respect to x with **Lipschitz constant** $k > 0$ (the size of k is not important here). Note that the constant k depends on the domain D, but not on the point $(t, x) \in D$.*

Exercise 1.16 *Which of the following single-variable function $f(x)$ is Lipschitz continuous with respect to x on the domain $[0, 1]$ (for some Lipschitz constant $k > 0$): (1). $f(x) = x^\alpha$, $0 < \alpha < 1$. (2). $f(x) = x^\alpha$, $\alpha \geq 1$. (3). $f(x) = \sin x$. (4). $f(x)$ is differentiable on $[0, 1]$ with bounded derivative $f'(x)$ on $[0, 1]$.*

We can now prove the uniqueness theorem:

Theorem 1.17 *(**Uniqueness of a solution.**) Assume that $f(t,x) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous on domain D and is **Lipschitz continuous** on D with respect to x for some constant $k > 0$. If $(t_0, x_0) \in D$ and $x(t)$, $\tilde{x}(t)$ are two solutions to $x' = f(t,x)$ on some common interval $I = (t_0 - h, t_0 + h)$, $h > 0$, with $x(t_0) = \tilde{x}(t_0) = x_0$, then $x(t) \equiv \tilde{x}(t)$ for all $t \in I$. In particular, any two **integral curves** (graphs of solutions) **cannot intersect** at any point of D.*

Remark 1.18 *Since $x(t)$, $\tilde{x}(t)$ are both solutions to the equation $x' = f(t,x)$ on I, by definition, we must have $(t, x(t)) \in D$ and $(t, \tilde{x}(t)) \in D$ for all $t \in I$,*

Proof. We have

$$x'(t) = f(t, x(t)) \quad \text{and} \quad \tilde{x}'(t) = f(t, \tilde{x}(t)), \quad \forall\, t \in I,$$

where $x(t_0) = x(t_0) = x_0$. We can rewrite the differential equations as **integral equations (equation involving $x(t)$ and its integral)**:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \quad \text{and} \quad \tilde{x}(t) = x_0 + \int_{t_0}^{t} f(s, \tilde{x}(s))\, ds, \quad t \in I. \tag{25}$$

Hence, for $t \in (t_0, t_0 + h)$, we have

$$0 \le |x(t) - \tilde{x}(t)| \le \int_{t_0}^{t} |f(s, x(s)) - f(s, \tilde{x}(s))|\, ds \le \int_{t_0}^{t} k\, |x(s) - \tilde{x}(s)|\, ds, \quad \forall\, t \in (t_0, t_0 + h)$$

By **Gronwall inequality**, we have $x(t) = \tilde{x}(t)$ on $(t_0, t_0 + h)$. The proof of $x(t) = \tilde{x}(t)$ on $(t_0 - h, t_0)$ is similar (see Remark 1.19 below). □

Remark 1.19 *In the above proof, for $t \in (t_0 - h, t_0)$, we have*

$$x(t) = x_0 - \int_{t}^{t_0} f(s, x(s))\, ds \quad \text{and} \quad \tilde{x}(t) = x_0 - \int_{t}^{t_0} f(s, \tilde{x}(s))\, ds, \quad t \in (t_0 - h, t_0)$$

and then

$$0 \le |x(t) - \tilde{x}(t)| \le \int_{t}^{t_0} |f(s, x(s)) - f(s, \tilde{x}(s))|\, ds \le \int_{t}^{t_0} k\, |x(s) - \tilde{x}(s)|\, ds, \quad \forall\, t \in (t_0 - h, t_0).$$

Now by the Gronwall inequality in the form in Remark 1.14, we have $x(t) = \tilde{x}(t)$ on $(t_0 - h, t_0)$.

Remark 1.20 *(See Remark 1.10 also.) The "**Lipschitz condition**" in the above theorem is necessary due to the **non-uniqueness** of a solution to the ivp on any interval $t \in (-\varepsilon, \varepsilon)$:*

$$\frac{dx}{dt} = x^{\frac{1}{3}}, \quad x(0) = 0. \tag{26}$$

Note that if we change the ivp as

$$\frac{dx}{dt} = x^{\frac{1}{3}}, \quad x(0) = x_0 > 0, \tag{27}$$

*then near $t = 0$ there exists a unique solution given by (here, since $x_0 > 0$, near x_0 we have $x > 0$ and one can **rewrite the equation** as $x^{-1/3} dx = dt$ and integrate both sides)*

$$x(t) = \left(x_0^{2/3} + \frac{2}{3} t \right)^{3/2} > 0, \quad t \in (-\varepsilon, \varepsilon) \text{ for some small } \varepsilon > 0. \tag{28}$$

*Another way to see uniqueness is that now the function $x^{1/3}$ is **Lipschitz continuous** near $x = x_0 > 0$ and we can apply Theorem 1.17 near $x = x_0$.*

Remark 1.21 *Let $f(x) : I \to \mathbb{R}$ be a continuously differentiable function, where $I \subset \mathbb{R}$ is an interval. Assume there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. Then we have*

$$|f(p) - f(q)| = |f'(\theta)(p-q)| \leq M|p-q|, \quad \forall\ p,\ q \in I.$$

*With this, we say $f(x)$ is a **Lipschitz continuous** function on I with Lipschitz constant M.*

Remark 1.22 *(**A simple way to test Lipschitz continuous condition.**) By Remark 1.21, if $f(t,x) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is **continuous** and $\frac{\partial f}{\partial x}(t,x) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is also **continuous,** where $D \subseteq \mathbb{R}^2$ is a domain, then for fixed $(t_0, x_0) \in D$ there exists a small **closed rectangle** R centered at (t_0, x_0) and a constant $M > 0$ such that*

$$\left|\frac{\partial f}{\partial x}(t,x)\right| \leq M, \quad \forall\ (t,x) \in R.$$

By the mean value theorem, we have

$$|f(t,x_1) - f(t,x_2)| = \left|\frac{\partial f}{\partial x}(t,\theta)\cdot(x_1-x_2)\right| \leq M|x_1-x_2|, \quad \forall\ (t,x_1),\ (t,x_2) \in R,$$

*where θ lies between x_1 and x_2 (θ may depend on t). This implies that f is **Lipschitz continuous on** R **with respect to** x. Therefore, the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$, has a unique solution $x(t)$ defined on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$. This is **Theorem 2.4.2 in p. 70 of the book.***

Remark 1.23 *(**Important.**) If a continuous function $f(t,x)$ is **not Lipschitz continuous** near a point $(t_0, x_0) \in D$, then it is **still possible** for the initial value problem*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{29}$$

*to **have a unique solution**. One example is (compare with (26))*

$$\frac{dx}{dt} = x^{\frac{1}{3}} + 1, \quad x(0) = 0. \tag{30}$$

*The function $f(x) = x^{\frac{1}{3}} + 1$ is **not Lipschitz continuous** near $(0,0)$. However, it has **unique solution** $x(t)$ **defined near** $t = 0$. You can prove the uniqueness by use of simple calculus argument or apply Lemma 1.24 below. Another interesting example for a **unique solution** is the following:*

$$\frac{dx}{dt} = f(x), \quad x(0) = 0, \quad \text{where} \quad f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{31}$$

*Note that $f(x)$ is continuous on $\mathbb{R}$ but **not** Lipschitz continuous near $x = 0$ (why ?). However, the above ivp has the unique solution $x(t) \equiv 0$, $t \in (-\infty, \infty)$. I will leave the proof as a homework problem.*

1.5 Another method to check the uniqueness of a solution for an autonomous ODE.

In case the ODE $x' = f(t,x)$ has the form $x' = f(x)$, we call it an **autonomous equation**. There is another way to determine uniqueness of a solution (do not have to involve the concept of "**Lipschitz continuity**") for an autonomous ODE. We first note the following simple situation:

Lemma 1.24 *(**The case** $f(x_0) \neq 0$.) Consider the ivp*

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0, \tag{32}$$

*where $f(x)$ is a **continuous function** defined on $I = (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$ and $f(x) > 0$ **on** I (or $f(x) < 0$ **on** I). Then if $x(t),\ \tilde{x}(t) : J \to I$ are two solutions to (32) on some interval $J = (t_0 - h, t_0 + h),\ h > 0$, then $x(t) \equiv \tilde{x}(t)$ for all $t \in J$. Here $f(x)$ **may not be** Lipschitz continuous on I.*

Remark 1.25 *The existence of a solution to (32), defined near $t = t_0$, is guaranteed due to Theorem 1.9.*

Remark 1.26 *(**Important.**) In case $f(x)$ is a **continuously differentiable function** on $I = (x_0 - \delta, x_0 + \delta)$, then by the mean value theorem, $f(x)$ is automatically **Lipschitz continuous** near $x = x_0$ and we have **uniqueness** of a solution. Therefore, in practical examples, the function $f(x)$ is **only continuous** near $x = x_0$ and **not Lipschitz continuous near** $x = x_0$. For example, the problem*

$$\frac{dx}{dt} = x^{\frac{1}{3}} + 1, \quad x(0) = 0, \tag{33}$$

*has a unique solution defined near $t = 0$ since the function $f(x) = x^{1/3} + 1$ is positive near $x = 0$ (it is **not Lipschitz continuous near** $x = 0$).*

Remark 1.27 *(**Important.**) By Lemma 1.24, the "**non-uniqueness**" of solutions can happen only when $f(x_0) = 0$.*

Proof. We have $x(t),\ \tilde{x}(t) : J \to I$ with $x(t_0) = \tilde{x}(t_0) = x_0$ and $f(x(t)) > 0,\ f(\tilde{x}(t)) > 0$ for all $t \in J$. Let $F(x) : I \to \mathbb{R}$ be the function

$$F(x) = \int_{x_0}^{x} \frac{1}{f(s)} ds, \quad x \in I = (x_0 - \delta, x_0 + \delta),$$

which is a continuously differentiable, **strictly increasing** function on I with $F(x_0) = 0$. We now have the two identities (use differentiation to verify them)

$$F(x(t)) = t - t_0 \quad \text{and} \quad F(\tilde{x}(t)) = t - t_0, \quad \forall\ t \in J = (t_0 - h, t_0 + h)$$

and so $F(x(t)) \equiv F(\tilde{x}(t))$ for all $t \in J$. As $F(x)$ is one-one on I, we have $x(t) \equiv \tilde{x}(t)$ for all $t \in J$. The proof is done. □

1.5.1 Using integral behavior to determine uniqueness of a solution.

By Lemma 1.24, it remains to deal with the case $f(x_0) = 0$. For convenience, we may assume $t_0 = x_0 = 0$ in (32) and now we look at the ivp:

$$\frac{dx}{dt} = f(x), \quad x(0) = 0, \quad f(0) = 0, \tag{34}$$

where $f(x)$ is a **continuous function** defined on $I = (-\delta, \delta)$ for some $\delta > 0$ with $f(0) = 0,\ f(x) > 0$ on $(0, \delta),\ f(x) < 0$ on $(-\delta, 0)$. Clearly the function $x(t) \equiv 0$ is an **equilibrium solution** to (34) (and no others). However, Lemma 1.24 is not applicable here due to $f(0) = 0$.

We have the following interesting result:

Lemma 1.28 *(**The case** $f(x_0) = 0$.) Let $f(x)$ be a continuous function only, defined near $x = 0$, and is described as in the above. Let $x(t) : J \to I$, $x(0) = 0$, be a solution to (34) on some interval $J = (-h, h)$, $h > 0$. Then if*

$$\int_{-\delta/2}^{0} \frac{1}{f(x)} dx = -\infty \quad \text{and} \quad \int_{0}^{\delta/2} \frac{1}{f(x)} dx = +\infty, \tag{35}$$

we must have $x(t) \equiv 0$ on J (i.e. we have uniqueness of a solution). Conversely, if the only solution to the ivp (34) is $x(t) \equiv 0$, then we have (35).

Remark 1.29 *Note that $f(x)$ may not be Lipschitz continuous on $I = (-\delta, \delta)$. In case $f(x)$ is Lipschitz continuous on $I = (-\delta, \delta)$, then we must have (35), and then we get uniqueness.*

Remark 1.30 *If we have*

$$\int_{-\delta/2}^{0} \frac{1}{f(x)} dx = -\infty, \quad \int_{0}^{\delta/2} \frac{1}{f(x)} dx \text{ converges,}$$

then the solution is not unique on the interval $(0, h)$. Similarly, if we have

$$\int_{-\delta/2}^{0} \frac{1}{f(x)} dx \text{ converges,} \quad \int_{0}^{\delta/2} \frac{1}{f(x)} dx = +\infty,$$

then the solution is not unique on the interval $(-h, 0)$.

Proof. Omit. □

Exercise 1.31 *Show that if $f(x)$ is Lipschitz continuous on $I = (-\delta, \delta)$, then we must have (35).*

Exercise 1.32 *Use Lemma 1.28 to show that the equation*

$$\frac{dx}{dt} = x^{1/3}, \quad x(0) = 0$$

has no unique solution.

Example 1.33 *(**Leave this as homework problem.**) Which of the following initial value problems have a unique solution $x(t)$ defined near $t = 0$?*

$$\begin{aligned}
(1). \quad & \frac{dx}{dt} = (x-1)^{\frac{1}{3}}, \quad x(0) = 1, \\
(2). \quad & \frac{dx}{dt} = x^{\frac{1}{3}} + 1, \quad x(0) = 0, \\
(3). \quad & \frac{dx}{dt} = x^{\frac{1}{3}} + 1, \quad x(0) = -1, \\
(4). \quad & \frac{dx}{dt} = f(x), \quad x(0) = 0,
\end{aligned}$$

where in (4) the function $f(x)$ is defined on $(-\delta, \delta)$ for some $\delta > 0$, given by

$$f(x) = \begin{cases} x\left(\sin \frac{1}{x} + 2\right), & x \neq 0 \\ 0, \quad x = 0. \end{cases}$$

Solution:

For (1), we note that the function $f(x) = (x-1)^{1/3}$ is not Lipschitz continuous near $x = 1$. If we let $y(t) = x(t) - 1$, we get

$$\frac{dy}{dt} = y^{\frac{1}{3}}, \quad y(0) = 0$$

and so it has no unique solution defined near $t = 0$. Hence the original equation has **no unique solution** defined near $t = 0$.

For (2), the function $f(x) = x^{1/3} + 1$ is **not Lipschitz continuous** near $x = 0$; however, $f(x)$ is **positive** near $x = 0$. By Lemma 1.24, it has a **unique** solution $x(t)$ defined near $t = 0$.

For (3), the function $f(x) = x^{1/3} + 1$ is **Lipschitz continuous** near $x = -1$ (since it is **differentiable** near $x = -1$). Hence it has a unique solution $x(t)$ defined for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. One can also use Lemma 1.28 to derive uniqueness. Let $y(t) = x(t) + 1$. We have

$$\frac{dy}{dt} = (y-1)^{\frac{1}{3}} + 1, \quad y(0) = 0.$$

The function $f(y) = (y-1)^{\frac{1}{3}} + 1$ satisfies $f(0) = 0$, $f(x) > 0$ on $(0, \infty)$, $f(x) < 0$ on $(-\infty, 0)$, and $f'(0) = 1/3$. Therefore, we have (pick $\delta = 1$)

$$\int_{-1/2}^{0} \frac{1}{f(y)} dy = -\infty \quad \text{and} \quad \int_{0}^{1/2} \frac{1}{f(y)} dy = +\infty, \tag{36}$$

and the uniqueness follows. The unique solution is $x(t) \equiv -1$.

For (4), we note that $f(x)$ is continuous on $(-\delta, \delta)$, not differentiable at $x = 0$, **not Lipschitz continuous** on $(-\delta, \delta)$ (verify this), with $f(0) = 0$, $f(x) > 0$ on $(0, \delta)$, $f(x) < 0$ on $(-\delta, 0)$. The best way is to use Lemma 1.28. We compute

$$\int_0^{\delta/2} \frac{1}{f(x)} dx = \int_0^{\delta/2} \frac{1}{x\left(\sin \frac{1}{x} + 2\right)} dx \geq \int_0^{\delta/2} \frac{1}{3x} dx = +\infty$$

and

$$\int_{-\delta/2}^{0} \frac{1}{f(x)} dx = \int_{-\delta/2}^{0} \frac{1}{x\left(\sin \frac{1}{x} + 2\right)} dx \leq \int_{-\delta/2}^{0} \frac{1}{3x} dx = -\infty.$$

Therefore, we have **unique solution** $x(t) \equiv 0$.

2 Chapter 2: First order differential equations.

2.1 First order linear differential equations (this is Section 2.1 of the book, see p. 31).

By a "**first order linear differential equation**" we mean it is an equation of the standard form (or can be **rewritten** in the form)

$$y' + p(t)\, y = q(t), \quad t \in (a, b), \tag{L}$$

where $p(t)$ and $q(t)$ are given continuous functions on (a, b) (some interval, usually it is $(-\infty, \infty)$) and the function $y(t)$ is to be solved.

Remark 2.1 *The reason of calling equation (L) a* ***linear equation*** *is that* $p(t)\, y + q(t)$ *is a linear expression of* y *(but not of* t*).*

Remark 2.2 *Another standard form for a* ***first order linear differential equation*** *is the following:*

$$a(t)\,y' + b(t)\,y + c(t) = 0, \quad t \in (a,b),$$

where $a(t),\ b(t),\ c(t)$ *are given continuous functions on* (a,b). *We will come back to it later on.*

For a linear equation (L), one can use the **method of integrating factor** (due to Leibniz) to find its **general solution formula (that means all solutions of (L) are included in this formula)**. Let

$$\mu(t) = e^{\int p(t)dt}, \quad t \in (a,b) \tag{37}$$

where in the indefinite integral $\int p(t)\,dt$, you **do not have to** add integration constant (**explain this !!!**). Multiply equation (L) by $\mu(t)$ to get

$$\mu(t)\left(y' + p(t)\,y\right) = \mu(t)\,q(t)$$

and note that the left hand side of the above is a "**total derivative**". Hence we get

$$\frac{d}{dt}\left[e^{\int p(t)dt} \cdot y(t)\right] = e^{\int p(t)dt} q(t)$$

and so

$$e^{\int p(t)dt} \cdot y(t) = \int \left(e^{\int p(t)dt} q(t)\right) dt + C \text{ (integration constant)}$$

and obtain the **general solution formula (all solutions are included in this formula with suitable choice of** C**):**

$$y(t) = e^{-\int p(t)dt}\left\{\int \left(e^{\int p(t)dt} q(t)\right) dt + C\right\}, \quad t \in (a,b) \tag{38}$$

where in the underlined integral you **have to** add integration constant C (in order to include all possible solutions).

Remark 2.3 *In case there is an initial condition* $y(t_0) = y_0,\ t_0 \in (a,b)$, *for equation (L), find its general solution (38) first (****if you can integrate the integrals in (38)****), then use the condition* $y(t_0) = y_0$ *to find the constant* C. *Another way (****if you cannot integrate the integrals in (38)****) is to use the formula*

$$y(t) = e^{-\int_{t_0}^{t} p(\theta)d\theta}\left\{\int_{t_0}^{t} \left(e^{\int_{t_0}^{s} p(\theta)d\theta} q(s)\right) ds + y_0\right\}, \quad t \in (a,b). \tag{39}$$

From it we see that $y(t_0) = y_0$. *Now we explain how to get (39). First we note that the function* $\mu(t) = \exp\left(\int_{t_0}^{t} p(\theta)\,d\theta\right)$ *is an integrating factor with* $\mu(t_0) = 1,\ \mu'(t) = p(t)\,\mu(t)$. *Hence the function* $g(t) = \mu(t)\,y(t)$ *satisfies* $g(t_0) = y_0$ *and*

$$g'(t) = \mu'(t)\,y(t) + \mu(t)\,y'(t) = p(t)\,\mu(t)\,y(t) + \mu(t)\left[-p(t)\,y(t) + q(t)\right] = \mu(t)\,q(t), \quad t \in (a,b),$$

and so

$$g(t) = \int_{t_0}^{t} \mu(s)\,q(s)\,ds + y_0 = \int_{t_0}^{t} \left(e^{\int_{t_0}^{s} p(\theta)d\theta} q(s)\right) ds + y_0, \quad t \in (a,b).$$

From the above we get (39).

From the general solution formula (38) one can see that $y(t)$ is also defined on the interval (a,b), which is the common domain of $p(t)$ and $q(t)$. Thus we can conclude the following **existence and uniqueness theorem** for **first order linear differential equations of the form (L):**

Theorem 2.4 *(**This is Theorem 2.4.1 in p. 69 of the book.**) Consider the linear equation (L), where $p(t)$ and $q(t)$ are given continuous functions defined on some interval (a,b). Then, for any $t_0 \in (a,b)$ and $y_0 \in \mathbb{R}$, the initial value problem*

$$\begin{cases} y' + p(t)\, y = q(t) \\ y(t_0) = y_0 \end{cases} \tag{40}$$

*has a **unique** solution $y(t)$ which is defined **on the whole interval** (a,b). Moreover, it is given by (38) for some unique constant C making $y(t_0) = y_0$ or by the formula (39).*

Remark 2.5 *(**Important.**) The general solution formula in (38) can be decomposed as*

$$y(t) = g_0(t) + h(t), \quad t \in (a,b)$$

where

$$g_0(t) = Ce^{-\int p(t)dt}$$

*is the **general solution** of the homogeneous linear equation $y' + p(t)\, y = 0$, and*

$$h(t) = e^{-\int p(t)dt}\left[\int \left(e^{\int p(t)dt} q(t)\right) dt + C\right]$$

*is a **particular solution** of the nonhomogeneous linear equation $y' + p(t)\, y = q(t)$. There will be a theory explaining this later on.*

Remark 2.6 *(**Important.**) In case a linear equation is of the form*

$$\alpha(t)\, y' + \beta(t)\, y = q(t), \quad t \in I = (a,b), \tag{41}$$

*where $\alpha(t)$, $\beta(t)$, $q(t)$ are given continuous functions on I with $a(t) \neq 0$ on I, then on I the equation (41) is **equivalent to** the linear equation*

$$y' + \frac{\beta(t)}{\alpha(t)} y = \frac{q(t)}{\alpha(t)}, \quad t \in I, \quad \alpha(t) \neq 0 \text{ on } I. \tag{42}$$

*If $\alpha(t_0) = 0$ at some $t_0 \in (a,b)$, then, **in general (but not always)**, solutions of (41) cannot be defined at the point $t = t_0$. For example, look at the equation*

$$ty'(t) = 1, \quad t \in (-\infty, \infty). \tag{43}$$

We see that it is impossible to have any function $y(t)$ differentiable at $t = 0$ such that

$$0y'(0) = 1.$$

Hence, it is impossible to have a solution $y(t)$ of (43) on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Any solution $y(t)$ to (43) has the form

$$y(t) = \log|t| + C, \quad t \in (-\infty, 0) \bigcup (0, \infty).$$

*It is not defined at $t = 0$. On the other hand, it is possible to have certain **particular solution** defined across $t = t_0$ even if $a(t_0) = 0$. For example, look at the equation*

$$ty' + 2y = 4t^2, \quad t \in (-\infty, \infty), \quad \alpha(t) = t = 0 \text{ at } t = 0. \tag{44}$$

*We see that $y(t) = t^2$ is a solution which is also defined on $(-\infty, \infty)$. **In fact, the only solution of (44) that is defined across** $t = 0$ **is the solution** $y(t) = t^2$ **and no others.** See the example below.*

Example 2.7 *(**This is Example 4 in book p. 37.**) Solve the initial value problem:*

$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2. \end{cases} \tag{45}$$

*Note that the equation is defined on $t \in (-\infty, \infty)$. However, the solution may not be defined on $t \in (-\infty, \infty)$ since the equation **degenerates** at $t = 0$.*

Remark 2.8 *Draw integral curves of the ODE. See p. 38.*

<u>**Solution**</u>:

If we confine to the interval $(-\infty, 0) \bigcup (0, \infty)$, then the equation is equivalent to

$$y' + \frac{2}{t}y = 4t, \quad t \in (-\infty, 0) \bigcup (0, \infty)$$

and its **general solution** is given by (the integrating factor is t^2)

$$y(t) = t^2 + \frac{C}{t^2}, \quad t \in (-\infty, 0) \bigcup (0, \infty), \quad C \text{ is an arbitrary constant} \tag{46}$$

and by the condition $y(1) = 2$ we get $1 + C = 2$, which gives the unique solution of (45):

$$y(t) = t^2 + \frac{1}{t^2}, \quad t \in (0, \infty) \text{ (note that } 1 \in (0, \infty) \text{ and we exclude the interval } (-\infty, 0)\text{)}.$$

Note that the particular solution is defined on $(0, \infty)$, **not on** $(-\infty, \infty)$ (even though $t \in (-\infty, \infty)$ is allowed in (45)). This is because when $t = 0$, equation (45) becomes **singular**.

We conclude that the **general solution** of the equation $ty' + 2y = 4t^2$ on $(0, \infty) \bigcup (-\infty, 0)$ is given by $y(t) = t^2 + \frac{C}{t^2}$, $C \in \mathbb{R}$. The only solution which is defined on $(-\infty, \infty)$ is $y(t) = t^2$ (when $C = 0$). It satisfies $y(0) = 0$. See the picture in p. 38 of the book. □

Example 2.9 *(**This is Example 5 in book p. 38.**) Solve the initial value problem:*

$$\begin{cases} 2y' + ty = 2, \quad t \in (-\infty, \infty) \\ y(0) = 1. \end{cases} \tag{47}$$

Remark 2.10 *Draw integral curves of the ODE. See p. 39.*

<u>**Solution**</u>:

The equation is equivalent to

$$y' + \frac{t}{2}y = 1$$

and so the general solution formula in (38) becomes

$$y(t) = e^{-\frac{t^2}{4}} \left(\int e^{\frac{t^2}{4}} dt + C \right) \tag{48}$$

and unfortunately we are not able to integrate $\int e^{\frac{t^2}{4}} dt$. **Hence, unlike what we usually do, we cannot plug in the condition** $y(0) = 1$ **to find** C. On the other hand, **one can also express the general solution formula as**

$$y(t) = e^{-\frac{t^2}{4}} \left(\int_0^t e^{\frac{s^2}{4}} ds + C \right). \tag{49}$$

Now one can plug in $t=0$ to find $C=1$. Thus the particular solution to the initial value problem is given by

$$y(t)=e^{-\frac{t^2}{4}}\left(\int_0^t e^{\frac{s^2}{4}}ds+1\right),\quad t\in(-\infty,\infty). \tag{50}$$

See the picture in p. 39 of the book. $\square$

Remark 2.11 *(**Important.**) By the **L'Hopital rule**, for any constant C in (49) we have*

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}\frac{\int_0^t e^{s^2/4}ds+C}{e^{t^2/4}}=\lim_{t\to\infty}\frac{e^{t^2/4}}{e^{t^2/4}\frac{t}{2}}=0. \tag{51}$$

*Note that the function $y(t)\equiv 0$ is **not** an equilibrium solution of the equation. For the general initial value problem (40), if the solution is defined on $(-\infty,\infty)$ with $y(t_0)=y_0$, then we have*

$$\begin{aligned}\lim_{t\to\infty}y(t)&=\lim_{t\to\infty}\frac{\int_{t_0}^t\left(e^{\int_{t_0}^s p(\theta)d\theta}q(s)\right)ds+y_0}{e^{\int_{t_0}^t p(\theta)d\theta}}\quad \textit{(assume it is of } \frac{\infty}{\infty} \textit{ form)}\\ &=\lim_{t\to\infty}\frac{e^{\int_{t_0}^t p(\theta)d\theta}q(t)}{e^{\int_{t_0}^t p(\theta)d\theta}p(t)}=\lim_{t\to\infty}\frac{q(t)}{p(t)}\quad \textit{(assume the limit exists)}.\end{aligned} \tag{52}$$

2.1.1 Bernoulli equations (this appears in p. 77 of the book).

This type of first order **nonlinear** equation can be converted to a first order **linear** equation by a suitable change of variables. This method was found by Leibniz in 1696. Consider the equation

$$y'+p(t)y=q(t)y^n,\quad n\in\mathbb{Z}. \tag{53}$$

When $n=0$ or 1, (53) is a linear equation and we know how to solve it. Otherwise, it is a **nonlinear equation**. When $n\neq 0$ or 1, we can let $v=y^{1-n}$ (on the interval when $y\neq 0$) to get

$$\begin{aligned}v'&=(1-n)y^{-n}y'=(1-n)y^{-n}\left[q(t)y^n-p(t)y\right]\\&=(1-n)q(t)-(1-n)p(t)v,\end{aligned}$$

i.e.,

$$v'+(1-n)p(t)v=(1-n)q(t). \tag{54}$$

Now (54) becomes a **linear equation** for $v(t)$ and one can solve it, and then one can find $y(t)$ by the identity $v=y^{1-n}$.

The following is a good example of Bernoulli equation:

Logistic equation (this appears in p. 80 of the book).

Example 2.12 *(**This appears in p. 80 of the book.**) Find the general solution of the **logistic equation (autonomous equation)**:*

$$y'=f(y):=r\left(1-\frac{y}{K}\right)y=-\frac{r}{K}y(y-K)=ry-\frac{r}{K}y^2,\quad r>0,\quad k>0 \textit{ are given constants.} \tag{55}$$

The function $f(y)$ satisfies $f(0)=f(K)=0$ with $f'(0)>0$, $f'(K)<0$.

Remark 2.13 *Note that equation (55) has **two equilibrium solutions** $y(t)\equiv 0$ and $y(t)\equiv K$ due to $f(0)=f(K)=0$.*

Remark 2.14 *See the pictures in p. 81 of the book.*

Solution:

Note that equation (**??**) is both a **separable** equation and a **Bernoulli** equation. Here we use **Bernoulli's method** to solve it. Let $v = \frac{1}{y}$. Then

$$v' = -\frac{1}{y^2}y' = -\frac{1}{y^2} \cdot r\left(1 - \frac{y}{K}\right)y = -\frac{1}{y}r\left(1 - \frac{y}{K}\right) = -rv + \frac{r}{K}$$

and we know that the general solution of the above equation for v is

$$v(t) = \frac{1}{K} + Ce^{-rt}, \quad C \text{ is integration const..} \tag{56}$$

Hence we get the general solution

$$y(t) = \frac{1}{\frac{1}{K} + Ce^{-rt}}, \quad \lim_{t\to\infty} y(t) = K \text{ (equilibrium solution).}$$

If we impose initial condition $y(0) = y_0$ (we assume $y_0 \neq 0$; otherwise we get **equilibrium solution** $y(t) \equiv 0$) in the equation, we get $C = \frac{1}{y_0} - \frac{1}{K}$ and so

$$y(t) = \frac{1}{\left(\frac{1}{y_0} - \frac{1}{K}\right)e^{-rt} + \frac{1}{K}} = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}, \quad y(0) = y_0. \tag{57}$$

The above solution is defined on a **maximal time interval** (a, b) on which $y_0 + (K - y_0)e^{-rt} \neq 0$ for all $t \in (a, b)$. We note that:

- If $y_0 \in (0, K)$, then $y(t)$ is defined on $(-\infty, \infty)$, **strictly increasing** on $(-\infty, \infty)$, with

$$\lim_{t\to-\infty} y(t) = 0 \text{ (equilibrium solution)}, \quad \lim_{t\to\infty} y(t) = K \text{ (equilibrium solution).} \tag{58}$$

  This kind of solutions **connect** two equilibrium solutions during the time interval $(-\infty, \infty)$.

- If $y_0 \in (K, \infty)$, then (denominator is positive on the interval (λ, ∞), $\lambda < 0$)

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} > K, \quad \forall\, t \in (\lambda, \infty) \tag{59}$$

  is defined on (λ, ∞), **strictly decreasing** on (λ, ∞), where

$$\lambda = -\frac{1}{r}\log\frac{y_0}{y_0 - K} < 0 \tag{60}$$

  and

$$\lim_{t\to\lambda^+} y(t) = \infty, \quad \lim_{t\to\infty} y(t) = K \text{ (equilibrium solution).} \tag{61}$$

- By (58) and (61), for any $y_0 \in (0, \infty)$, the solution $y(t)$ in (57) satisfies $\lim_{t\to\infty} y(t) = K$. By this, we say the equilibrium solution $y(t) \equiv K$ is an **asymptotically stable solution** of the equation (55) (if we confine $y_0 \in (0, \infty)$) or say it is a **locally stable solution** of the equation (55) (if we confine $y_0 \in (K - \delta, K + \delta)$ for some $\delta > 0$). In contrast, the equilibrium solution $y(t) \equiv 0$ is an **unstable solution** of the equation (55). In terms of derivatives, we see that $f'(0) > 0$ (**unstable**) and $f'(K) < 0$ (**stable or more precisely "locally stable"**). We call this the **derivative test for stability** (explain more on this !!!). See the picture on p. 81 of the book.

- If $y_0 \in (-\infty, 0)$, then (denominator is positive on the interval $(-\infty, \lambda)$, $\lambda > 0$)

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0) e^{-rt}} < 0, \quad \forall\, t \in (-\infty, \lambda), \tag{62}$$

is defined on $(-\infty, \lambda)$, **strictly decreasing** on $(-\infty, \lambda)$, where now

$$\lambda = -\frac{1}{r} \log \frac{y_0}{y_0 - K} > 0 \tag{63}$$

and

$$\lim_{t \to -\infty} y(t) = 0 \text{ (equilibrium solution)}, \quad \lim_{t \to \lambda^-} y(t) = -\infty. \tag{64}$$

Again, the equilibrium solution $y(t) \equiv 0$ is an **unstable solution** of the equation (55). We have $f'(0) > 0$ (**unstable**).

The solution is complete. □

Remark 2.15 *See the pictures in p. 81 of the book.*

Remark 2.16 *We can also use* ***separable equation method*** *(see next section) to solve (55). We have*

$$\int \frac{dy}{y\left(1 - \frac{y}{K}\right)} = \int r dt = rt + c,$$

where by

$$\int \frac{dy}{y\left(1 - \frac{y}{K}\right)} = \int \frac{1}{y} dy - \int \frac{1}{y - K} dy = \ln \left| \frac{y}{y - K} \right|,$$

we get

$$\left| \frac{y}{y - K} \right| = Ce^{rt}, \quad C = e^c > 0$$

and so

$$\frac{y}{y - K} = Ce^{rt}, \quad C \neq 0 \quad \left(\text{same as } y = \frac{KCe^{rt}}{K + Ce^{rt}} \right).$$

To satisfy the initial condition $y(0) = y_0$, we need $C = \frac{y_0}{y_0 - K}$, and conclude the solution of (55):

$$\frac{y}{y - K} = \frac{y_0}{y_0 - K} e^{rt},$$

which is the same as

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0) e^{-rt}}, \quad y(0) = y_0. \tag{65}$$

Exercise 2.17 *(**Put this as a HW problem.**) Consider the nonlinear ODE*

$$x'(t) + e^{x(t)} = \sin t. \tag{66}$$

Convert it into a Bernoulli equation and find its general solution.

2.2 Separable equations (this is Section 2.2 of the book, see p. 42).

The easiest type of a first order ODE is probably the following:

Definition 2.18 *Consider the equation* $x' = f(t,x)$. *If* $f(t,x)$ *has the form* $f(t,x) = g(t)h(x)$ *for some continuous functions* $g(t)$ *and* $h(x)$ *defined on some intervals* $I,\ J \subset \mathbb{R}$ *respectively, then the equation*

$$x' = f(t,x) = g(t)h(x), \quad x = x(t) \tag{67}$$

is called a ***separable*** *differential equation (because the function* $f(t,x)$ *is in separable form). A separable equation can be solved using a standard method.*

Remark 2.19 *If* $h(x_0) = 0$ *at some* $x_0 \in J$, *then the function* $x(t) \equiv x_0$, *defined on* $t \in I$, *is an* ***equilibrium solution*** *of the ODE (67).*

Remark 2.20 *Unlike a first order linear equation, a solution* $x(t)$ *of (67)* ***may not*** *be defined on the whole interval* $t \in I$ *even if* $g(t)$ *is defined on* I. *Its domain also depends on the form of* $h(x)$ *and the initial condition.* ***In general, the domain*** $\tilde{I}$ ***of*** $x(t)$ ***is only a subinterval of*** I.

Remark 2.21 *Any first-order equation which can be* ***rewritten*** *in the form (67) is also called a* ***separable*** *differential equation. For example, the following are all separable equations:*

$$h(x)x' = g(t), \quad g(t)x' = h(x), \quad g(t)h(x)x' = 1,$$

because one can rewrite them as

$$x' = \frac{g(t)}{h(x)}, \quad x' = \frac{h(x)}{g(t)}, \quad x' = \frac{1}{g(t)h(x)}.$$

However, there are some ***minor differences*** *due to the "***zero denominator***" problem (****however, in this elementary course, we will not pay too much attention on this !!!****). For example, compare the two equations*

$$(1).\ tx' = x, \quad (2).\ x' = \frac{x}{t}, \quad x = x(t).$$

In the first equation, the value $t = 0$ *is allowed (****although the equation will degenerate****); however, in the second equation, the value* $t = 0$ *is* ***not*** *allowed. So the first equation is defined on* $t \in (-\infty,\infty)$, *and the second equation is defined only on* $t \in (-\infty,\infty) \setminus \{0\}$. *The general solution for the first equation is* $x(t) = Kt,\ t \in (-\infty,\infty)$, *where* K *is an arbitrary constant. Each solution* $x(t) = Kt$ *is a differentiable function defined on* $t \in (-\infty,\infty)$. *On the other hand, the general solution for the second equation is*

$$x(t) = \begin{cases} K_1 t, & t \in (-\infty,0) \\ K_2 t, & t \in (0,\infty), \end{cases} \tag{68}$$

where K_1 *and* K_1 *are two arbitrary constants, which can be different !! Each solution* $x(t)$ *given by (68) is a differentiable function defined on* $t \in (-\infty,\infty) \setminus \{0\}$.

There is a routine method to solve a separable equation (67) (as long as you can find indefinite integrals). For each $(t_0,x_0) \in I \times J$, by existence theory, there is a solution $x(t)$ of (67) defined on $(t_0-\varepsilon, t_0+\varepsilon)$ satisfying $x(t_0) = x_0$. In case $h(x_0) \neq 0$ (**this is crucial**), by making $\varepsilon > 0$ smaller, we can have $x(t) \neq 0$ for all $t \in (t_0-\varepsilon, t_0+\varepsilon)$. Hence we can rewrite the identity $x'(t) = g(t)h(x(t)),\ t \in (t_0-\varepsilon, t_0+\varepsilon)$, as

$$\frac{x'(t)}{h(x(t))} = g(t), \quad t \in (t_0-\varepsilon, t_0+\varepsilon) \quad \text{(this is to "}\textbf{separate}\text{" } x \text{ and } t) \tag{69}$$

and integrate both sides of (69) with respect to $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ to get

$$\int \frac{x'(t)}{h(x(t))} dt = \int g(t)\, dt. \tag{70}$$

If we let $H(x) = \int \frac{1}{h(x)} dx$ be the antiderivative of $\frac{1}{h(x)}$ (which is defined at least near x_0 since $h(x_0) \neq 0$), the above will give us the identity

$$H(x(t)) = G(t) + C, \quad \text{where } G'(t) = g(t), \quad \forall\, t \in (t_0 - \varepsilon, t_0 + \varepsilon), \tag{71}$$

i.e. $x(t)$, near the point (t_0, x_0), will satisfy the identity

$$H(x) = G(t) + C, \tag{72}$$

where C is some integration constant determined by the condition $x(t_0) = x_0$. Therefore, one can solve $x = x(t)$ from the above equation (72) and it will be a solution of the equation (67). In summary, as long as $h(x(t)) \neq 0$, you can always use the above method to find $x(t)$.

More precisely, we have the following Lemma 2.22, Lemma 2.24, and Theorem 2.25:

Lemma 2.22 *Let I, J be two open intervals in $\mathbb{R}$, where $g(t)$ is a continuous function defined on I and $h(x)$ is a continuous function defined on J. Assume $h(x) \neq 0$ on J. If $x(t) : \tilde{I} \subset I \to J$ ($\tilde{I}$ is a subinterval of I) is a solution to the equation*

$$x' = g(t)\, h(x), \quad t \in \tilde{I},$$

then $x(t)$, $t \in \tilde{I}$, will satisfy the equation

$$H(x) = G(t) + C, \quad t \in \tilde{I} \tag{73}$$

for some constant C. Here $G(t)$ is an antiderivative of $g(t)$ on I and $H(x)$ is an antiderivative of $\frac{1}{h(x)}$ on J.

Remark 2.23 *If $h(x_0) = 0$ at some point x_0 in its domain, then the equation $x' = g(t)\, h(x)$ has an equilibrium solution $x(t) \equiv x_0$, $t \in (-\infty, \infty)$.*

Proof. We have

$$x'(t) = g(t)\, h(x(t)), \quad \forall\, t \in \tilde{I}, \quad \text{where} \quad x(t) : \tilde{I} \to J$$

and so (note that $h(x) \neq 0$ on J)

$$\frac{x'(t)}{h(x(t))} = g(t), \quad \forall\, t \in \tilde{I}.$$

By integration with respect to $t \in \tilde{I}$, we get

$$\int \frac{x'(t)}{h(x(t))} dt = \int g(t)\, dt.$$

Hence $x(t)$, $t \in \tilde{I}$, satisfies the equation

$$H(x(t)) = G(t) + C, \quad \forall\, t \in \tilde{I}$$

for some integration constant C. The proof is done. □

Conversely, we have:

Lemma 2.24 *Assume $h(x) \neq 0$ on J. Let $G(t)$ and $H(x)$ be antiderivatives of $g(t)$ and $\frac{1}{h(x)}$ on $I,\ J$ respectively. If $x = x(t) : \tilde{I} \subset I \to J$ is a differentiable function which satisfies the equation*

$$H(x) = G(t) + C, \quad t \in \tilde{I}$$

for some constant C, then $x(t)$ is a solution to the ODE $x' = g(t)h(x)$ on $\tilde{I}$.

Proof. We have

$$H(x(t)) = G(t) + C, \quad t \in \tilde{I}$$

and by differentiation we get

$$\frac{x'(t)}{h(x(t))} = g(t), \quad t \in \tilde{I}.$$

Therefore $x'(t) = g(t)h(x(t))$ for all $t \in \tilde{I}$. The proof is done. □

By the above two lemmas, we conclude the following:

Theorem 2.25 *Assume $h(x) \neq 0$ is a continuous function on J, and $g(t)$ is a continuous function on I, and $x(t) : \tilde{I} \subset I \to J$ is a differentiable function. Then $x(t) : \tilde{I} \to J$ is a solution to the separable equation*

$$x' = g(t)h(x), \quad t \in \tilde{I} \tag{74}$$

if and only if it satisfies the equation

$$H(x) = G(t) + C, \quad t \in \tilde{I}, \tag{75}$$

for some constant C. Here $G(t)$ is an antiderivative of $g(t)$ on I and $H(x)$ is an antiderivative of $\frac{1}{h(x)}$ on J.

Remark 2.26 *By allowing C to be any possible constant, the formula (75) can describe **all possible** solutions of the equation (74). We say the **general solution** of the ODE (74) is given by the equation (75). From (75), if you can solve x as a function of $t \in I$ explicitly, then you get **general explicit solutions**; otherwise, you will get **general implicit solutions**.*

Remark 2.27 *If there is an initial condition $x(t_0) = x_0$ for the equation, then one can substitute the condition $x(t_0) = x_0$ into (75) to solve the constant C or use the formula*

$$\int_{x_0}^{x} \frac{1}{h(s)} ds = \int_{t_0}^{t} g(s)\, ds \tag{76}$$

*to get the unique solution satisfying $x(t_0) = x_0$ (**in implicit form**).*

Example 2.28 *Find the general solution of the equation*

$$\frac{dx}{dt} = tx. \tag{77}$$

<u>**Solution**</u>:

We first note that $x(t) \equiv 0$ is an equilibrium solution, defined on $t \in (-\infty, \infty)$. If $x(t)$ is a solution defined on some interval I and $x(t) \neq 0$ on some interval I, then we have

$$\int \frac{x'(t)}{x(t)} dt = \int t dt, \quad t \in I$$

and so

$$\ln|x(t)| = \frac{t^2}{2} + C,$$

which gives $|x(t)| = Ke^{t^2/2}$, $K = e^C > 0$, i.e. $x(t) = \tilde{K}e^{t^2/2}$, where $\tilde{K} = \pm e^C \neq 0$. Finally, we note that $\tilde{K} = 0$ will also give rise to a solution (the equilibrium solution). Hence the general solution of the equation is given by $x(t) = \tilde{K}e^{t^2/2}$, where $\tilde{K}$ is an arbitrary constant. Note that for any $\tilde{K}$ the solution $x(t) = \tilde{K}e^{t^2/2}$ is defined on $t \in (-\infty, \infty)$. □

Example 2.29 *Find the solution of the equation*

$$\frac{dx}{dt} = tx^2, \qquad x(0) = 1, \tag{78}$$

where $x = x(t)$ is a function of t.

<u>**Solution**</u>:

We first note that $x(t) \equiv 0$ is an equilibrium solution of $\frac{dx}{dt} = tx^2$, defined on $t \in (-\infty, \infty)$. For the ivp, we have

$$\int \frac{dx}{x^2} = \int t dt, \quad \frac{-1}{x} = \frac{t^2}{2} + C,$$

which gives the **general solution**

$$x(t) = \frac{1}{K - \frac{t^2}{2}}, \quad K \text{ is a constant.}$$

and by the condition $x(0) = 1$, we have $K = 1$ and so

$$x(t) = \frac{1}{1 - \frac{t^2}{2}}, \quad t \in \left(-\sqrt{2}, \sqrt{2}\right).$$

It is defined on the maximal time interval $\left(-\sqrt{2}, \sqrt{2}\right)$ only (even if $f(t,x) = tx^2$ is defined on $(-\infty, \infty) \times (-\infty, \infty)$). The maximal time interval is finite is due to the term x^2, not the term t. □

Remark 2.30 *Note that the domain interval of the solution $x(t) = \left(1 - \frac{t^2}{2}\right)^{-1}$ has to contain the number $t = 0$ due to the initial condition $x(0) = 1$. Therefore, the interval is $\left(-\sqrt{2}, \sqrt{2}\right)$.*

Remark 2.31 *If we change the initial condition as*

$$\frac{dx}{dt} = tx^2, \qquad x(0) = -1, \tag{79}$$

then the solution is given by

$$x(t) = -\frac{2}{t^2 + 2}, \quad t \in (-\infty, \infty).$$

Surprisingly, the solution is now defined on $(-\infty, \infty)$. As $t \to \pm\infty$, the solution $x(t)$ converges to the equilibrium solution $x(t) \equiv 0$.

Example 2.32 *(**This is Example 1 in p. 43.**) Find the general solution of the equation*

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}, \tag{80}$$

where $y = y(x)$ is a function of x.

Remark 2.33 *(Important.)* ***This is intuitive feeling:*** *Since $1-y^2$ appears in the denominator, a solution $y(x)$ of (80) is defined on some maximal interval $(a,b)\subseteq\mathbb{R}$ such that $y(x)\to\pm\infty$ or $y(x)\to\pm1$ as $x\to a^+$. The same behavior applies for $y(x)$ as $x\to b^-$.*

<u>**Solution**</u>:

We can formally rewrite the equation as (assume $y(x)$ is a solution defined on I and $y(x)\neq\pm1$ on I)

$$\left(1-y^2\right)dy = x^2dx \quad \text{(this is to "separate" } x \text{ and } y)$$

and integrate both sides to get

$$\int\left(1-y^2\right)dy = \int x^2dx,$$

and obtain the identity

$$y^3-3y+x^3=K \tag{81}$$

for some constant K. Since it is difficult to solve $y(x)$ explicitly and difficult to find the domain of $y(x)$, the solution is defined **implicitly** by the identity (81).

The integral curve C passing through (x_0,y_0) satisfies the equation

$$f(x,y)=y^3-3y+x^3=y_0^3-3y_0+x_0^3 \tag{82}$$

and at $(x_0,y_0)\in C$ we have

$$\frac{\partial f}{\partial y}(x_0,y_0)=3y_0^2-3. \tag{83}$$

At any point $(x_0,y_0)\in C$, as long as $y_0\neq\pm1$ (same as $\frac{\partial f}{\partial y}(x_0,y_0)\neq0$), then by the **implicit function theorem**, there exists a unique solution $y(x)$ of the equation (82) defined on $(x_0-\varepsilon,x_0+\varepsilon)$ for some $\varepsilon>0$ and satisfies $y(x_0)=y_0$. Note that the ODE is undefined at $y_0=\pm1$ due to the function $1-y^2$ in the denominator. The slope $y'(x)$ will tend to $\pm\infty$ as $y\to\pm1$. □

Remark 2.34 *If we impose the initial condition $y(0)=0$ (i.e. $(x_0,y_0)=(0,0)$), then we get the equation of integral curve C, which passes through the point $(0,0)$:*

$$C: y^3-3y+x^3=0. \tag{84}$$

One can see the picture of C on p. 44 of the book. Along the curve $C: y^3-3y+x^3=0$ we have

$$\frac{\partial f}{\partial y}(x,y)=3y^2-3=0 \text{ (if and only if } y=\pm1). \tag{85}$$

When $y=\pm1$, the corresponding x-coordinates are $x=\pm2^{1/3}$. Therefore we have $\left(\pm2^{1/3},\pm1\right)\in C$. As $x\to\pm2^{1/3}$, we have $y\to\pm1$, and so

$$\lim_{x\to\pm2^{1/3}} y'(x)=\lim_{y\to\pm1}\frac{x^2}{1-y^2}=\pm\infty.$$

Therefore the domain of the solution $y(x)$ on C cannot contain the point $x_0=2^{1/3}$ or $x_0=-2^{1/3}$. The domain of the solution $y(x)$ on C passing through $(0,0)$ is $x\in\left(-2^{1/3},2^{1/3}\right)$. One can see the picture on p. 44 of the book to confirm this.

Exercise 2.35 *Find the domain of the solution $y(x)$ of the equation (80) passing through the point $\left(0,\sqrt{3}\right)$ and the domain of the solution $y(x)$ passing through the point $\left(0,-\sqrt{3}\right)$.*

<u>**Solution**</u>:

The answer for the first question is $\left(-\infty,2^{1/3}\right)$ and the answer for the second question is $\left(-2^{1/3},\infty\right)$. □

Example 2.36 *(**This is Example 2 in p. 45.**) Find the solution of the equation*

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1, \tag{86}$$

where $y = y(x)$ is a function of x.

Solution:

First note that the equation has no equilibrium solution. Since it is a separable equation, as long as $y \neq 1$, the equation can be solved by the separation method and we get

$$\int 2(y-1)\,dy = \int \left(3x^2+4x+2\right) dx$$

and then

$$y^2 - 2y = x^3 + 2x^2 + 2x + C,$$

and by $y(0) = -1$ we have $C = 3$. By the **quadratic formula** we see that

$$y(x) = 1 - \sqrt{x^3+2x^2+2x+4} \tag{87}$$

is the **explicit solution** to the ivp (86). Since we must require

$$x^3 + 2x^2 + 2x + 4 = (x+2)\left(x^2+2\right) \geq 0$$

we know that the **domain** of $y(x)$ is $x \in (-2, \infty)$ with $y(-2) = 1$ and $y(x)$ is **not differentiable** at $x = -2$. This corresponds to the fact that $y = 1$ is **undefined** on the right hand side of the equation. □

Remark 2.37 *See the picture in p. 46. The solution (87) is unique, in explicit form, and is defined on $x \in (-2, \infty)$. As $x > -2$ is close to -2, $y(x) < 1$ is close to 1 and so the slope*

$$\frac{3x^2+4x+2}{2(y-1)}$$

is close to $-\infty$. This explains why $y(x)$ cannot be differentiable at $x = -2$.

Exercise 2.38 *Find the solution $y(x)$ of the equation (86) with the initial condition $y(0) = 3$. What is the domain of $y(x)$?*

Solution:

The answer is $y(x) = 1 + \sqrt{x^3+2x^2+2x+4}$ with domain $(-2, \infty)$. □

Exercise 2.39 *Find the solution $y(x)$ of the equation (86) with the initial condition $y(0) = 0$. What is the domain of $y(x)$?*

Solution:

The answer is $y(x) = 1 - \sqrt{x^3+2x^2+2x+1}$ with domain $(-1, \infty)$. □

2.2.1 Homogeneous equations (this is in p. 49 of the book).

Definition 2.40 *A function $f(t,y)$ is called* ***homogeneous*** *if we have*

$$f(\lambda t, \lambda x) = f(t,x)$$

for any constant $\lambda \neq 0$ and any $(t,x) \in \mathbb{R}^2$ such that both $(\lambda t, \lambda x)$ and (t,x) are in the domain of f.

For example, the following are all homogeneous:

$$f(t,x) = \frac{at+bx}{ct+dx}, \quad a,\ b,\ c,\ d \ \text{ are constants}$$

or

$$f(t,x) = \frac{t}{x}\sin\left(\frac{x}{t}\right), \qquad f(t,x) = \log x - \log t = \log\frac{x}{t}, \qquad f(t,x) = \frac{2tx}{t^2+x^2},$$

etc.

Remark 2.41 *By choosing $\lambda = \frac{1}{t}$, a* ***homogeneous*** *function $f(t,x)$ will satisfy the identity*

$$f(t,x) = f\left(1, \frac{x}{t}\right) \quad (\text{whenever } t \neq 0). \tag{88}$$

Thus one can also say that a function $f(t,x)$ is homogeneous if and only if it can be expressed as ***a function of the variable*** *$\frac{x}{t}$. That is*

$$f(t,x) = F\left(\frac{x}{t}\right)$$

for some ***one-variable function*** *F. For example, we have*

$$f(t,x) = \frac{2tx}{t^2+x^2} = \frac{2\frac{x}{t}}{1+\left(\frac{x}{t}\right)^2} = F\left(\frac{x}{t}\right), \quad F(v) = \frac{2v}{1+v^2}.$$

Remark 2.42 *In some homogeneous equation, solutions cannot be defined across $t = 0$ if the variable t appears in the denominator. See (90) below.*

A first order homogeneous ODE

$$\frac{dx}{dt} = f(t,x) = F\left(\frac{x}{t}\right) \tag{89}$$

can be converted into a **separable** equation. Let

$$v = \frac{x}{t}, \quad v = v(t).$$

Then $x = tv$ and so $\frac{dx}{dt} = v + t\frac{dv}{dt}$. Hence (89) becomes a **separable** equation:

$$v + t\frac{dv}{dt} = F(v)$$

and then one can separate t and v as

$$\frac{dv}{F(v)-v} = \frac{1}{t}dt.$$

Example 2.43 *Solve the equation*

$$\frac{dx}{dt} = \frac{t^2 + 3x^2}{2tx}, \tag{90}$$

which is clearly ***not*** *a separable equation. However, it is a* ***homogeneous*** *equation.*

Solution:

First note that any solution $x(t)$ of (90) cannot be defined across $t = 0$, nor can we have $x(t_0) = 0$ at some $t_0 \neq 0$. Note that the function $f(t, x) = \frac{t^2+3x^2}{2tx}$ is homogeneous. We can write it as

$$\frac{dx}{dt} = \frac{1 + 3\left(\frac{x}{t}\right)^2}{2\left(\frac{x}{t}\right)}$$

and, according to the method, get

$$v + t\frac{dv}{dt} = \frac{1 + 3v^2}{2v}.$$

Hence

$$\int \frac{2vdv}{1 + v^2} = \int \frac{1}{t}dt$$

and then

$$\ln\left(1 + v^2\right) = \ln|t| + C$$

which gives

$$1 + v^2 = 1 + \left(\frac{x}{t}\right)^2 = K|t|, \quad K = e^C > 0$$

and then

$$x = \pm t\sqrt{K|t| - 1}, \quad t \neq 0 \tag{91}$$

is the general solution. Here $K > 0$ is an arbitrary constant.

Note that the domain of $x(t)$ is $(-\infty, -1/K) \bigcup (1/K, \infty)$ (at the endpoint $t = \pm 1/K$ we have $x(t) = 0$, which is not allowed since x appears in the denominator of the equation). □

Example 2.44 *Find the solution of the ivp:*

$$\frac{dx}{dt} = \log x - \log t, \quad x(3) = 18, \quad t \in (0, \infty).$$

Solution:

We have $\frac{dx}{dt} = \log\frac{x}{t}$ and so it is a homogeneous equation. By the method, we let $v(t) = \frac{x}{t}$ and get

$$v + t\frac{dv}{dt} = \log v, \quad \int \frac{dv}{\log v - v} = \int \frac{1}{t}dt, \quad v(3) = \frac{x(3)}{3} = \frac{18}{3} = 6.$$

Since we cannot integrate $\int \frac{dv}{\log v - v}$, the solution for $v(t)$ is defined implicitly by the identity

$$\int_6^{v(t)} \frac{ds}{\log s - s} = \int_3^t \frac{1}{s}dt$$

and the unique solution $x(t)$ is given by $x(t) = tv(t)$ for t lying on the interval $(3 - \delta, 3 + \delta)$ for some $\delta > 0$. □

Example 2.45 *Solve the equation*

$$\frac{dx}{dt} = \frac{2t + 3x + 5}{3t - 7x - 4}, \quad x = x(t). \tag{92}$$

Solution:This equation does not look like a homogeneous equation. But if we let s be the new independent variable given by $s = t + \alpha$ for some suitable constant α and let $y(s)$ be the new function satisfying $y(s) = x(t) + \beta$ (which means $y(t+\alpha) = x(t) + \beta$) for another suitable constant β, then one can convert it into a **homogeneous equation** for the function $y(s)$. We can write the original equation as

$$\frac{dx}{dt}(t) = \frac{2(t+\alpha) + 3(x+\beta) + (5 - 2\alpha - 3\beta)}{3(t+\alpha) - 7(x+\beta) + (-4 - 3\alpha + 7\beta)}, \quad x = x(t)$$

and if we choose $\alpha = \beta = 1$, the above becomes

$$\frac{dx}{dt}(t) = \frac{2s + 3y(s)}{3s - 7y(s)}, \quad y(s) = x(t) + 1, \tag{93}$$

which is not a self-contained ODE. However, by the chain rule, we know that

$$\frac{dy}{ds}(s) = \frac{dx}{dt}(t) \cdot \frac{dt}{ds} = \frac{dx}{dt}(t). \tag{94}$$

Hence, the self-contained ODE for $y(s)$ is given by

$$\frac{dy}{ds}(s) = \frac{2s + 3y(s)}{3s - 7y(s)}, \tag{95}$$

which has become a **homogeneous equation**. Letting $v = \frac{y}{s}$ (v is a function of s), we get the separable equation for $v(s)$, which is

$$v + s\frac{dv}{ds} = \frac{2+3v}{3-7v} \quad \left(\text{same as } s\frac{dv}{ds} = \frac{2+7v^2}{3-7v}\right), \quad v = v(s),$$

and then

$$\int \frac{3-7v}{2+7v^2} dv = \int \frac{1}{s} ds = \log|s| + C \quad (C \text{ is integ. const.}),$$

where we note that (recall the formula $\frac{d}{dx}\left(\frac{1}{\lambda}\tan^{-1}(\lambda x)\right) = \frac{1}{1+(\lambda x)^2}$)

$$\int \frac{3}{2+7v^2} dv = \frac{3}{2}\int \frac{1}{1+\frac{7}{2}v^2} dv = \frac{3}{2}\sqrt{\frac{2}{7}}\tan^{-1}\left(\sqrt{\frac{7}{2}}v\right)$$

and

$$-\int \frac{7v}{2+7v^2} dv = -\frac{1}{2}\int \frac{14v}{2+7v^2} dv = -\frac{1}{2}\log\left(2+7v^2\right).$$

We conclude the identity

$$\frac{3}{2}\sqrt{\frac{2}{7}}\tan^{-1}\left(\sqrt{\frac{7}{2}}v\right) - \frac{1}{2}\log\left(2+7v^2\right) = \log|s| + C, \quad v(s) = \frac{y(s)}{s} = \frac{x(t)+1}{t+1}$$

and, back to the variable t and the function $x(t)$, we get the identity

$$\frac{3}{2}\sqrt{\frac{2}{7}}\tan^{-1}\left(\sqrt{\frac{7}{2}}\frac{x+1}{t+1}\right) - \frac{1}{2}\log\left(2+7\left(\frac{x+1}{t+1}\right)^2\right) - \log|t+1| = C \tag{96}$$

for arbitrary integration constant C. The above defines the general solution of the equation (92) **implicitly**. □

2.3 Section 2.3 Example 4 in p. 58; escape velocity.

Example 2.46 *(**This is Example 4 in p. 58. of the textbook**.) ... See book statement ...*

Remark 2.47 *My method of solution for this example is slightly different from the book method.*

Solution:

According to the statement of the problem, the equation to be solved is

$$m\frac{d^2x}{dt^2} = -\frac{mgR^2}{(R+x)^2}, \quad \frac{dx}{dt}(0) = v_0, \quad x(0) = 0, \quad x = x(t). \tag{97}$$

This is a second order differential equation (therefore we need **two** initial conditions). There is a trick to solve (97) (also see Remark 2.48 below). **We can multiply the equation by** $\frac{dx}{dt}(t)$, **after that it can be reduced to a first order separable equation (this trick can be applied to any equation of the form** $\frac{d^2x}{dt^2} = f(x)$). We have

$$m\frac{dx}{dt}(t)\frac{d^2x}{dt^2}(t) = -\frac{mgR^2}{(R+x(t))^2}\frac{dx}{dt}(t)$$

which is same as

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}(t)\right)^2\right] = \frac{d}{dt}\left[\frac{gR^2}{R+x(t)}\right]$$

and so

$$\frac{1}{2}\left(\frac{dx}{dt}(t)\right)^2 = \frac{gR^2}{R+x(t)} + C_1 \tag{98}$$

for some integration constant C_1. In view of the condition $\frac{dx}{dt}(0) = v_0$ and $x(0) = 0$, we get $C_1 = \frac{1}{2}v_0^2 - gR$. Hence (98) becomes

$$\left(\frac{dx}{dt}(t)\right)^2 = \frac{2gR^2}{R+x(t)} + v_0^2 - 2gR$$

and then

$$\frac{dx}{dt}(t) = \pm\sqrt{\frac{2gR^2}{R+x(t)} + v_0^2 - 2gR}, \tag{99}$$

where in the above we choose the **plus sign** if the body is **rising**, and the **minus sign** if it is **falling back to the earth**. Note that now (99) is a first order **separable equation** which can be solved by the integration

$$\int_0^x \frac{dx}{\sqrt{\frac{2gR^2}{R+x} + v_0^2 - 2gR}} = \pm\int_0^t dt. \tag{100}$$

In general, it can be difficult to find the integral on the LHS (left-hand side). However, we do not have to integrate it here. We can use (99) to get the relation between **the maximum height** and the **initial velocity** (note that velocity $\frac{dx}{dt}(t)$ is **zero** at some time $t = T$ at the maximum height ξ, where $x(T) = \xi$). We have

$$0 = \frac{dx}{dt}(T) = \frac{2gR^2}{R+x(T)} + v_0^2 - 2gR = \frac{2gR^2}{R+\xi} + v_0^2 - 2gR, \quad \xi = x(T) = \text{maximal height},$$

which gives the relation between the maximum height ξ and the initial velocity v_0 :

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}} \qquad \left(\text{same as } \xi = \frac{v_0^2 R}{2gR - v_0^2}\right). \tag{101}$$

By letting $\xi \to \infty$ (we want the maximum height as large as possible), the **escape velocity** v_e of the projectile is

$$v_0 = v_e = \sqrt{2gR}. \tag{102}$$

In this situation, equation (99) becomes the **separable** equation

$$\frac{dx}{dt}(t) = \sqrt{\frac{2gR^2}{R+x(t)}}, \quad x(0) = 0, \qquad \int \sqrt{R+x}dx = \int \sqrt{2gR^2}dt \tag{103}$$

and can be integrated to get

$$x(t) = \left(\frac{3}{2}\sqrt{2gR^2}t + R^{\frac{3}{2}}\right)^{\frac{2}{3}} - R, \quad x(0) = 0, \quad t \in [0, \infty),$$

with

$$\frac{dx}{dt}(t) = \left(\frac{3}{2}\sqrt{2gR^2}t + R^{\frac{3}{2}}\right)^{-\frac{1}{3}} \sqrt{2gR^2} > 0, \quad \forall\, t \in [0, \infty)\ .$$

It satisfies $\lim_{t\to\infty} x(t) = \infty,\ \lim_{t\to\infty} \frac{dx}{dt}(t) = 0$ (the projectile stops at $\xi = \infty$ at time $T = \infty$). $\square$

Remark 2.48 *(**This is a remark on book method.**) In the textbook, the equation is in terms of $v(t)$ $(v(t) = dx/dt)$, given by*

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}.$$

*But then the equation does not seem to be self-contained. The book says that one can view $v(t)$ as a function of x (**in the physical situation here, before the body starts to fall back to the earth, this is allowed**) and get the identity:*

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{104}$$

and conclude the self-contained equation

$$mv\frac{dv}{dx} = -\frac{mgR^2}{(R+x)^2},$$

which gives

$$\frac{v^2(x)}{2} = \frac{gR^2}{R+x} + c$$

for some constant c. When $t = 0$, we have $x = 0$ and $v(0) = v_0$, we get

$$\frac{v^2(x)}{2} = \frac{gR^2}{R+x} + \frac{1}{2}v_0^2 - gR,$$

which is the same as the above.

Remark 2.49 *In fact the integral on the LHS of (100) is of the form*

$$\int \sqrt{\frac{a+bx}{c+dx}}dx$$

for some constants $a,\ b,\ c,\ d$. One can find its indefinite integral formula from calculus books.

Example 2.50 *(Leave this as homework problem.)* ***(This is Exercise 25 in p. 65.)*** *A body of mass m is projected vertically into the space with an initial velocity $v_0 > 0$ and with a resistance force $k|v|$ due to the air resistance. Here $k > 0$ is a constant. Assume we have constant gravitational force mg. Find the maximum height x_m attained by the object and the time t_m at which the maximum height is reached.*

Solution:

Let $v = dx/dt$. It has the equation

$$m\frac{dv}{dt} = -mg - kv, \quad v(0) = v_0 > 0. \tag{105}$$

We see that $v(t)$ is decreasing in time. We get the identity

$$\frac{m}{k}\log\left(\frac{mg+kv}{mg+kv_0}\right) = -t$$

and so

$$mg + kv = e^{-\frac{k}{m}t}\cdot(mg+kv_0).$$

At time t_m we have $v(t_m) = 0$. Hence we get the identity

$$t_m = \frac{m}{k}\log\left(\frac{mg+kv_0}{mg}\right) = \frac{m}{k}\log\left(1+\frac{kv_0}{mg}\right). \tag{106}$$

To get x_m, we view v as a function of x and equation (105) becomes

$$mv\frac{dv}{dx} = -mg - kv, \quad v(0) = v_0 > 0. \tag{107}$$

Hence

$$\int\frac{mv}{mg+kv}dv = \int\frac{\frac{m}{k}(mg+kv) - \frac{m^2g}{k}}{mg+kv}dv = -\int dx$$

and by the fact that at $x = 0,\ v(0) = v_0$, we get

$$\begin{aligned} x_m &= \int_0^{v_0}\frac{mv}{mg+kv}dv = \int_0^{v_0}\left(\frac{\frac{m}{k}(mg+kv) - \frac{m^2g}{k}}{mg+kv}\right)dv \\ &= \frac{m}{k}v_0 - \frac{m^2g}{k^2}\log\left(\frac{mg+kv_0}{mg}\right) = \frac{m}{k}v_0 - \frac{m^2g}{k^2}\log\left(1+\frac{kv_0}{mg}\right). \end{aligned} \tag{108}$$

We conclude the answer as

$$t_m = \frac{m}{k}\log\left(1+\frac{kv_0}{mg}\right), \quad x_m = \frac{m}{k}v_0 - \frac{mg}{k}\frac{m}{k}\log\left(1+\frac{kv_0}{mg}\right).$$

Finally, to obtain the Taylor series expansion of t_m and x_m, we use the expansion formula

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad \text{valid for } -1 < x \le 1.$$

$\square$

2.4 Section 2.5: Exponential growth and logistic growth (did not cover this section in class).

2.4.1 Exponential growth.

Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = ry, \\ y(0) = y_0 \in (-\infty, \infty), \end{cases} \tag{109}$$

where $r \in \mathbb{R},\ r \neq 0$, is a given constant. The number r is called the *growth rate* (if $r > 0$) of *decline rate* (if $r < 0$) of the problem. It $y(t)$ represents the quantity of something (for example, population) at time t, then the equation $dy/dt = ry$ means that the *rate of change* dy/dt is **proportional** to the *current value* of y, with r as *the constant of proportionality.* Note that the equation is both separable and linear.

The general solution of (109) is given by

$$y(t) = y_0 e^{rt}, \quad t \in (-\infty, \infty) \tag{110}$$

and the only equilibrium solution of the equation is $y(t) \equiv 0$ (same as $y_0 = 0$). Note that if $r > 0$, then as long as $y_0 \neq 0$ (no matter how small it is), we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_0 e^{rt}\ \ (r > 0) = \begin{cases} +\infty, & \text{if} \quad y_0 > 0 \\ -\infty, & \text{if} \quad y_0 < 0. \end{cases} \tag{111}$$

In such a case we say $y(t) \equiv 0$ is an **unstable** equilibrium solution. On the other hand, if $r < 0$, then for any $y_0 \neq 0$ (no matter how large it is), we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_0 e^{rt}\ \ (r < 0) = 0. \tag{112}$$

In such a case we say $y(t) \equiv 0$ is an **asymptotically stable** equilibrium solution.

If we let $f(y) = ry$, then the equation becomes $dy/dt = f(y)$. The only zero of the function $f(y)$ is $y = 0$, which means that the only equilibrium solution of the equation is $y(t) \equiv 0$. Moreover, if the derivative $f'(y)$ at $y = 0$ is **positive** (same as $r > 0$), then we have **unstable** equilibrium solution, and if the derivative $f'(y)$ at $y = 0$ is **negative** (same as $r < 0$), then we have **asymptotically stable** equilibrium solution. This observation can be applied to equation $dy/dt = f(y)$ for **arbitrary function** $f(y)$.

Remark 2.51 *Draw two pictures on blackboard.*

2.4.2 Logistic growth (see Example 2.12.)

2.5 Section 2.6: Exact equations and integrating factors (this is in p. 95 of the book).

In this section we shall write a first order ODE in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \quad (x,y) \in D \tag{113}$$

where $D \subseteq \mathbb{R}^2$ is an **open and connected** set in $\mathbb{R}^2$ (in topology, an open connected set is also called a **domain**). It is the common domain of the continuous functions $M(x,y)$ and $N(x,y)$. In this section we shall also assume that $M(x,y)$ and $N(x,y)$ have continuous first order partial derivatives with respect to x and y, i.e. $M(x,y)$ and $N(x,y)$ are C^1 functions on D.

Definition 2.52 *If there exists a C^2 function $\psi(x,y) : D \to \mathbb{R}$ such that*

$$\psi_x(x,y) = M(x,y) \quad \text{and} \quad \psi_y(x,y) = N(x,y), \quad \forall\ (x,y) \in D, \tag{114}$$

then we call (113) an ***exact*** *differential equation on D.*

Remark 2.53 *There will be a specific method to solve an exact equation.*

Remark 2.54 *It is possible that equation (113) is not exact on D, but exact on a smaller domain $\tilde{D} \subset D$. In this case, the method is valid only on $\tilde{D}$.*

Lemma 2.55 *If (113) is an exact differential equation on $D \subseteq \mathbb{R}^2$, then we have*

$$M_y(x,y) = N_x(x,y), \quad \forall\ (x,y) \in D. \tag{115}$$

Here the domain $D \subseteq \mathbb{R}^2$ can be arbitrary.

Remark 2.56 *Note that (115) is a* ***necessary condition*** *of an exact equation. It is not a sufficient condition.*

Proof. This is obvious since, by definition, there exists a C^2 function $\psi(x,y) : D \to \mathbb{R}$ satisfying (114) on D, and then

$$M_y(x,y) = \psi_{xy}(x,y), \qquad N_x(x,y) = \psi_{yx}(x,y), \quad (x,y) \in D$$

and by the identity $\psi_{xy}(x,y) = \psi_{yx}(x,y)$ on D for a C^2 function, we have (115). □

Example 2.57 *Consider the equation*

$$3xy + \left(x^2 + y^2\right)\frac{dy}{dx} = 0, \quad (x,y) \in D \subset \mathbb{R}^2.$$

The equation is not exact since it does not satisfy (115) on D. However, the equation

$$2xy + \left(x^2 + y^2\right)\frac{dy}{dx} = 0, \quad (x,y) \in D \subset \mathbb{R}^2$$

satisfies the ***necessary condition*** *$M_y(x,y) = N_x(x,y)$ for all $(x,y) \in D$. Hence it is very likely to be an exact equation (no applicable lemma at this moment). Since we can find*

$$\psi(x,y) = x^2 y + \frac{y^3}{3} + C$$

satisfying

$$\psi_x(x,y) = 2xy \quad \text{and} \quad \psi_y(x,y) = x^2 + y^2, \quad \forall\ (x,y) \in D, \tag{116}$$

the equation is indeed exact on D.

The bad thing is that if we have (115) on D, then in general it **may not** imply that the equation is exact (for example, D is a **ring-shaped** domain). However, if D is a **rectangle** in the plane, then (115) will imply that (113) is exact on D. More precisely, we have:

Theorem 2.58 *(****Theorem 2.6.1. in p. 96.****) Assume $M(x,y)$, $N(x,y)$ are C^1 functions on a* ***rectangle*** *$R = (\alpha,\beta) \times (\gamma,\delta) \subseteq \mathbb{R}^2$ satisfying*

$$M_y(x,y) = N_x(x,y), \quad \forall\ (x,y) \in R. \tag{117}$$

Then there exists a C^2 function $\psi(x,y)$ on R such that

$$\psi_x(x,y) = M(x,y) \quad \text{and} \quad \psi_y(x,y) = N(x,y), \quad \forall\ (x,y) \in R. \tag{118}$$

Therefore, on a ***rectangular domain****, the equation $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is* ***exact*** *on R if and only if (117) is satisfied.*

Remark 2.59 *One can replace the rectangle R by any* ***simply-connected domain*** *in $\mathbb{R}^2$. Also, the* ***whole plane*** *$\mathbb{R}^2$ can be regarded as a rectangle. Similarly, any* ***half-plane*** *can be regarded as a rectangle.*

Proof. **(The proof is different from the book proof and is easier).** The direction ($\Longrightarrow$) is due to Lemma 2.55.

For the proof of the direction ($\Longleftarrow$), we use the method in p. 101, Exercise 17. Fix some $(x_0, y_0) \in R$ and for any $(x, y) \in R$ we can consider the line segment L_x going from (x_0, y_0) to (x, y_0) and then the line segment L_y going from (x, y_0) to (x, y). Because R is a rectangle, we have $L_x \bigcup L_y \subset R$. Now define

$$\psi(x, y) = \int_{x_0}^{x} M(s, y_0)\, ds + \int_{y_0}^{y} N(x, t)\, dt, \quad (x, y) \in R, \tag{119}$$

which is a C^2 function on R. We first have $\psi_y(x, y) = N(x, y)$ for all $(x, y) \in R$. Next we have (here one can move differentiation $\frac{\partial}{\partial x}$ into the integral sign)

$$\begin{aligned}
\psi_x(x, y) &= M(x, y_0) + \frac{\partial}{\partial x} \int_{y_0}^{y} N(x, t)\, dt \\
&= M(x, y_0) + \int_{y_0}^{y} N_x(x, t)\, dt = M(x, y_0) + \int_{y_0}^{y} M_t(x, t)\, dt \\
&= M(x, y_0) + M(x, y) - M(x, y_0) = M(x, y), \quad \forall\ (x, y) \in R.
\end{aligned}$$

Hence we have found a C^2 function $\psi(x, y)$ defined on R satisfying $\psi_x(x, y) = M(x, y)$, $\psi_y(x, y) = N(x, y)$ for all $(x, y) \in R$. The proof is done. □

Remark 2.60 *(**Important.**) If $D \subseteq \mathbb{R}^2$ is a domain with the property that for some fixed $(x_0, y_0) \in D$ and for any $(x, y) \in D$ one can connect (x_0, y_0) and (x, y) by the line segments: $(x_0, y_0) \to (x, y_0) \to (x, y)$, then if we have $M_y = N_x$ on D, the equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ must be exact on D. For example D can be an* ***open disc*** *in $\mathbb{R}^2$ or other shapes. Similarly, if we can connect (x_0, y_0) and (x, y) by the line segments: $(x_0, y_0) \to (x_0, y) \to (x, y)$, then the function (compare with (119))*

$$\psi(x, y) = \int_{y_0}^{y} N(x_0, t)\, dt + \int_{x_0}^{x} M(s, y)\, ds \tag{120}$$

will satisfy $\psi_x(x, y) = M(x, y)$, $\psi_y(x, y) = N(x, y)$ for all $(x, y) \in D$. Hence the equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ is exact on D if and only if we have $M_y = N_x$ on D.

Theorem 2.61 *(**Solving an exact equation on** D.) Assume the ODE*

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0, \quad (x, y) \in D \ \ (\text{an arbitrary domain in } \mathbb{R}^2) \tag{121}$$

is ***exact*** *on D and $\psi(x, y)$ is any function defined on D satisfying*

$$\begin{cases} \psi_x(x, y) = M(x, y) \\ \psi_y(x, y) = N(x, y) \end{cases}, \quad \forall\ (x, y) \in D. \tag{122}$$

Then $y(x)$ is a solution to (121) on interval $I = (a, b)$ (hence $(x, y(x)) \in D$ for all $x \in I$) ***if and only if*** *it satisfies*

$$\psi(x, y(x)) = C, \quad \forall\ x \in (a, b) \tag{123}$$

for some constant C.

Remark 2.62 *Roughly speaking, the above lemma says that the* ***level curves*** *of the function* $\psi(x,y)$ *are* ***integral curves*** *of the ODE (121).*

Remark 2.63 ***(Important.)*** *For equation (121) with an initial condition* $y(x_0) = y_0$*, one can find a small rectangle* R *with* $(x_0, y_0) \in R \subset D$*. If on* R *we have* $M_y(x,y) = N_x(x,y)$ *for all* $(x,y) \in R$*, then near* (x_0, y_0) *the ODE is* ***exact*** *and we can use the above theorem to solve it (at least near* (x_0, y_0)*).*

Remark 2.64 ***(Important.)*** *Hence to solve an exact equation it suffices to find* $\psi(x,y)$ *from (122), and solve the algebraic equation* $\psi(x,y) = C$ *(solve* y *in terms of* x*,* C*). The method of finding* $\psi(x,y)$ *from (122) is: integrate* $\psi_x(x,y) = M(x,y)$ *first and get an* ***"integration constant function"*** $h(y)$*. Then substitute it into* $\psi_y(x,y) = N(x,y)$ *to find* $h(y) = \cdots + C$*, where* C *is an integration constant.*

Proof. Since equation (121) is exact, there exists a C^2 function $\psi(x,y)$ satisfying (122) on D. If $y(x)$ is a solution on $I = (a,b)$, then $(x, y(x)) \in D$ for all $x \in (a,b)$ and by **chain rule** we have

$$\frac{d}{dx}\psi(x,y(x)) \\ = \psi_x(x,y(x)) + \psi_y(x,y(x))\, y'(x) = M(x,y(x)) + N(x,y(x))\, y'(x) = 0, \quad \forall\, x \in (a,b).$$

Hence $\psi(x,y(x)) = C$ for some constant and for all $x \in I$.

On the other hand, if (123) holds, then we have

$$0 = \frac{d}{dx}\psi(x,y(x)) = M(x,y(x)) + N(x,y(x))\, y'(x), \quad \forall\, x \in (a,b),$$

which means that $y(x)$ is a solution to (121) on the interval $I = (a,b)$. The proof is done. □

Example 2.65 ***(This is Example 1 in p. 95.)*** *Solve (find the general solution) the equation*

$$2x + y^2 + 2xyy' = 0. \tag{124}$$

<u>**Solution**</u>:

For this problem, we have $M(x,y) = 2x + y^2$, $N(x,y) = 2xy$, both are defined on $\mathbb{R}^2$ (rectangle) with $M_y = N_x = 2y$. Hence it is an **exact equation on** $\mathbb{R}^2$. We need to find $\psi(x,y)$ satisfying

$$\begin{cases} \psi_x(x,y) = 2x + y^2 \\ \psi_y(x,y) = 2xy \end{cases}, \quad \forall\ (x,y) \in \mathbb{R}^2.$$

By the first equation we have $\psi(x,y) = x^2 + xy^2 + h(y)$ for some integration function $h(y)$. Substitute this $\psi(x,y)$ into the second equation to get

$$\psi_y(x,y) = 2xy + h'(y) = 2xy.$$

Therefore, $h(y)$ is a constant function. We conclude $\psi(x,y) = x^2 + xy^2 + C$ and any solution $y(x)$ to the equation (124) on some interval $x \in I$ is included in the general equation

$$\psi(x,y) = x^2 + xy^2 + C = K \quad (\text{same as } x^2 + xy^2 = C)$$

for arbitrary constants C and K. Thus the general solution of (124) is defined **implicitly** by the equation $x^2 + xy^2 = C$, where now y can be solved explicitly in terms of x and C as

$$y(x) = \pm\sqrt{\frac{C}{x} - x}, \quad x \neq 0. \tag{125}$$

The domain I of $y(x)$ depends on the constant C and we need $\frac{C}{x} - x > 0$. For example, if $C > 0$, then we need $\frac{C}{x} > x$. In case $x > 0$, we have $C > x^2$, giving $0 < x < \sqrt{C}$. In case $x < 0$, we have $C < x^2$, giving $-\infty < x < -\sqrt{C}$. Thus for $C > 0$, the domain of $y(x)$ is $\left(-\infty, -\sqrt{C}\right) \bigcup \left(0, \sqrt{C}\right)$. One can do similar discussions for $C < 0$ and for $C = 0$. In any case, the solution $y(x)$ cannot be defined at $x = 0$ (even for $C = 0$ since $y(x) = \pm\sqrt{-x}$ is not differentiable at $x = 0$).

Remark 2.66 ***(Interesting.)*** *One can rewrite (124) as*

$$y' + \frac{1}{2x}y = -\frac{1}{y},$$

which is a ***Bernoulli*** *equation. One can also use Bernoulli's method to solve it.*

Example 2.67 ***(This is Example 2 in p. 98.)*** *Solve the equation*

$$\left(y\cos x + 2xe^y\right) + \left(\sin x + x^2 e^y - 1\right) y' = 0. \tag{126}$$

<u>**Solution**</u>:

For this problem, we have $M(x,y) = y\cos x + 2xe^y$, $N(x,y) = \sin x + x^2e^y - 1$, both are defined on $\mathbb{R}^2$ (rectangle) with $M_y = N_x = \cos x + 2xe^y$. Hence it is an **exact equation on** $\mathbb{R}^2$. We need to find $\psi(x,y)$ satisfying

$$\begin{cases} \psi_x(x,y) = y\cos x + 2xe^y \\ \psi_y(x,y) = \sin x + x^2e^y - 1 \end{cases}, \quad \forall\ (x,y) \in \mathbb{R}^2.$$

We get $\psi(x,y) = y\sin x + x^2e^y + h(y)$ for some integration function $h(y)$. Substitute this $\psi(x,y)$ into the second equation to get

$$\psi_y(x,y) = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1,$$

which implies $h(y) = -y + C$ is a constant function. We conclude that the general solution of the equation is defined implicitly by the equation

$$y\sin x + x^2e^y - y = C, \tag{127}$$

where C is an arbitrary constant. For this equation it seems very difficult to solve y in terms of x and C explicitly. One can only obtain general implicit solutions. □

Remark 2.68 *A* ***separable*** *equation (it can be written in the form $M(x) + N(y)\frac{dy}{dx} = 0$) is an exact equation with*

$$\psi(x,y) = \int M(x)\,dx + \int N(y)\,dy.$$

The solution defined by the identity

$$\psi(x,y) = \int M(x)\,dx + \int N(y)\,dy = C$$

is the same as the solution obtained by the previous separable method.

2.5.1 Integrating factors for non-exact equations.

The idea is to multiply a non-exact equation $M+N\frac{dy}{dx}=0$ by some function $\mu(x,y)$ (**an integrating factor**) so that the new equation becomes an **exact** equation. More precisely, the new equation

$$\mu(x,y)\,M(x,y)+\mu(x,y)\,N(x,y)\,\frac{dy}{dx}=0 \tag{128}$$

is exact if we have the identity (here we assume that equation (128) is defined on some rectangular region in $\mathbb{R}^2$)

$$[\mu(x,y)\,M(x,y)]_y=[\mu(x,y)\,N(x,y)]_x\,,$$

which is the same as

$$\mu(x,y)\,[M_y(x,y)-N_x(x,y)]=\mu_x(x,y)\,N(x,y)-\mu_y(x,y)\,M(x,y)\,. \tag{129}$$

Since it is not easy to find $\mu(x,y)$ satisfying (129), **we assume that** $\mu(x,y)$ **depends only on** x **or only on** y. In the first case, (129) becomes

$$\mu(x)\,[M_y(x,y)-N_x(x,y)]=\mu_x(x)\,N(x,y)\,, \tag{130}$$

which we rewrite it as

$$\mu_x(x)=\underbrace{\frac{M_y(x,y)-N_x(x,y)}{N(x,y)}\mu(x)}\,. \tag{131}$$

Therefore, if $\frac{M_y-N_x}{N}$ **depends only on** x, i.e. $M_y-N_x=Q(x)\,N$ for some function $Q(x)$, then $\mu(x)$ can be found and equation (128) will become exact.

On the other hand, in the second case, (129) becomes

$$\mu(y)\,[M_y(x,y)-N_x(x,y)]=-\mu_y(y)\,M(x,y)\,, \tag{132}$$

which we rewrite it as

$$\mu_y(y)=\underbrace{\frac{N_x(x,y)-M_y(x,y)}{M(x,y)}\mu(y)}\,. \tag{133}$$

Therefore, if $\frac{N_x-M_y}{M}$ **depends only on** y, i.e. $N_x-M_y=H(y)\,M$ for some function $H(y)$, then $\mu(y)$ can be found and equation (128) will become exact.

We now can state the following result:

Lemma 2.69 *Consider the equation*

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0. \tag{134}$$

Assume that it is not exact. Then if

$$M_y-N_x=Q(x)\,N \tag{135}$$

for some $Q(x)$, *which is a function of* x *only, then (134) has an integrating factor* $\mu(x)$ *of the form*

$$\mu(x)=\exp\int Q(x)\,dx. \tag{136}$$

Moreover, the solution to the equation (134) is the same as the solution to the new exact equation

$$\mu(x)\,M(x,y)+\mu(x)\,N(x,y)\,\frac{dy}{dx}=0. \tag{137}$$

Similarly, if

$$N_x - M_y = H(y) M \tag{138}$$

for some $H(y)$, *which is a function of* y *only, then (134) has an integrating factor* $\mu(y)$ *of the form*

$$\mu(y) = \exp \int H(y)\, dy. \tag{139}$$

Moreover, the solution to the equation (134) is the same as the solution to the new exact equation

$$\mu(y) M(x,y) + \mu(y) N(x,y) \frac{dy}{dx} = 0. \tag{140}$$

Proof. Assume (135) and multiply equation (134) by $\mu(x)$ of the form (136). We get

$$\mu(x) M(x,y) + \mu(x) N(x,y) \frac{dy}{dx} = 0, \quad \text{where} \quad \mu(x) = \exp \int Q(x)\, dx. \tag{141}$$

For this equation we have

$$\frac{\partial [\mu(x) M(x,y)]}{\partial y} = \frac{\partial [\mu(x) N(x,y)]}{\partial x}.$$

Hence equation (141) is exact and we can solve it to get general solution $y(x)$. Note that this general solution $y(x)$ is also a solution of the original equation (134) since $\mu(x) \neq 0$ **everywhere in its domain**. The proof for the second case is similar. □

Example 2.70 ***(This is Example 4 in p. 100.)*** *Solve the equation*

$$\left(3xy + y^2\right) + \left(x^2 + xy\right) y' = 0. \tag{142}$$

<u>Solution</u>:

For this problem, we have $M(x,y) = 3xy + y^2$, $N(x,y) = x^2 + xy$, both are defined on $\mathbb{R}^2$ (rectangle) with

$$M_y = 3x + 2y, \qquad N_x = 2x + y, \qquad M_y \neq N_x.$$

Hence, equation (142) is **not exact**. However, by

$$M_y - N_x = x + y, \quad M = y(3x + y), \quad N = x(x + y),$$

we see that $M_y - N_x = Q(x) N$, where $Q(x) = 1/x$, which suggests that there is an integrating factor for the equation of the form (we focus on $x > 0$; the discussion for $x < 0$ is similar)

$$\mu(x) = \exp \int Q(x)\, dx = \exp \int \frac{1}{x} dx = \exp \log |x| = |x| = x \quad \text{for} \quad x > 0$$

Now if we multiply equation by $\mu(x) = x$, we get

$$\left(3x^2 y + xy^2\right) + \left(x^3 + x^2 y\right) y' = 0, \quad x > 0$$

which becomes an **exact equation**. The function $\psi(x,y)$ has the form

$$\psi(x,y) = x^3 y + \frac{1}{2} x^2 y^2$$

and the general solution of the equation (142) is given implicitly by

$$x^3 y + \frac{1}{2} x^2 y^2 = C, \quad x > 0$$

for arbitrary constant C. One can solve $y(x)$ explicitly to get

$$y(x) = \frac{-x^2 \pm \sqrt{x^4 + 2C}}{x}, \quad x > 0.$$

The domain of $y(x)$ is those $x > 0$ such that $x^4 + 2C > 0$. □

Remark 2.71 *We can also rewrite (142) as*

$$y' = -\frac{3xy + y^2}{x^2 + xy}, \tag{143}$$

which is a homogeneous equation. Therefore, we can use the method of homogeneous equation to solve it.

2.6 An example of non-exact equation satisfying $M_y = N_x$.

There is a theory saying that on a **simply-connected domain** $D \subset \mathbb{R}^2$ (see topology book for its definition; **rectangles** and **discs** are both simply-connected; the domain $\mathbb{R}^2 \setminus \{(0,0)\}$ is **not simply-connected**), the ODE

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0, \quad (x, y) \in D \tag{144}$$

is **exact** on D **if and only if** we have $M_y = N_x$ on D. On the other hand, if $D \subset \mathbb{R}^2$ is **not** simply-connected but we have $M_y = N_x$ on D, then the ODE (144) **may or may not be exact on** D. That is: the identity $M_y = N_x$ on D is not equivalent to the ODE being **exact** on D if $D \subset \mathbb{R}^2$ is not simply-connected. However, if we restrict the ODE (144) to a **smaller simply-connected domain** $\tilde{D} \subset D$, then it is exact on $\tilde{D}$ and we can use Theorem 2.61 to solve it on $\tilde{D}$ (but not on D).

We recall the following three elementary functions:

1. The **radial function** $r(x, y) = \sqrt{x^2 + y^2}$, $(x, y) \in \mathbb{R}^2$. This function is defined on $\mathbb{R}^2$, **continuous** on $\mathbb{R}^2$, **differentiable** on $\mathbb{R}^2 \setminus \{(0,0)\}$. It is not differentiable at $(0,0)$. We have $r(x, y) \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$, with

$$\frac{\partial r}{\partial x}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial r}{\partial y}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{145}$$

   The gradient vector field $\nabla r(x, y) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ is also pointing in the radial direction (draw a picture).

2. The **angle function** $\theta(x, y)$, which is the polar angle of the segment from $(0,0)$ to the point $(x, y) \in \mathbb{R}^2 \{(0,0)\}$ with respect to the positive x-axis. $\theta(0,0)$ is undefined. This function is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ (here we choose $\theta(x, y) \in [0, 2\pi)$), **continuous** on $\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$, **differentiable** on $\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$, with $\theta(x, y) \in C^1(\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\})$ and

$$\frac{\partial \theta}{\partial x}(x, y) = \frac{-y}{x^2 + y^2}, \quad \frac{\partial \theta}{\partial y}(x, y) = \frac{x}{x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}. \tag{146}$$

   The gradient vector field $\nabla \theta(x, y) : \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\} \to \mathbb{R}^2$ is also pointing in the angle direction (draw a picture) with $\nabla \theta(x, y) \perp \nabla r(x, y)$. To express $\theta(x, y)$ in terms of elementary function, we need to decompose $\mathbb{R}^2 \setminus \{(0,0)\}$ into several regions. We have (here $\tan^{-1}(y/x)$ is taken to lie between $-\pi/2$ and $\pi/2$)

$$\theta(x, y) = \begin{cases} \tan^{-1} \frac{y}{x}, & x > 0, \quad y \geq 0 \\ \pi + \tan^{-1} \frac{y}{x}, & x < 0 \\ 2\pi + \tan^{-1} \frac{y}{x}, & x > 0, \quad y < 0 \\ \frac{\pi}{2}, & x = 0, \quad y > 0 \\ \frac{3\pi}{2}, & x = 0, \quad y < 0. \end{cases} \tag{147}$$

3. The function $f(x,y) = \frac{1}{2}\log(x^2+y^2)$ is defined on $\mathbb{R}^2 \backslash \{(0,0)\}$, **differentiable** on $\mathbb{R}^2 \backslash \{(0,0)\}$. We have $\log(x^2+y^2) \in C^1(\mathbb{R}^2 \backslash \{(0,0)\})$ with

$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{x^2+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{y}{x^2+y^2}, \quad (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}. \tag{148}$$

Example 2.72 *($M_y = N_x$ **on non-simply-connected domain** $\mathbb{R}^2 \backslash \{(0,0)\}$ **and the equation is exact on** $\mathbb{R}^2 \backslash \{(0,0)\}$.) Show that the equation*

$$\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}y' = 0 \tag{149}$$

is exact on $\mathbb{R}^2 \backslash \{(0,0)\}$ and find its general solution.

<u>**Solution**</u>:

The equation is defined on $\mathbb{R}^2 \backslash \{(0,0)\}$ and we have

$$M_y = N_x = \frac{-2xy}{(x^2+y^2)^2}, \quad \forall\ (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}.$$

But since the domain $\mathbb{R}^2 \backslash \{(0,0)\}$ is **not a "rectangle" domain (nor a "simply-connected" domain)**, it is **not clear** if the equation is exact on $\mathbb{R}^2 \backslash \{(0,0)\}$ or not. However, by

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2+y^2)\right) = \frac{x}{x^2+y^2}, \\ \frac{\partial}{\partial y}\left(\frac{1}{2}\log(x^2+y^2)\right) = \frac{y}{x^2+y^2}, \end{cases} \quad \forall\ (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\},$$

we know that the equation is exact on $\mathbb{R}^2 \backslash \{(0,0)\}$. Its general solution is given by

$$\frac{1}{2}\log(x^2+y^2) = C$$

for arbitrary constant $C \in (-\infty, \infty)$, which is same as

$$x^2 + y^2 = K, \quad K = e^{2C} > 0. \tag{150}$$

Therefore

$$y(x) = \pm\sqrt{K - x^2}, \quad x \in (-K, K), \quad K > 0$$

is the **general solution** of the equation (149). Note that equation (150) describes a family of circles filling the domain $\mathbb{R}^2 \backslash \{(0,0)\}$. □

Remark 2.73 *(**Simple observation.**) In fact, on $\mathbb{R}^2 \backslash \{(0,0)\}$ the equation $\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}y' = 0$ is equivalent to*

$$x + yy' = 0 \quad \left(\textit{same as } \frac{d}{dx}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = 0\right).$$

Therefore, we get the identity (150).

Example 2.74 *($M_y = N_x$ **on non-simply-connected domain** $\mathbb{R}^2 \backslash \{(0,0)\}$, **but the equation is not exact on** $\mathbb{R}^2 \backslash \{(0,0)\}$.) Show that the equation*

$$-\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}y' = 0 \tag{151}$$

*is **not** exact on $\mathbb{R}^2 \backslash \{(0,0)\}$. Show that it is **exact** on $\mathbb{R}^2 \backslash \{(x,0) : x \geq 0\}$ or on $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$.*

Remark 2.75 *In topology, the two domains $\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$ and $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ are **simply connected**, and the equation is exact on such domains if and only if we have $M_y = N_x$ on them. In this example, one can find explicit $\psi(x,y)$ (angle function) on such domains.*

Remark 2.76 *(**Important.**) In vector analysis, the above says that the vector field*

$$V(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

*is **not a gradient vector field (not a conservative vector field)** on $\mathbb{R}^2 \setminus \{(0,0)\}$ even if it is **divergence-free** (i.e. $M_y = N_x$ on $\mathbb{R}^2 \setminus \{(0,0)\}$). However, it is a gradient vector field on $\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$ and on $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$.*

<u>**Solution**</u>:

The equation is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ and it does satisfy the necessary condition:

$$M_y = N_x = \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad \forall\ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Same as the previous example, since the domain $\mathbb{R}^2 \setminus \{(0,0)\}$ is not a "rectangle" domain (nor a "simply-connected" domain), it is **not clear** if the equation is exact on $\mathbb{R}^2 \setminus \{(0,0)\}$ or not. Now we claim that **the equation is not exact on** $\mathbb{R}^2 \setminus \{(0,0)\}$ (however, **it is exact on some smaller domains**). To see this, assume it is exact on $\mathbb{R}^2 \setminus \{(0,0)\}$ and we have a C^2 function $\psi(x,y)$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ satisfying the requirement. We restrict $\psi(x,y)$ to the unit circle $(\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$, and compute

$$\begin{aligned}
&\frac{d}{d\theta}\psi(\cos\theta, \sin\theta) \\
&= \psi_x(\cos\theta, \sin\theta)(-\sin\theta) + \psi_y(\cos\theta, \sin\theta)(\cos\theta) \\
&= M(\cos\theta, \sin\theta)(-\sin\theta) + N(\cos\theta, \sin\theta)(\cos\theta) \\
&= \frac{-\sin\theta}{\cos^2\theta + \sin^2\theta}(-\sin\theta) + \frac{\cos\theta}{\cos^2\theta + \sin^2\theta}(\cos\theta) = 1, \quad \forall\ \theta \in [0, 2\pi],
\end{aligned}$$

which implies that

$$0 = \psi(\cos 2\pi, \sin 2\pi) - \psi(\cos 0, \sin 0) = 2\pi - 0 = 2\pi.$$

We get a contradiction. Therefore, equation (151) is not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$.

On the domain $\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$, the function $\theta(x,y)$ is differentiable with

$$\theta_x(x,y) = M(x,y), \quad \theta_x(x,y) = N(x,y).$$

Hence equation (151) is exact on $\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$. The general solution on the domain $\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$ is given by

$$\theta = \theta_0, \quad \theta_0 \in (0, 2\pi),$$

which gives the family of lines:

$$y = Cx, \quad C \in (-\infty, \infty), \quad (x,y) \in \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}.$$

Similarly, on the domain $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$, the function $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$ is differentiable with

$$\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x} = \frac{-y}{x^2+y^2}, \qquad \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} = \frac{x}{x^2+y^2}, \quad \forall\ (x,y) \in \mathbb{R}^2_+.$$

Hence equation (151) is exact on $\mathbb{R}^2_+$. □

Remark 2.77 ***(Important.)*** *Although the equation* $\frac{-y}{x^2+y^2}+\frac{x}{x^2+y^2}y'=0$ *is not exact on* $\mathbb{R}^2\setminus\{(0,0)\}$*, it is exact on many smaller domains* $D\subset\mathbb{R}^2\setminus\{(0,0)\}$*. Therefore, if we restrict the equation to such* D *and if* $y(x)$*,* $x\in(a,b)$*, is a solution of the ODE with* $(x,y(x))\in D$ *for all* $x\in(a,b)$*, then we can still use Theorem 2.61 to solve it. In view of this, if* $y(x)$*,* $x\in(a,b)$*, is a solution of the ODE lying on* $\mathbb{R}^2\setminus\{(x,0):x\geq 0\}$ *or on* $\{(x,y)\in\mathbb{R}^2:x>0\}$*, then it lies on the ray* $\theta=\theta_0,\ \theta_0\in(0,2\pi)$*, or* $\theta=\theta_0,\ \theta_0\in(-\pi/2,\pi/2)$*.*

Remark 2.78 ***(Important.)*** *In fact, on* $\mathbb{R}^2\setminus\{(0,0)\}$ *the equation* $\frac{-y}{x^2+y^2}+\frac{x}{x^2+y^2}y'=0$ *is equivalent to*

$$-y+xy'=0. \tag{152}$$

The general solution of (152) is given by $y(x)=Cx$ *for any constant* $C\in(-\infty,\infty)$*, defined on* $x\in(-\infty,\infty)$*. The original equation has general solution given by* $y(x)=Cx$ *for any constant* $C\in(-\infty,\infty)$*, defined on* $x\in(-\infty,\infty)\setminus\{0\}$*. Along each solution, the angle function* $\theta(x,y)$ *is a constant.*

3 Chapter 3: Second order linear equations.

3.1 General second order equations.

A general second order ODE for $y(t)$ is an equation of the form

$$y''=\frac{d^2y}{dt^2}=f(t,y(t),y'(t)) \tag{153}$$

for some function $f(t,y,z)$ of three variables defined on some domain $D\subseteq\mathbb{R}^3$.

If there are no initial conditions in (153), there will be **two** integration constants in its general solution formula. To determine a unique solution of (153) we need initial conditions of the form:

$$y(t_0)=y_0,\quad y'(t_0)=z_0, \tag{154}$$

where the point $(t_0,y_0,z_0)\in D$. **If** $f(t,y,z)\in C^1(D)$, **then (153) together with (154) have existence and uniqueness property (keep this in mind). We will not prove this in class.** This is because $f(t,y,z)$ is continuous on D and satisfies certain Lipschitz continuous condition on D.

There are some interesting **special cases** so that (153) can be converted into a **first order** ODE.

3.2 Second order equations in special forms.

Case 1: $y''(t)=f(t,y'(t))$ **(missing** $y(t)$**).**

In this case, let $w(t)=y'(t)$. Then it becomes a **first order** equation

$$\frac{dw}{dt}=f(t,w).$$

If $w(t)$ can be solved (say separable, homogeneous, linear, Bernoulli, exact, etc.), then $y(t)$ is given by

$$y(t)=\int w(t)\,dt.$$

If there is an initial condition $y(t_0)=y_0,\ y'(t_0)=z_0$, then $y(t)$ is given by

$$y(t)=y_0+\int_{t_0}^{t}w(s)\,ds,\quad \text{where}\quad w(t_0)=z_0,\quad w'(t)=f(t,w).$$

Example 3.1 *Solve the equation*

$$y''(t) = ty'(t).$$

<u>**Solution**</u>:

Let $w(t) = y'(t)$. Then $w'(t) = tw(t)$, which gives $w(t) = Ke^{t^2/2}$, $K \in (-\infty, \infty)$. Hence we need to solve $y'(t) = Ke^{t^2/2}$ and get the general solution

$$y(t) = K \int e^{\frac{t^2}{2}} dt + C$$

for arbitrary constants K and C. □

<u>**Case 2**</u>**:** $y''(t) = f(y(t), y'(t))$ **(missing** t**).**

Again let $w(t) = y'(t)$. Then it becomes

$$\frac{dw}{dt} = f(y, w),$$

which is **not self-contained** because its right hand side is not of the form $f(t, w)$. However, a trick is to express $w(t)$ as a function of y (assume this is possible, i.e., **during the period when time** t **and the position** y **are convertible; in fact, as long as** $w(t) = y'(t)$ **is not zero, then the conversion is possible**). Then by the **chain rule** we have

$$\frac{dw}{dt} \text{ (view } w \text{ as a func. of } t) = \frac{dw}{dy}\frac{dy}{dt} = w\frac{dw}{dy} \text{ (view } w \text{ as a func. of } y)$$

and so the equation becomes

$$w\frac{dw}{dy} = f(y, w), \quad w = w(y) \tag{155}$$

which is a **first order** equation for $w(y)$. If $w(y)$ can be solved, then we can solve the **separable** equation

$$\frac{dy}{dt} = w(y) \tag{156}$$

to get $y(t)$.

Remark 3.2 *For example, let* $y(t) = t^3$, $t > 0$. *Since* $y(t)$ *is a strictly increasing function of* $t \in (0, \infty)$, y *and* t *are convertible with* $t = y^{1/3}$, $y > 0$. *Now if* $w = y'(t) = 3t^2$ *(it is not zero as long as* $t > 0$*), then*

$$\frac{dw}{dt} = 6t = 6y^{1/3}.$$

On the other hand, one can express w *as a function of* y*, given by* $w = 3t^2 = 3y^{2/3}$. *Now*

$$w\frac{dw}{dy} = 3y^{2/3} \cdot 3\frac{2}{3}y^{-1/3} = 6y^{1/3} = \frac{dw}{dt}.$$

Thus we do have the identity $dw/dt = w \cdot dw/dy$. *Now the ODE for* $w(y)$ *is* $w\frac{dw}{dy} = 6y^{1/3}$ *and we can solve it to get*

$$\frac{1}{2}w^2 = \frac{9}{2}y^{4/3} + C$$

and if we choose $C = 0$ *and take the plus sign in square root, we get* $w = 3y^{2/3} = 3t^2$.

Example 3.3 *Solve the equation*

$$y''(t) = y(t)\, y'(t). \tag{157}$$

Solution:

Let $w = y'$ to get $w'(t) = y(t)\,w(t)$ and use the identity $w'(t) = w(y)\,w'(y)$ to convert it into an ODE of y, given by

$$w(y)\,w'(y) = yw(y)\,.$$

Hence $w(y) \equiv 0$ is an equilibrium solution, which gives $y(t) \equiv C$ for any constant $C \in (-\infty, \infty)$. For $w \neq 0$, we have $w'(y) = y$ and so $w(y) = y^2/2 + K$. Then solve

$$y'(t) = w(y) = \frac{y^2}{2} + K \text{ (same as } 2y'(t) = y^2 + K,\ K \in (-\infty, \infty) \text{ is a const.)} \tag{158}$$

to get

$$\int \frac{dy}{y^2 + K} = \frac{1}{2} \int dt. \tag{159}$$

Remark 3.4 *We can also write equation (157) as*

$$\frac{d}{dt} y'(t) = \frac{d}{dt} \left(\frac{y^2}{2} \right)$$

and get (158).

Case (1)**:** $K > 0$.

For $K > 0$, we can write it as $K = \lambda^2,\ \lambda = \sqrt{K} > 0$, and obtain $2y'(t) = y^2 + \lambda^2$. By

$$\int \frac{dy}{y^2 + \lambda^2} = \frac{1}{\lambda} \tan^{-1} \left(\frac{y}{\lambda} \right), \quad \lambda = \sqrt{K} > 0,$$

we conclude

$$\frac{1}{\sqrt{K}} \tan^{-1} \left(\frac{y}{\sqrt{K}} \right) = \frac{1}{2} (t + C), \quad C \in \mathbb{R},$$

which gives the solution

$$y(t) = \sqrt{K} \tan \left[\frac{\sqrt{K}}{2} (t + C) \right], \quad K > 0, \quad t \in \left(-C - \frac{\pi}{\sqrt{K}}, -C + \frac{\pi}{\sqrt{K}} \right). \tag{160}$$

Case (2)**:** $K = 0$.

For $K = 0$, we have $y' = y^2/2$, and so

$$y(t) = \frac{-2}{t + C}, \quad C \in \mathbb{R}, \quad t \in (-\infty, \infty) \setminus \{-C\}\,. \tag{161}$$

Note that we also have an equilibrium solution $y \equiv 0$, which has been included before.

Case (3)**:** $K < 0$.

For $K < 0$, we can write it as $K = -\lambda^2,\ \lambda = \sqrt{-K} > 0$, and obtain $2y'(t) = y^2 - \lambda^2$. It has two equilibrium solutions $y \equiv \lambda$ and $y \equiv -\lambda$ (they are included already). By

$$\int \frac{dy}{y^2 - \lambda^2} = \frac{1}{2\lambda} \int \left(\frac{1}{y - \lambda} - \frac{1}{y + \lambda} \right) = \frac{1}{2\lambda} \log \left| \frac{y - \lambda}{y + \lambda} \right|,$$

we get

$$\frac{1}{2\lambda} \log \left| \frac{y - \lambda}{y + \lambda} \right| = \frac{1}{2} \int dt = \frac{1}{2} (t + C), \quad y \neq \lambda, \quad y \neq -\lambda. \tag{162}$$

To go on, recall that in calculus there are the identities:

$$\begin{cases} \tanh^{-1} x = \dfrac{1}{2} \log \left(\dfrac{1+x}{1-x} \right), & x \in (-1,1), \\ \dfrac{d}{dx} \left(\tanh^{-1} x \right) = \dfrac{1}{1-x^2}, & x \in (-1,1). \end{cases} \tag{163}$$

Therefore, if y lies in the integral $(-\lambda, \lambda)$, (162) gives

$$\frac{1}{2\lambda} \log \left| \frac{y-\lambda}{y+\lambda} \right| = -\frac{1}{2\lambda} \log \left(\frac{1+\frac{y}{\lambda}}{1-\frac{y}{\lambda}} \right) = -\frac{1}{\lambda} \tanh^{-1} \left(\frac{y}{\lambda} \right) = \frac{1}{2} (t+C), \quad y \in (-\lambda, \lambda)$$

and we get the solution

$$y(t) = \lambda \tanh \left(-\frac{\lambda}{2} (t+C) \right), \quad t \in (-\infty, \infty), \quad y(t) \in (-\lambda, \lambda). \tag{164}$$

If $y \in (-\infty, -\lambda) \bigcup (\lambda, \infty)$, $\lambda > 0$, we must have $(y-\lambda)/(y+\lambda) > 0$ and (162) gives

$$\frac{1}{\lambda} \log \left(\frac{y-\lambda}{y+\lambda} \right) = t + C, \quad C \in \mathbb{R}, \quad y \in (-\infty, -\lambda) \bigcup (\lambda, \infty)$$

and then

$$\frac{y-\lambda}{y+\lambda} = \tilde{C} e^{\lambda t}, \quad \tilde{C} = e^{\lambda C} > 0$$

and we conclude

$$y(t) = \lambda \left(\frac{1+\tilde{C} e^{\lambda t}}{1-\tilde{C} e^{\lambda t}} \right), \quad t \in (-\infty, \infty) \setminus \left\{ \lambda^{-1} \log \left(1/\tilde{C} \right) \right\}, \quad y(t) \in (-\infty, -\lambda) \bigcup (\lambda, \infty). \tag{165}$$

The proof is done. □

Remark 3.5 ***(Useful observation ...)*** *In the above four solutions (160), (161), (164), (165), we have $w(t) = y'(t)$ either positive everywhere or negative everywhere on its domain interval I. Therefore, $y(t)$ is a one-one function of $t \in I$ and so one can express t in terms of y. This is consistent with our method* ***which assumes that time t and y are convertible****. However, the equilibrium solution $w(y) \equiv 0$ (same as $y(t) \equiv C \in (-\infty, \infty)$) is an exception (the only exception).*

Case 3: $y''(t) = f(y(t))$ (missing t and $y'(t)$).

Strictly speaking, this case is a special case of Case 2. However, it deserves more attention and now we can solve (155). We have

$$w \frac{dw}{dy} = f(y), \quad w = w(y),$$

which gives

$$\frac{w^2}{2} = \int f(y)\, dy = g(y) + C_1, \quad C_1 \in \mathbb{R} \tag{166}$$

and then we can get (on the interval of y where $g(y) + C_1 > 0$)

$$w(y) = \pm \sqrt{2g(y) + 2C_1}.$$

Finally we solve

$$w = \frac{dy}{dt} = \pm \sqrt{2g(y) + 2C_1},$$

which is a **separable** equation for $y(t)$, and then

$$\pm\int\frac{dy}{\sqrt{2g(y)+2C_1}}=t+C_2.$$

The constants $C_1,\ C_2$ are determined by the initial conditions.

Remark 3.6 ***(Important.)*** ***(Another equivalent method for Case 3.)*** *If we multiply the equation $y''(t)=f(y(t))$ by $y'(t)$, then we can solve it. We will obtain*

$$y'(t)\,y''(t)=f(y(t))\,y'(t)$$

and then

$$\frac{d}{dt}\left(\frac{(y'(t))^2}{2}\right)=\frac{d}{dt}g(y(t)),\quad \text{where}\quad g(y)=\int f(y)\,dy.$$

This gives

$$\frac{(y'(t))^2}{2}=g(y(t))+C_1, \tag{167}$$

which is same as (166). This method is same as the method above. One can rewrite (167) as

$$\frac{(y'(t))^2}{2}-g(y(t))=C_1. \tag{168}$$

In particle physics (classical mechanics), if $y(t)$ describes the position of a particle (which satisfies an equation of motion of the form $y''(t)=f(y(t))$), the quantity $(y'(t))^2/2$ is called ***kinetic energy*** *and the quantity $-g(y(t))$ is called* ***potential energy.*** *(168) says that there is a* ***conservation law of the total energy.*** *The constant C_1 can be determined from the initial position and the initial velocity.*

We conclude the following:

Lemma 3.7 *For the general second order equation*

$$y''(t)=f(t,y(t),y'(t)),$$

if we are in one of the following special cases

$$y''=f(t,y'),\qquad y''=f(y,y'),\qquad y''=f(y),$$

then we can convert it into first order equations.

Remark 3.8 ***(Important.)*** *If equation is of the form $y''(t)=f(t,y(t))$ (missing $y'(t)$), then* ***in general*** *we* ***cannot*** *reduce it to a first order ODE. For example, the equation*

$$y''(t)=t+y(t)$$

cannot be reduced to a first order ODE (convince yourself of this !!).

Example 3.9 *Solve the equation*

$$y''(t)=y^2(t).$$

Solution:

Multiply it by $y'(t)$ and integrate to get

$$\frac{(y'(t))^2}{2} = \frac{y^3}{3} + C_1$$

and then

$$(y'(t))^2 = \frac{2}{3}y^3 + 2C_1 > 0$$

and so

$$\pm \int \frac{dy}{\sqrt{\frac{2}{3}y^3 + 2C_1}} = t + C_2.$$

The solution $y(t)$ is defined ***implicitly*** by the above identity. □

Example 3.10 *Solve the equation*

$$y''(t) + y(t) = 0. \tag{169}$$

Remark 3.11 *We will see in Section 3.3 that the general solution of the above ODE is*

$$y(t) = A\sin t + B\cos t, \quad t \in (-\infty, \infty), \tag{170}$$

for arbitrary constants $A,\ B.$

Solution:

Multiply it by $y'(t)$ to get

$$y'(t)\,y''(t) + y(t)\,y'(t) = \frac{1}{2}\frac{d}{dt}\left((y'(t))^2 + y^2(t)\right) = 0,$$

which gives

$$(y'(t))^2 + y^2(t) = C, \quad C \geq 0 \text{ is a constant.}$$

If $C = 0$, we get the **equilibrium solution** $y(t) \equiv 0$. For $C > 0$, we can write $C = \lambda^2$ $(\lambda > 0)$ and the value of $y(t)$ must lie in the interval $(-\lambda, \lambda)$ and we conclude

$$\frac{dy}{dt} = \pm\sqrt{\lambda^2 - y^2}, \quad y \in (-\lambda, \lambda), \quad \lambda > 0. \tag{171}$$

If we choose the **plus sign**, we get the identity

$$\int \frac{dy}{\sqrt{\lambda^2 - y^2}} = \sin^{-1}\left(\frac{y}{\lambda}\right) = t + K, \quad y \in (-\lambda, \lambda), \tag{172}$$

where $K \in (-\infty, \infty)$ is an arbitrary constant. Since the value of the function $\sin^{-1} x,\ x \in (-1, 1)$, lies in the interval $(-\pi/2, \pi/2)$, in the above identity, we need to require $t + K \in (-\pi/2, \pi/2)$. Hence, at this moment, we conclude

$$y(t) = \lambda \sin(t + K), \quad t \in \left(-\frac{\pi}{2} - K, \frac{\pi}{2} - K\right). \tag{173}$$

We note that on the interval $(-\pi/2 - K, \pi/2 - K)$, the function $y(t)$ has **positive derivative (increasing)**, which matches with the equation $dy/dt = \sqrt{\lambda^2 - y^2}$ (we choose **plus sign** here).

If we choose the **minus sign**, we get the identity

$$-\int \frac{dy}{\sqrt{\lambda^2 - y^2}} = -\sin^{-1}\left(\frac{y}{\lambda}\right) = t + K, \quad y \in (-\lambda, \lambda), \quad \lambda > 0, \tag{174}$$

where $K \in (-\infty, \infty)$ is an arbitrary constant and again we require $t + K \in (-\pi/2, \pi/2)$ (we still have $-\sin^{-1} x,\ x \in (-1,1)$, lies in the interval $(-\pi/2, \pi/2)$). Now we conclude

$$y(t) = \lambda \sin(-(t+K)) = -\lambda \sin(t+K), \quad t \in \left(-\frac{\pi}{2} - K, \frac{\pi}{2} - K\right). \tag{175}$$

We note that on the interval $(-\pi/2 - K, \pi/2 - K)$, the function $y(t)$ has **negative derivative (decreasing)**, which matches with the equation $dy/dt = -\sqrt{\lambda^2 - y^2}$ (we choose **minus sign** here).

Since the two constants λ and K in the above can be arbitrary, one can use them to generate all solutions of the form (note that, although the value $y = \lambda$ is not allowed in the integral (172), it is actually allowed in the original equation (171)):

$$y(t) = A \sin t + B \cos t, \quad t \in (-\infty, \infty), \tag{176}$$

where $A,\ B$ are arbitrary constants. We leave the details to you (not difficult at all). □

Example 3.12 *Solve the equation*

$$y''(t) = y(t). \tag{177}$$

Remark 3.13 *We will see in Section 3.3 that the general solution of the above ODE is*

$$y(t) = Ae^t + Be^{-t}, \quad t \in (-\infty, \infty), \tag{178}$$

for arbitrary constants $A,\ B$.

<u>**Solution**</u>:

Multiply it by $y'(t)$ to get

$$y'(t)\, y''(t) - y(t)\, y'(t) = \frac{1}{2}\frac{d}{dt}\left((y'(t))^2 - y^2(t)\right) = 0,$$

which gives

$$(y'(t))^2 = y^2(t) + C, \quad C \in (-\infty, \infty) \text{ is a constant}$$

for all time in the domain of $y(t)$. If $C = 0$, we get

$$\text{either } \ y'(t) = y(t) \ \text{ or } \ y'(t) = -y(t)$$

for all time in the domain of $y(t)$. We get either $y(t) = C_1 e^t$ or $y(t) = C_1 e^{-t}$ for arbitrary constant C_1.

If $C > 0$, we write $C = \lambda^2,\ \lambda > 0$, and get

$$y'(t) = \pm\sqrt{y^2(t) + \lambda^2}, \qquad \int \frac{dy}{\sqrt{y^2 + \lambda^2}} = \log\left(y + \sqrt{y^2 + \lambda^2}\right) = \pm \int dt$$

(we always have $y + \sqrt{y^2 + \lambda^2} > 0$, so no need to add absolute value sign) and conclude

$$\log\left(y + \sqrt{y^2 + \lambda^2}\right) = \pm t + \tilde{C}, \qquad y + \sqrt{y^2 + \lambda^2} = Ke^{\pm t}, \quad K = e^{\tilde{C}} > 0$$

and then

$$y^2 + \lambda^2 = \left(Ke^{\pm t} - y\right)^2 = y^2 - 2Ke^{\pm t} y + \left(Ke^{\pm t}\right)^2,$$

which gives

$$y = \frac{\left(Ke^{\pm t}\right)^2 - \lambda^2}{2Ke^{\pm t}} = \frac{Ke^{\pm t}}{2} - \frac{\lambda^2}{2Ke^{\pm t}}$$

We arrive at

$$y=\frac{K}{2}e^{t}-\frac{\lambda^{2}}{2K}e^{-t}\quad\text{or}\quad y=-\frac{\lambda^{2}}{2K}e^{t}+\frac{K}{2}e^{-t},\quad K>0,\quad\lambda>0 \tag{179}$$

for arbitrary $K>0$ and $\lambda>0$.

Finally, for $C<0$, we write $C=-\lambda^2,\ \lambda>0$, and get

$$y'(t)=\pm\sqrt{y^{2}(t)-\lambda^{2}},\qquad\int\frac{dy}{\sqrt{y^{2}-\lambda^{2}}}=\log\left|y+\sqrt{y^{2}-\lambda^{2}}\right|=\pm\int dt \tag{180}$$

(note that it is possible to have $y+\sqrt{y^2-\lambda^2}<0$, so here we need to add absolute value sign in (180)) and conclude

$$\log\left|y+\sqrt{y^{2}-\lambda^{2}}\right|=\pm t+\tilde{C},\qquad\left|y+\sqrt{y^{2}-\lambda^{2}}\right|=Ke^{\pm t},\quad K=e^{\tilde{C}}>0$$

and then

$$y+\sqrt{y^{2}-\lambda^{2}}=\tilde{K}e^{\pm t},\quad\tilde{K}=\pm K\neq0$$

and

$$y^{2}-\lambda^{2}=\left(\tilde{K}e^{\pm t}-y\right)^{2}=y^{2}-2\tilde{K}e^{\pm t}y+\left(\tilde{K}e^{\pm t}\right)^{2},\quad\tilde{K}\neq0$$

which gives

$$y=\frac{\left(\tilde{K}e^{\pm t}\right)^{2}+\lambda^{2}}{2\tilde{K}e^{\pm t}}=\frac{\tilde{K}e^{\pm t}}{2}+\frac{\lambda^{2}}{2\tilde{K}e^{\pm t}}.$$

We arrive at

$$y=\frac{\tilde{K}}{2}e^{t}+\frac{\lambda^{2}}{2\tilde{K}}e^{-t}\quad\text{or}\quad y=\frac{\tilde{K}}{2}e^{-t}+\frac{\lambda^{2}}{2\tilde{K}}e^{t},\quad\tilde{K}\neq0,\quad\lambda>0 \tag{181}$$

for arbitrary $\tilde{K}\neq0$ and $\lambda>0$.

By (179) and (181), we obtain the general solution

$$y(t)=Ae^{t}+Be^{-t},\quad t\in(-\infty,\infty), \tag{182}$$

for arbitrary constants $A,\ B.$ □

3.3 Second order homogeneous linear equations with constant coefficients (this is Section 3.1 of the book; see p. 137).

We say equation $y''=f(t,y(t),y'(t))$ is a **linear** second order ODE if $f(t,y,y')$ is **linear** in y and y' (but not linear in t), i.e., if

$$f(t,y,y')=a(t)+b(t)\,y+c(t)\,y'$$

for some functions $a(t),\ b(t),\ c(t)$. In conclusion, a linear second order ODE can be written in the standard form as

$$y''(t)+p(t)\,y'(t)+q(t)\,y(t)=g(t),\quad t\in I \tag{183}$$

where $p(t),\ q(t)$ and $g(t)$ are given **continuous** functions (this is minimal requirement) defined on some common interval I.

We say equation (183) is **homogeneous** if $g(t)\equiv0$ everywhere, i.e., we have

$$y''(t)+p(t)\,y'(t)+q(t)\,y(t)=0,\quad t\in I. \tag{184}$$

Otherwise we say equation (183) is **nonhomogeneous**.

Sometimes we consider a linear second order ODE of the more general form

$$P(t)\,y''(t)+Q(t)\,y'(t)+R(t)\,y(t)=G(t),\quad t\in I \tag{185}$$

where $P(t),\ Q(t),\ R(t),\ G(t)$ are continuous on I, with $P(t)\neq0$ on I.

Theorem 3.14 *Consider equation (185), where $P(t)$, $Q(t)$, $R(t)$, $G(t)$ are continuous on I, with $P(t) \neq 0$ on I. We have the following properties:*

1. *Any solution $y(t)$ to equation (185) is defined on I.*

2. *We have* ***existence and uniqueness*** *result for equation (185) with the initial condition*

$$y(t_0) = y_0, \quad y'(t_0) = z_0,$$

where $t_0 \in I$ and y_0, z_0 are two given arbitrary numbers.

Proof. We will prove the theorem later on. □

Remark 3.15 *(**Important.**) In the above theorem, the condition $P(t) \neq 0$ on I is essential. If $P(t_0) = 0$ for some $t_0 \in I$, then a solution $y(t)$ to equation (185)* ***may or may not*** *be defined on the whole interval I. For example, the general solution of the Euler equation*

$$t^2 y''(t) + ty'(t) + y(t) = 0, \quad t \in (-\infty, \infty)$$

is

$$y(t) = C_1 \cos(\log|t|) + C_2 \sin(\log|t|), \quad t \in (-\infty, 0) \bigcup (0, \infty),$$

which cannot be defined across $t = 0$. On the other hand, the general solution of the equation

$$(t-1) y''(t) - ty'(t) + y(t) = 0, \quad t \in (-\infty, \infty),$$

is

$$y(t) = C_1 t + C_2 e^t, \quad t \in (-\infty, \infty),$$

which can be defined across $t = 1$.

Another two important properties are:

Lemma 3.16 *Consider the* ***homogeneous*** *linear equation (it means that the right hand side of the equation is zero)*

$$P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = 0, \quad t \in I, \tag{186}$$

where $P(t) \neq 0$ on I. If $y_1(t)$ and $y_2(t)$ are both solutions to (186) on I, so is the ***linear combination*** *$c_1 y_1(t) + c_2 y_2(t)$ for any constants c_1 and c_2.*

Remark 3.17 *Hence the solution space has a* ***vector space structure.***

Proof. This is a simple exercise. □

Lemma 3.18 *Consider the* ***nonhomogeneous*** *linear equation (it means that the right hand side of the equation is nonzero)*

$$P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = G(t), \quad t \in I, \tag{187}$$

where $P(t) \neq 0$ on I. If $y_1(t)$ and $y_2(t)$ are both solutions to (187) on I, then $y_2(t)$ can be expressed as

$$y_2(t) = y(t) + y_1(t), \quad t \in I,$$

for some function $y(t)$, which is a solution of the ***homogeneous*** *equation (186) on I. Therefore, the* ***general solution*** *$y_g(t)$ of (186) on I is given by*

$$y_g(t) = y_h(t) + y_p(t), \quad t \in I, \tag{188}$$

where $y_p(t)$ is some ***particular solution*** *of (187) on I ($y_p(t)$ has no integration constant) and $y_h(t)$ is the* ***general solution*** *of the* ***homogeneous*** *equation (186) on I ($y_h(t)$ has two integration constants).*

Proof. Just let $y(t) = y_2(t) - y_1(t)$ and see that $y(t)$ is a solution of the homogeneous equation. □

We shall discuss the existence and uniqueness of a solution to a general second order ODE (185) later on. For now, we want to look at some special cases first.

Solving a linear equation of the form (185) can be very difficult. Hence we first focus on the **homogeneous** case with **constant coefficients**. That is, equation of the form

$$ay''(t) + by'(t) + cy(t) = 0 \tag{189}$$

where $a,\ b,\ c$ are real constants, $a \neq 0$. By Theorem 3.14, we know that any solution $y(t)$ of (189) is defined on $(-\infty, \infty)$.

The following observation is useful: if we try $y(t) = e^{rt}$ in (189), we will get

$$ay''(t) + by'(t) + cy(t) = \left(ar^2 + br + c\right) e^{rt}.$$

In particular, if the number r satisfies $ar^2 + br + c = 0$, then $y(t) = e^{rt}$ will be a solution of (189).

In most cases, the equation $ar^2 + br + c = 0$ has two distinct roots $r_1,\ r_2$. By this, we can obtain the following result:

Theorem 3.19 *Assume that the equation (call it* ***characteristic equation*** *of the differential equation)*

$$ar^2 + br + c = 0, \quad a \neq 0, \quad a,\ b,\ c \in \mathbb{R} \tag{190}$$

has two ***distinct real roots*** $r_1,\ r_2$*. Then the* ***general solution*** $y(t)$ *to equation (189) is given by*

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad t \in (-\infty, \infty), \quad r_1 \neq r_2 \tag{191}$$

for arbitrary constants $c_1,\ c_2$*.*

Proof. It is easy to check that (191) is a solution to (189) defined on $t \in (-\infty, \infty)$. On the other hand, we need to show that any solution $y(t)$ of (189) is defined on $(-\infty, \infty)$ and has the form (191) for some constants $c_1,\ c_2$. The idea is to that we can decompose the **second** order equation (189) into **two** first order equations if the characteristic equation **has two distinct real roots** $r_1,\ r_2$.

For convenience of notation, we let D denote the differential operator d/dt. We can write $y'(t)$ as $Dy(t)$ and write $y''(t)$ as $D(Dy(t))$ or $D^2 y(t)$. By analogy, we can write

$$\begin{aligned}
3y'(t) - 5y(t) &= 3Dy(t) - 5y(t) = (3D - 5)\, y(t) \\
3y'(t) - 6y(t) &= 3Dy(t) - 6y(t) = (3D - 6)\, y(t) = 3\,(Dy(t) - 2y(t)) = 3\,(D - 2)\, y(t) \\
3y''(t) + 4y'(t) + 5y(t) &= \left(3D^2 + 4D + 5\right) y(t), \quad \text{... etc.}
\end{aligned}$$

Now we can write (189) as

$$ay''(t) + by'(t) + cy(t) = \left(aD^2 + bD + c\right) y(t) = 0.$$

We first claim the following:

Lemma 3.20 ***(Decomposing second order ODE into two first order ODE.)*** *Let* $a,\ b,\ c$ *be real constants with* $a \neq 0$*. If* $r_1,\ r_2$ *are two roots (****real (repeated or not) or complex****) of the polynomial* $aD^2 + bD + c = 0$*, then we have*

$$\left(aD^2 + bD + c\right) y(t) = a\,(D - r_1)\, w(t), \tag{192}$$

where

$$w(t) = (D - r_2)\, y(t). \tag{193}$$

Proof. We have

$$
\begin{aligned}
a(D-r_1)w(t) &= a(D-r_1)[(D-r_2)y(t)] = a(D-r_1)\underbrace{(y'(t)-r_2y(t))} \\
&= a\left[D\underbrace{(y'(t)-r_2y(t))}-r_1\underbrace{(y'(t)-r_2y(t))}\right] = a[y''(t)-r_2y'(t)-r_1y'(t)+r_1r_2y(t)] \\
&= a[y''(t)-(r_1+r_2)y'(t)+r_1r_2y(t)] = ay''(t)+by'(t)+cy(t),
\end{aligned}
$$

where we have used the identity $r_1+r_2=-b/a,\ r_1r_2=c/a.$ □

Come back to the proof of Theorem 3.19:

If $r_1\neq r_2$ are two real roots of $ar^2+br+c=0$, then we have

$$0=ay''(t)+by'(t)+cy(t)=a(D-r_1)w(t),\quad \text{where } w(t)=(D-r_2)y(t).$$

Hence $w(t)$ satisfies the **first order** equation $a(D-r_1)w(t)=0$ and its general solution is given by $w(t)=\lambda e^{r_1t}$, $\lambda\in\mathbb{R}$. But since $w(t)=(D-r_2)y(t)$, we need to solve $y(t)$ satisfying

$$(D-r_2)y(t)=\lambda e^{r_1t}\quad (\text{same as } y'(t)-r_2y(t)=\lambda e^{r_1t}).$$

The general solution for $y(t)$ is

$$
\begin{aligned}
y(t) &= e^{\int r_2dt}\left\{\int\left(e^{\int -r_2dt}\lambda e^{r_1t}\right)dt+C\right\} = e^{r_2t}\left\{\lambda\int e^{(r_1-r_2)t}dt+C\right\} \\
&= e^{r_2t}\left\{\frac{\lambda}{r_1-r_2}e^{(r_1-r_2)t}+C\right\} = c_1e^{r_1t}+c_2e^{r_2t},\quad t\in(-\infty,\infty)
\end{aligned}
$$

for some constants $c_1,\ c_2$. Hence we see that any solution $y(t)$ to equation (189) is given by (191). The proof is done. □

Remark 3.21 *In case there are initial conditions for ODE (189), given by*

$$y(t_0)=y_0,\quad y'(t_0)=z_0,$$

*then one can **always** solve for c_1 and c_2 to fulfill them. We need to solve the system*

$$
\begin{cases}
c_1e^{r_1t_0}+c_2e^{r_2t_0}=y_0 \\
c_1r_1e^{r_1t_0}+c_2r_2e^{r_2t_0}=z_0.
\end{cases}
$$

Since the coefficients determinant is nonzero, given by

$$
\begin{vmatrix}
e^{r_1t_0} & e^{r_2t_0} \\
r_1e^{r_1t_0} & r_2e^{r_2t_0}
\end{vmatrix}
=(r_2-r_1)e^{(r_1+r_2)t_0}\neq 0,\quad r_1\neq r_2,
$$

we can solve for c_1 and c_2 uniquely.

Similarly, we have:

Theorem 3.22 *Assume that the characteristic equation*

$$ar^2+br+c=0,\quad a\neq 0,\quad a,\ b,\ c\in\mathbb{R} \tag{194}$$

*has two **repeated** real roots $r_1=r_2$ (call it r). Then any solution $y(t)$ to equation (189) is given by*

$$y(t)=c_1e^{rt}+c_2te^{rt},\quad t\in(-\infty,\infty) \tag{195}$$

for some constants $c_1,\ c_2$.

Proof. It is easy to check that any function of the form (195) is a solution of equation (189) (you need to use the fact that $2ar + b = 0$, i.e. $r = -b/2a$). On the other hand, if $y(t)$ satisfies (189), then by

$$0 = ay''(t) + by'(t) + cy(t) = a(D - r)w(t), \quad \text{where } w(t) = (D - r)y(t),$$

we see that $w(t)$ must have the form $w(t) = \lambda e^{rt}$, $\lambda \in \mathbb{R}$. Hence $y(t)$ must satisfy

$$(D - r)y(t) = \lambda e^{rt}.$$

This implies

$$y(t) = e^{\int r dt}\left\{\int \left(e^{\int -r dt}\lambda e^{rt}\right)dt + C\right\} = e^{rt}(\lambda t + C) = c_1 e^{rt} + c_2 t e^{rt}, \quad t \in (-\infty, \infty)$$

for some constants $c_1,\ c_2$. The proof is done. □

Remark 3.23 *In case there are initial conditions given by*

$$y(t_0) = y_0, \quad y'(t_0) = z_0,$$

then one can ***always*** *solve for c_1 and c_2 to fulfill them. We need to solve the system*

$$\begin{cases} c_1 e^{rt_0} + c_2 t_0 e^{rt_0} = y_0 \\ c_1 r e^{rt_0} + c_2 (1 + rt_0) e^{rt_0} = z_0. \end{cases}$$

Since the coefficients determinant is nonzero, given by

$$\begin{vmatrix} e^{rt_0} & t_0 e^{rt_0} \\ r e^{rt_0} & (1 + rt_0) e^{rt_0} \end{vmatrix} = e^{2rt_0} \neq 0 \quad \text{for any} \quad t_0,$$

we can solve for c_1 and c_2 uniquely.

The last case is the following:

Theorem 3.24 *Assume that the characteristic equation*

$$ar^2 + br + c = 0, \quad a \neq 0, \quad a,\ b,\ c \in \mathbb{R} \tag{196}$$

has two complex roots $r = \alpha + i\beta,\ \bar{r} = \alpha - i\beta,\ \alpha,\ \beta \in \mathbb{R},\ \beta > 0$. Then any (real) solution $y(t)$ to equation (189) is given by

$$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t, \quad t \in (-\infty, \infty) \tag{197}$$

for some (real) constants $c_1,\ c_2$.

Exercise 3.25 *By direct computation, verify that $y(t)$ given by (197) is a real solution of the equation $ay''(t) + by'(t) + cy(t) = 0$ defined on $(-\infty, \infty)$.*

To prove the above theorem we need to introduce **complex exponential functions**, defined by

$$e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t), \quad \alpha,\ \beta \in \mathbb{R}, \quad t \in (-\infty, \infty). \tag{198}$$

In particular, when $\beta = 0$, it coincides with our usual real exponential function.

Remark 3.26 *Another motivation for the definition (198) is that we first define (use Taylor series expansion to see this)*

$$e^{i\theta} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = \cos\theta + i\sin\theta, \quad i^2 = -1$$

and then use exponential law to define $e^{(\alpha+i\beta)t}$ *as* $e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t}$. *This gives the formula (198).*

For complex exponential functions they satisfy the following exponential law and differentiation formula (same as the real exponential function):

- $e^{(\alpha+i\beta)t} \cdot e^{(\gamma+i\delta)t} = e^{[(\alpha+i\beta)+(\gamma+i\delta)]t}, \quad \forall\ \alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}.$
- We have

$$\frac{d}{dt}e^{(\alpha+i\beta)t} \text{ (this means we differentiate real part and imaginary part respectively)}$$
$$= (\alpha + i\beta)\, e^{(\alpha+i\beta)t}, \quad \forall\ \alpha,\ \beta \in \mathbb{R}.$$

Hence if $r \in \mathbb{C}$ is a complex number and $y(t) = e^{rt}$, we still have

$$ay''(t) + by'(t) + cy(t) = \left(ar^2 + br + c\right)e^{rt}, \quad a,\ b,\ c \in \mathbb{R}. \tag{199}$$

If $r = \alpha + i\beta$ is a complex root to the equation $ar^2 + br + c = 0$, then the complex function $y(t) = e^{rt} = e^{(\alpha+i\beta)t}$ is a **complex solution** to (189). Conversely if $y(t) = e^{(\alpha+i\beta)t}$ is a complex solution to (189), then $r = \alpha + i\beta$ is a complex root to the equation $ar^2 + br + c = 0$.

We note the following useful facts. They are all easy to verify by yourself.

1. If $y(t) = R(t) + iI(t)$ is a complex function, then $y'(t)$ means differentiation with respect to its real part and imaginary part respectively. If $y(t) = R(t) + iI(t)$ is a **complex** solution to (189), then its real part $R(t)$ and imaginary part $I(t)$ are both **real** solutions to (189).

2. Since $a,\ b,\ c$ are real numbers, if a complex function $y(t) = e^{(\alpha+i\beta)t},\ \alpha,\ \beta \in \mathbb{R},\ \beta > 0$, is a solution to (189), so is its **conjugate** function $\bar{y}(t) = e^{(\alpha-i\beta)t}$ (this is because $\alpha - i\beta$ is also a root if $\alpha + i\beta$ is).

3. If $y_1(t)$ and $y_2(t)$ are two complex solutions to (189), so is their **complex** linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1,\ c_2 \in \mathbb{C}.$$

4. The **general complex solution** to the ODE

$$y'(t) - (\alpha + i\beta)\, y(y) = 0$$

is given by $y(y) = \lambda e^{(\alpha+i\beta)t},\ \lambda \in \mathbb{C}$ (one can multiply the equation by integration factor to see this). Also the **general complex solution** to the ODE

$$y'(t) - (\alpha - i\beta)\, y(y) = \lambda e^{(\alpha+i\beta)t}$$

is given by $y(y) = C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t},\ C_1,\ C_2 \in \mathbb{C}.$

5. If $ar^2 + br + c = 0$ has roots $r = \alpha + i\beta$ and $\bar{r} = \alpha - i\beta,\ \alpha,\ \beta \in \mathbb{R},\ \beta > 0$, we have

$$0 = ay''(t) + by'(t) + cy(t) = a(D - r)\, w(t), \quad \text{where } w(t) = (D - \bar{r})\, y(t), \quad r = \alpha + i\beta,$$

for any complex function $y(t)$.

6. By the above, we see that the **general complex solution** to (189) is given by

$$y(t) = C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t}, \qquad C_1,\ C_2 \in \mathbb{C}.$$

If we let $C_1 = a + ib,\ C_2 = c + id$, we would have

$$\begin{aligned} &(a+ib)\, e^{(\alpha+i\beta)t} + (c+id)\, e^{(\alpha-i\beta)t} \\ &= (a+ib)\, e^{\alpha t}(\cos\beta t + i\sin\beta t) + (c+id)\, e^{\alpha t}(\cos\beta t - i\sin\beta t) \\ &= \left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right) + i\left(Ce^{\alpha t}\cos\beta t + De^{\alpha t}\sin\beta t\right) \end{aligned}$$

for some real numbers $A,\ B,\ C,\ D$ depending on $a,\ b,\ c,\ d$. Hence the **general complex solution** to the ODE (189) is given by

$$y(t) = \left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right) + i\left(Ce^{\alpha t}\cos\beta t + De^{\alpha t}\sin\beta t\right) \tag{200}$$

where $A,\ B,\ C,\ D$ are arbitrary real constants. In particular, the **general real solution** to the ODE is given by (choose $C = D = 0$)

$$y(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t \tag{201}$$

for some real constants $A,\ B$.

By the above facts, Theorem 3.24 is proved. □

Remark 3.27 *Again in case there is initial conditions for ODE (189), given by*

$$y(t_0) = y_0, \quad y'(t_0) = z_0$$

then one can ***always*** *solve for c_1 and c_2 to fulfill them. We need to solve*

$$\begin{cases} c_1 e^{\alpha t_0}\cos\beta t_0 + c_2 e^{\alpha t_0}\sin\beta t_0 = y_0 \\ c_1\left(\alpha e^{\alpha t_0}\cos\beta t_0 - \beta e^{\alpha t_0}\sin\beta t_0\right) + c_2\left(\alpha e^{\alpha t_0}\sin\beta t_0 + \beta e^{\alpha t_0}\cos\beta t_0\right) = y_0' \end{cases}$$

and the coefficients determinant is nonzero, given by

$$\begin{vmatrix} e^{\alpha t_0}\cos\beta t_0 & e^{\alpha t_0}\sin\beta t_0 \\ \alpha e^{\alpha t_0}\cos\beta t_0 - \beta e^{\alpha t_0}\sin\beta t_0 & \alpha e^{\alpha t_0}\sin\beta t_0 + \beta e^{\alpha t_0}\cos\beta t_0 \end{vmatrix} = \beta e^{2\alpha t_0} \neq 0 \quad \textit{for any} \quad t_0,$$

due to $\beta > 0$.

At this moment, we can summarize the following:

Theorem 3.28 *The ODE*

$$\begin{cases} ay''(t) + by'(t) + cy(t) = 0 \\ y(t_0) = y_0, \quad y'(t_0) = z_0 \end{cases} \tag{202}$$

where $a,\ b,\ c$ are real constants, $a \neq 0$, has a ***unique real*** *solution $y(t)$ defined on $(-\infty, \infty)$. Moreover, we know how to find the solution explicitly.*

3.4 Theory of second order linear homogeneous equation with variable coefficients; the Wronskian (this is Section 3.2 of the book).

In this section we look at a second order linear homogeneous equation with **variable coefficients** of the form

$$P(t)\,y''(t) + Q(t)\,y'(t) + R(t)\,y(t) = 0, \quad t \in I \tag{203}$$

where $P(t),\ Q(t),\ R(t)$ are real-valued continuous functions on I, with $P(t) \neq 0$ on I. For convenience, we can divide the equation by $P(t)$ and it becomes

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0, \quad t \in I, \tag{204}$$

where $p(t),\ q(t)$ are continuous on I. By Theorem 3.14, we know that if we have initial conditions $y(t_0) = y_0,\ y'(t_0) = z_0$, for equation (204), where $t_0 \in I$ and $y_0,\ z_0$ are two given numbers, then the solution $y(t)$ exists, unique, and is defined on the whole interval I.

Unlike the case of constant coefficients, it is, in general, very difficult to solve equation (204). However, we can ask the following interesting question: if $y_1(t)$ and $y_2(t)$ are two known solutions of (204) on I, **is it true that any other solution** $y(t)$ **of (204) can be expressed as**

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad \forall\, t \in I, \tag{205}$$

for some constants c_1 **and** c_2**?** If yes, then the **general solution** of (204) on I is given by (205).

To answer the above question, we need the concept of **Wronskian**, defined by the following:

Definition 3.29 *Consider the ODE (homogeneous)*

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{206}$$

where $p(t),\ q(t)$ are continuous functions defined on open interval I. If $y_1(t)$ and $y_2(t)$ are two solutions on I, then the function

$$W(y_1, y_2)(t) := \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t), \quad t \in I,$$

is called the ***Wronskian*** *of $y_1(t)$ and $y_2(t)$ on I.*

Remark 3.30 *In the above definition, the coefficient of $y''(t)$ is* 1*. From now on, when we talk about the Wronskian $W(y_1, y_2)(t)$ of two functions $y_1(t)$ and $y_2(t)$ on I, we always assume that $y_1(t)$ and $y_2(t)$ are two* ***solutions*** *of the ODE (206) on I.*

The following theorem implies that the Wronskian $W(y_1, y_2)(t)$ of two solutions on I is **either everywhere zero or everywhere nonzero** on I. More precisely, we have:

Theorem 3.31 *(**Abel.**) (**This is Theorem 3.2.7 in p. 154.**) Consider the ODE (homogeneous)*

$$y'' + p(t)\,y' + q(t)\,y = 0, \quad t \in I \tag{207}$$

where $p(t),\ q(t)$ are continuous functions defined on open interval I. If $y_1(t)$ and $y_2(t)$ are two solutions on I, then we have

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}, \quad t \in I \tag{208}$$

for some constant c.

Remark 3.32 *Note that the ODE in the above theorem has coefficient* 1*.*

Proof. For convenience, denote $W(y_1, y_2)(t)$ as $W(t)$. Compute

$$\begin{aligned} W'(t) &= y_1(t)\, y_2''(t) - y_2(t)\, y_1''(t) \\ &= y_1(t)\left[-p(t)\, y_2'(t) - q(t)\, y_2(t)\right] - y_2(t)\left[-p(t)\, y_1'(t) - q(t)\, y_1(t)\right] \\ &= -p(t)\left[y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t)\right] = -p(t)\, W(t), \quad t \in I. \end{aligned}$$

Thus $W(t)$ satisfies the first order linear ODE

$$W'(t) + p(t)\, W(t) = 0, \quad t \in I.$$

Hence it is given by

$$W(t) = ce^{-\int p(t)dt}, \quad t \in I,$$

for some constant c. □

Remark 3.33 *If there is an initial condition $W(y_1, y_2)(t_0) = c_0$, then one can express (208) as*

$$W(y_1, y_2)(t) = c_0 e^{-\int_{t_0}^{t} p(s)ds}. \tag{209}$$

It satisfies

$$W'(t) + p(t)\, W(t) = 0, \quad W(t_0) = c_0, \quad t_0,\ t \in I.$$

Corollary 3.34 *By (208), we see that $W(t)$ is either $W(t) \equiv 0$ on I (if $c = 0$) or $W(t)$ is never zero on I (if $c \neq 0$). In particular, if $W(t_0) > 0$ at some $t_0 \in I$, then $W(t) > 0$ everywhere on I. Similarly, if $W(t_0) < 0$ at some $t_0 \in I$, then $W(t) < 0$ everywhere on I.*

Proof. This is a direct consequence of Theorem 3.31. □

The most important theorem in this section is the following:

Theorem 3.35 *(**This is Theorem 3.2.4 in p. 149.**) Consider the ODE (homogeneous)*

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{210}$$

where $p(t)$, $q(t)$ are continuous functions defined on open interval I. If $y_1(t)$ and $y_2(t)$ are two solutions on I, then the family of solutions

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad t \in I \tag{211}$$

*with **arbitrary** coefficients c_1, c_2 includes **every solution** of (210) on I **if and only if** $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$ (hence $W(y_1, y_2)(t) \neq 0$ for all $t \in I$).*

Proof. Assume $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Let $\varphi(t)$ be a solution of (210) on I with

$$\varphi(t_0) = a, \quad \varphi'(t_0) = b.$$

One can find unique constants c_1 and c_2 such that

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = a \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = b \end{cases}$$

due to $W(y_1, y_2)(t_0) \neq 0$. Now the linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution of the ODE on I with

$$y(t_0) = a, \quad y'(t_0) = b.$$

Uniqueness implies that $y(t) \equiv \varphi(t)$ for all $t \in I$.

Conversely, assume that **every solution** $\varphi(t)$ of (210) on I can be expressed in the form

$$\varphi(t) = c_1 y_1(t) + c_2 y_2(t), \quad t \in I,$$

for some coefficients $c_1,\ c_2$. If $W(y_1, y_2)(t) \equiv 0$ on I, we will get a contradiction. Fix some $t_0 \in I$ and look at the equations

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = a \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = b. \end{cases} \tag{212}$$

There are **some** $a,\ b \in \mathbb{R}$ such that (212) has **no** solutions for c_1 and c_2 (since $W(y_1, y_2)(t_0) = 0$). For $\varphi(t)$ satisfying the ODE with $\varphi(t_0) = a,\ \varphi'(t_0) = b$, by the assumption, it can be expressed as $\varphi(t) = \lambda_1 y_1(t) + \lambda_2 y_2(t)$ for some $\lambda_1,\ \lambda_2$, for all $t \in I$. This implies that $\lambda_1,\ \lambda_2$ is a solution of (212), a contradiction. □

By the above theorem, we define the following:

Definition 3.36 *Consider the ODE*

$$y'' + p(t) y' + q(t) y = 0, \quad p(t),\ q(t) \text{ continuous on } I. \tag{213}$$

If $y_1(t)$ and $y_2(t)$ are two solutions on I such that $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$ (hence $W(y_1, y_2)(t) \neq 0$ for all $t \in I$), then the ***general solution of (213) on*** *I is given by*

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad t \in I,$$

where $c_1,\ c_2$ are arbitrary constants. In this case, we call $y_1(t)$ and $y_2(t)$ a ***fundamental set*** *of solutions of ODE (213) on I. Note that a given ODE on I has infinitely many* ***fundamental set*** *of solutions on I.*

Example 3.37 *Given the linear equation*

$$y''(t) + p(t) y'(t) + q(t) y(t) = 0, \quad t \in (-\infty, \infty) \tag{214}$$

where $p(t)$ and $q(t)$ are continuous on the interval $(-\infty, \infty)$. Is it possible that both t and t^2 are solutions (for some continuous $p(t)$ and $q(t)$ on $(-\infty, \infty)$) to the equation on $(-\infty, \infty)$? Give your reasons.

Solution:

It is impossible. The Wronskian of t and t^2 is

$$W(t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2, \quad t \in (-\infty, \infty).$$

On the interval $I = (-\infty, \infty)$ we have $W(t) = 0$ at $t = 0$, but $W(t) \neq 0$ for $t \neq 0$. It violates Theorem 3.31 □

Example 3.38 *Find an ODE of the form*

$$y''(t) + p(t) y'(t) + q(t) y(t) = 0, \quad t \in (0, \infty), \tag{215}$$

where $p(t)$ and $q(t)$ are continuous functions on the interval $(0, \infty)$ so that both t and t^2 are solutions to the equation on $(0, \infty)$.

Solution:

It is possible for both t and t^2 to be solutions to the equation on $(0,\infty)$ since its Wronskian $W(t) = t^2 \neq 0$ on $(0,\infty)$. If we want t and t^2 to be solutions of the ODE (215) on $t \in (0,\infty)$, we must require

$$\begin{cases} p(t) + q(t)\,t = 0 \\ 2 + 2tp(t) + q(t)\,t^2 = 0, \quad t \in (0,\infty), \end{cases}$$

which gives $q(t) = 2/t^2$ and $p(t) = -2/t$. Both $p(t)$ and $q(t)$ are continuous on the interval $(0,\infty)$. Therefore, the equation

$$y''(t) - \frac{2}{t}y'(t) + \frac{2}{t^2}y(t) = 0, \quad t \in (0,\infty) \tag{216}$$

has both t and t^2 as solutions on the interval $(0,\infty)$. By Theorem 3.35, they form a **fundamental set** of solutions on $(0,\infty)$. The **general solution** of the ODE (216) is given by

$$y(t) = c_1 t + c_2 t^2, \quad t \in (0,\infty)$$

for arbitrary real constants c_1 and c_2. □

Example 3.39 *Consider the equation*

$$y'' - 3y' + 2y = 0.$$

We know that $y_1(t) = e^t$ *and* $y_2(t) = e^{2t}$ *are two solutions of it defined on* $t \in (-\infty,\infty)$. *By*

$$W(y_1, y_2)(t) = e^{3t} > 0 \quad \text{for all} \quad t \in (-\infty,\infty),$$

we know that they form a ***fundamental set of solutions*** *and every solution* $y(t)$ *of the equation has the form*

$$y(t) = c_1 e^t + c_2 e^{2t}, \quad t \in (-\infty,\infty).$$

We also knew this fact before by ***decomposing the second order ODE into two first order ODEs.***

Example 3.40 *Consider the equation*

$$y'' + p(t)\,y' + q(t)\,y = 0, \quad p(t),\ q(t) \text{ continuous on } I$$

and fix some $t_0 \in I$. *Let* $y_1(t)$ *be the solution satisfying*

$$y_1(t_0) = 1, \quad y_1'(t_0) = 0$$

and let $y_2(t)$ *be the solution satisfying*

$$y_2(t_0) = 0, \quad y_2'(t_0) = 1.$$

Then $y_1(t)$ *and* $y_2(t)$ *form a* ***fundamental set*** *of solutions (since* $W(y_1, y_2)(t_0) = 1 \neq 0$*).*

Example 3.41 *Do Example 6 in p. 152.*

In general, it is very difficult to solve an equation of the form $y'' + p(t)\,y' + q(t)\,y = 0$ (or the form $P(t)\,y''(t) + Q(t)\,y'(t) + R(t)\,y(t) = 0$). However, there is a special case we can solve it, which is known as **Euler equation**.

3.5 Euler equations (this is Exercise 34 in p. 166).

We consider a second order **linear** equation of the form (call it **Euler equation**)

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \quad t \in (0,\infty), \quad \alpha,\ \beta \text{ constants.} \tag{217}$$

One can use change of variables to convert it into a **linear equation with constant coefficients**. Let $x = \ln t,\ t \in (0,\infty)$, (same as $t = e^x$), where $x \in (-\infty,\infty)$ will be the new variable. The function $y(t)$ will become a function of x, which we denote it as $\tilde{y}(x)$. That is, $\tilde{y}(x) = y(t)$. We have

$$\frac{dy}{dt} = \frac{d\tilde{y}}{dx}\frac{dx}{dt} = \frac{d\tilde{y}}{dx}\frac{1}{t} = e^{-x}\frac{d\tilde{y}}{dx} \quad \text{(in short: } \frac{d}{dt} = e^{-x}\frac{d}{dx}\text{)},$$

which is same as

$$\frac{d\tilde{y}}{dx} = t\frac{dy}{dt} \quad \text{(in short: } \frac{d}{dx} = t\frac{d}{dt}\text{)}$$

and

$$\frac{d^2y}{dt^2} = e^{-x}\frac{d}{dx}\left(e^{-x}\frac{d\tilde{y}}{dx}\right) = -e^{-2x}\frac{d\tilde{y}}{dx} + e^{-2x}\frac{d^2\tilde{y}}{dx^2}.$$

Hence

$$\begin{aligned}
& t^2 y''(t) + \alpha t y'(t) + \beta y(t) \\
&= e^{2x}\left(-e^{-2x}\frac{d\tilde{y}}{dx} + e^{-2x}\frac{d^2\tilde{y}}{dx^2}\right) + \alpha e^x\left(e^{-x}\frac{d\tilde{y}}{dx}\right) + \beta\tilde{y}(x) \\
&= \frac{d^2\tilde{y}}{dx^2} + (\alpha-1)\frac{d\tilde{y}}{dx} + \beta\tilde{y}(x), \quad x = \ln t \in (-\infty,\infty), \quad t \in (0,\infty)
\end{aligned}$$

and, in terms of the new variable x, the function $\tilde{y}(x)$ will satisfy the linear equation with constant coefficients

$$\frac{d^2\tilde{y}}{dx^2} + (\alpha-1)\frac{d\tilde{y}}{dx} + \beta\tilde{y}(x) = 0. \tag{218}$$

One can solve $\tilde{y}(x)$ for equation (218) on the interval $x \in (-\infty,\infty)$ and then obtain solution $y(t)$ for equation (217) on the interval $t \in (0,\infty)$.

Remark 3.42 *In case the Euler equation has the form*

$$At^2 y''(t) + Bty'(t) + Cy(t) = 0, \quad t \in (0,\infty), \quad A \neq 0,\ B,\ C \text{ are constants,}$$

then equation (218) becomes

$$A\frac{d^2\tilde{y}}{dx^2} + (B-A)\frac{d\tilde{y}}{dx} + C\tilde{y}(x) = 0. \tag{219}$$

Remark 3.43 *(**Motivation for Euler equation**.) Euler equation is very special in the sense that if we have a general equation of the form*

$$P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0, \quad t \in I,$$

and we do change of variables:

$$x = h(t), \quad t = g(x), \quad y(t) = \tilde{y}(x), \quad h(g(x)) = x, \quad t = g(h(t)),$$

then by the chain rule we get

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d\tilde{y}}{dx}\frac{dx}{dt} = h'(t)\frac{d\tilde{y}}{dx} = \frac{1}{g'(x)}\frac{d\tilde{y}}{dx} \\
y''(t) &= \frac{1}{g'(x)}\frac{d}{dx}\left(\frac{1}{g'(x)}\frac{d\tilde{y}}{dx}\right) = \frac{1}{(g'(x))^2}\frac{d^2\tilde{y}}{dx^2} - \frac{g''(x)}{(g'(x))^3}\frac{d\tilde{y}}{dx}
\end{aligned}$$

and the new equation for $\tilde{y}(x)$ becomes

$$\begin{aligned}
&P(t)\,y''(t)+Q(t)\,y'(t)+R(t)\,y(t)\\
&=P(g(x))\left(\frac{1}{(g'(x))^2}\frac{d^2\tilde{y}}{dx^2}-\frac{g''(x)}{(g'(x))^3}\frac{d\tilde{y}}{dx}\right)+Q(g(x))\left(\frac{1}{g'(x)}\frac{d\tilde{y}}{dx}\right)+R(g(x))\,\tilde{y}(x)\\
&=\left(P(g(x))\frac{1}{(g'(x))^2}\right)\frac{d^2\tilde{y}}{dx^2}+\left(-P(g(x))\frac{g''(x)}{(g'(x))^3}+Q(g(x))\frac{1}{g'(x)}\right)\frac{d\tilde{y}}{dx}+R(g(x))\,\tilde{y}(x).
\end{aligned}$$

For the case of Euler equation, we have

$$\begin{aligned}
P(t)&=t^2, \quad x=h(t)=\log t, \quad t=g(x)=e^x\\
P(g(x))&=g^2(x), \quad g'(x)=g(x), \quad g''(x)=g(x)\\
Q(t)&=\alpha t, \quad Q(g(x))=\alpha g(x)\\
R(t)&=\beta, \qquad \alpha,\ \beta \text{ are constants.}
\end{aligned}$$

and so

$$\begin{aligned}
&\left(P(g(x))\frac{1}{(g'(x))^2}\right)\frac{d^2\tilde{y}}{dx^2}+\left(-P(g(x))\frac{g''(x)}{(g'(x))^3}+Q(g(x))\frac{1}{g'(x)}\right)\frac{d\tilde{y}}{dx}+R(g(x))\,\tilde{y}(x)\\
&=\frac{d^2\tilde{y}}{dx^2}+(\alpha-1)\frac{d\tilde{y}}{dx}+\beta\tilde{y},
\end{aligned}$$

which is an ODE with constant coefficients.

Example 3.44 *(This is problem 35, p. 166.) Find the general solution of the Euler equation*

$$t^2y''(t)+ty'(t)+4y(t)=0, \quad t\in(0,\infty). \tag{220}$$

<u>**Solution**</u>:

By the change of variables $x=\ln t,\ t\in(0,\infty)$, the new equation for $\tilde{y}(x)$ is

$$\tilde{y}''(x)+4\tilde{y}(x)=0,$$

which has general solution

$$\tilde{y}(x)=c_1\cos(2x)+c_2\sin(2x), \quad x\in(-\infty,\infty).$$

Back to $y(t)$, we get

$$y(t)=c_1\cos(2\log t)+c_2\sin(2\log t), \quad t\in(0,\infty),$$

which is the general solution of the Euler equation. Here c_1 and c_2 are arbitrary constants. □

3.6 The method of "reduction of order" (this is part of Section 3.4).

Consider the second order linear homogeneous equation

$$y''(t)+p(t)\,y'(t)+q(t)\,y(t)=0, \quad t\in I \tag{221}$$

and assume that **we already know one nonzero solution** $y_1(t)$. We can use the following "**reduction method**" to find another solution $y_2(t)$.

Remark 3.45 *Note that here the coefficient of $y''(t)$ is 1 in (221).*

Let

$$y_2(t) = y_1(t)\, v(t), \quad t \in I.$$

The idea is to find suitable $v(t)$ so that the above $y_2(t)$ will be a solution of (221) different from $y_1(t)$. We substitute $y_2(t)$ into the equation to get

$$\begin{aligned} & y_2''(t) + p(t)\, y_2'(t) + q(t)\, y_2(t) \\ &= y_1(t)\, v''(t) + \left[2y_1'(t) + p(t)\, y_1(t)\right] v'(t) + \left[\underbrace{y_1''(t) + p(t)\, y_1'(t) + q(t)\, y_1(t)}\right] v(t) \\ &= y_1(t)\, v''(t) + \left[2y_1'(t) + p(t)\, y_1(t)\right] v'(t). \end{aligned} \tag{222}$$

Therefore, if $v(t)$ satisfies the equation

$$y_1(t)\, v''(t) + \left[2y_1'(t) + p(t)\, y_1(t)\right] v'(t) = 0 \tag{223}$$

and is not a constant solution, then $y_2(t) = y_1(t)\, v(t)$ will be a new solution of (221). Note that (223) is a **first order linear equation** for $w(t) := v'(t)$ and one can solve $v'(t)$ (and then $v(t)$). After that, one can find a different solution $y_2(t)$. Finally we check the Wronskian $W(y_1, y_2)(t)$ and if there is one point $t_0 \in I$ such that $W(y_1, y_2)(t_0) \neq 0$, then every solution $y(t)$ to the linear equation (221) is of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1,\ c_2 \text{ are arbitrary constants.}$$

Remark 3.46 *(**Be careful.**) In case equation (221) has the form*

$$P(t)\, y''(t) + Q(t)\, y'(t) + R(t)\, y(t) = 0, \quad t \in I, \quad P(t) \neq 0 \text{ on } I, \tag{224}$$

then you can either rewrite it as the form (221) and use the equation (223) for $v(t)$ or you can maintain the original equation (224) and now equation (223) becomes

$$P(t)\, y_1(t)\, v''(t) + \left[2P(t)\, y_1'(t) + Q(t)\, y_1(t)\right] v'(t) = 0. \tag{225}$$

You can use either way.

Example 3.47 *Use **method of reduction** to find the general solution of*

$$y''(t) + 4y'(t) + 4y(t) = 0.$$

<u>**Solution**</u>:

We first know that $y_1(t) = e^{-2t}$ is one solution. To find the second solution, let

$$y_2(t) = v(t)\, e^{-2t}$$

and plug it into the equation to get $y_1(t)\, v''(t) + \left[2y_1'(t) + p(t)\, y_1(t)\right] v'(t) = 0$, i.e.

$$e^{-2t} v''(t) = 0. \tag{226}$$

Thus

$$v(t) = at + b, \quad a,\ b \text{ are arbitrary const}$$

and $y_2(t) = (at + b)\, e^{-2t}$, which gives a new solution te^{-2t}. Therefore, the general solution is given by (since e^{-2t} and te^{-2t} have nonzero Wronskian)

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t},$$

same as before. $\square$

3.6.1 Reduction method for nonhomogeneous equations.

The **reduction method** can also be used to solve the **nonhomogeneous** equation

$$y''(t) + p(t) y'(t) + q(t) y(t) = g(t), \quad t \in I. \tag{227}$$

Assume that $y_1(t)$ is a nonzero solution to the **homogeneous** equation $y''(t) + p(t) y'(t) + q(t) y(t) = 0$ on I. Let $y_2(t) = v(t) y_1(t)$ to get

$$y_1(t) v''(t) + [2y_1'(t) + p(t) y_1(t)] v'(t) = g(t) \quad \text{(compare with (223))}, \tag{228}$$

which is a first order linear ODE for $w(t) := v'(t)$. By solving (228), one can get $y_2(t)$ and $Y(t)$, where $y_2(t)$ is a solution of $y''(t)+py'+qy=0$ and $Y(t)$ is a solution of $y''+py'+qy=g$. Now the general solution of (227) is given by (assume that $y_1(t)$ and $y_2(t)$ have **nonzero** Wronskian)

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \quad t \in I.$$

We use the above reduction method to solve the nonhomogeneous equation

$$y''(t) + 4y'(t) + 4y(t) = e^{3t}. \tag{229}$$

We know $y_1(t) = e^{-2t}$ is one solution of $y''(t) + 4y'(t) + 4y(t) = 0$. Let

$$y_2(t) = v(t) e^{-2t}$$

and plug it into the nonhomogeneous equation (229) to get

$$e^{-2t} v''(t) = e^{3t} \quad \text{(compare with (226))}.$$

We obtain

$$v(t) = \frac{1}{25} e^{5t} + c_1 t + c_2,$$

which gives

$$y_2(t) = \left(\frac{1}{25} e^{5t} + c_1 t + c_2\right) e^{-2t} = \frac{1}{25} e^{3t} + \left(c_1 t e^{-2t} + c_2 e^{-2t}\right).$$

Note that $Y(t) = \frac{1}{25} e^{3t}$ is a particular solution of the nonhomogeneous equation (229) and $y_2(t) = te^{-2t}$ is another solution of the corresponding homogeneous equation. Thus the general solution of (229) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{25} e^{3t}$$

for arbitrary constants c_1, c_2.

Remark 3.48 ***(Be careful ...)*** *In case we know a particular solution $Y(t)$ to the equation $y''(t) + p(t) y'(t) + q(t) y(t) = g(t)$ on I. Then if we try to use reduction method,* ***it will not work in general****. To see this, let $y_2(t) = v(t) Y(t)$ and plug it into the nonhomogeneous equation (227) to get (see (222) first)*

$$\begin{aligned} & y_2''(t) + p(t) y_2'(t) + q(t) y_2(t) \\ &= Y(t) v''(t) + [2Y'(t) + p(t) Y(t)] v'(t) + \left[\underbrace{Y''(t) + p(t) Y'(t) + q(t) Y(t)}\right] v(t) \\ &= Y(t) v''(t) + [2Y'(t) + p(t) Y(t)] v'(t) + \underbrace{g(t) v(t)} = g(t). \end{aligned} \tag{230}$$

Unfortunately for this situation, we are ***not able*** *to solve $v(t)$ in general, because it cannot be reduced to a first order equation.*

Example 3.49 *One can see that $y_1(t) = t$ is a solution of the following ODE on $t \in (1, \infty)$. Use* ***method of reduction*** *to find the general solution of the equation*

$$(t-1)\, y''(t) - t y'(t) + y(t) = 0, \quad t \in (1, \infty),$$

is given by

$$y(t) = C_1 t + C_2 e^t, \quad t \in (1, \infty), \tag{231}$$

Remark 3.50 *Note that the general solution in (231) is actually defined for all $t \in (-\infty, \infty)$, i.e. they are defined across $t = 1$.*

Solution:

We first rewrite the equation as the form (221):

$$y''(t) - \frac{t}{t-1} y'(t) + \frac{1}{t-1} y(t) = 0.$$

Let $y_2(t) = y_1(t)\, v(t) = t v(t)$. We will have

$$y_1(t)\, v''(t) + \left[2y_1'(t) + p(t)\, y_1(t)\right] v'(t) = 0, \quad y_1(t) = t,$$

which is

$$t v''(t) + \left(2 - \frac{t^2}{t-1}\right) v'(t) = 0,$$

i.e.

$$w'(t) + \left(\frac{2}{t} - \frac{t}{t-1}\right) w(t) = 0, \quad w(t) = v'(t).$$

We get

$$w(t) = c_1 e^{-\int\left(\frac{2}{t} - \frac{t}{t-1}\right)dt}, \quad c_1 \text{ is arbitrary constant}$$

$$-\int \left(\frac{2}{t} - \frac{t}{t-1}\right) dt = -2\log t + t + \log(t-1) = t + \log\left(\frac{t-1}{t^2}\right)$$

and so

$$w(t) = c_1 e^t \left(\frac{t-1}{t^2}\right) = v'(t).$$

Finally, we get

$$v(t) = c_1 \frac{e^t}{t} + c_2, \quad c_1,\ c_2 \text{ are arbitrary constants,}$$

which gives

$$y_2(t) = t v = t\left(c_1 \frac{e^t}{t} + c_2\right) = c_1 e^t + c_2 t.$$

The proof is done. □

3.7 Use Wronskian to solve a second order linear equation (Wronskian method) (see p. 174, Exercise 32).

Consider the second order linear homogeneous equation

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0, \quad t \in I \tag{232}$$

and assume that **we already know one "nonzero" solution** $y_1(t)$. Again, we note that the coefficient of $y''(t)$ is 1 in (232).

Let $y_2(t)$ be another solution of the above equation. The Wronskian $W(t)$ of $y_1(t)$, $y_2(t)$ satisfies

$$W(t) = Ce^{-\int p(t)dt}, \quad C \neq 0. \tag{233}$$

Therefore, we get

$$y_1(t) y_2'(t) - y_1'(t) y_2(t) = Ce^{-\int p(t)dt}, \quad C \neq 0 \tag{234}$$

This gives a first order **linear** equation for $y_2(t)$ and one can solve $y_2(t)$. From $y_2(t)$ one can obtain a new solution different from $y_1(t)$ and then obtain a fundamental set of solutions (note that since we require the constant $C \neq 0$ in (234), $\{y_1(t), y_2(t)\}$ forms a **fundamental set** of solutions for the ODE (232) on I and every solution $y(t)$ on I is a linear combination of them).

This method is as good as the reduction method.

Remark 3.51 *(**Important.**) Note that the Wronskian method is valid only for **homogeneous equations.***

Remark 3.52 *(**Important.**) Be careful that when you use the Wronskian method, make sure you rewrite the equation into the form $y'' + p(t)y' + q(t)y = 0$ first.*

Example 3.53 *(**Example 3 in p. 172.**) We shall use reduction method and Wronskian method to solve the equation*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0,$$

given that $y(t) = 1/t$ is a solution of it.

Remark 3.54 *This equation is, in fact, an Euler equation. So we know how to obtain its general solution. However, here we want to use different methods and see how they work.*

<u>**Solution**</u>:

1. Reduction method:

We first rewrite the equation as

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0, \quad t > 0$$

and let $y(t) = t^{-1}v(t)$ and by the formula

$$y_1(t) v''(t) + [2y_1'(t) + p(t) y_1(t)] v'(t) = 0,$$

we get

$$w' - \frac{1}{2t}w = 0, \quad w = v', \quad t > 0,$$

which gives

$$w(t) = v'(t) = Ce^{\int \frac{1}{2t}dt} = Ct^{1/2}.$$

Hence $v(t) = \frac{2}{3}Ct^{3/2} + k$ and

$$y(t) = t^{-1}v(t) = \frac{2}{3}Ct^{1/2} + kt^{-1}.$$

Thus the general solution (easy to see that $t^{1/2}$ and t^{-1} is a **fundamental set** of solutions on $(0, \infty)$) is given by

$$y(t) = C_1t^{1/2} + C_2t^{-1}.$$

2. Wronskian method:

We first rewrite the equation as

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0, \quad t > 0.$$

Let $y_1(t) = t^{-1}$ and we want to find $y_2(t)$. By Abel's Theorem, the Wronskian $W(t)$ of $y_1(t)$ and $y_2(t)$ is given by

$$W(t) = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) = Ce^{-\int p(t)dt} = Ce^{-\int \frac{3}{2t}dt} = Ct^{-3/2}, \quad C \text{ is a const..}$$

Hence we get

$$\frac{1}{t}y_2'(t) + \frac{1}{t^2}y_2(t) = Ct^{-3/2},$$

i.e.,

$$y_2'(t) + \frac{1}{t}y_2(t) = Ct^{-1/2}.$$

We obtain

$$\begin{aligned} y_2(t) &= e^{-\int \frac{1}{t}dt}\left[\int \left(e^{\int \frac{1}{t}dt}Ct^{-1/2}\right)dt + \tilde{C}\right], \quad t > 0, \quad \tilde{C} \text{ is another const.} \\ &= \frac{1}{t}\left(C\int t^{1/2}dt + \tilde{C}\right) = \frac{1}{t}\left(\frac{2}{3}Ct^{3/2} + \tilde{C}\right) = \frac{2}{3}Ct^{1/2} + \tilde{C}t^{-1}, \quad t > 0. \end{aligned}$$

Thus the general solution is given by

$$y(t) = C_1 t^{1/2} + C_2 t^{-1}, \quad t > 0.$$

We get the same result as in the reduction method. □

3.8 Method of undetermined coefficients (this is Section 3.5 of the book).

In this section, we consider a nonhomogeneous second order linear equation with constant coefficients, given by

$$ay''(t) + by'(t) + cy(t) = g(t), \quad a \neq 0, \quad t \in (-\infty, \infty), \tag{235}$$

where $g(t)$ has the form $P_n(t)\, e^{\lambda t}$ or $P_n(t)\, e^{\alpha t}\cos\beta t$ or $P_n(t)\, e^{\alpha t}\sin\beta t$. Here $P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n$ is a polynomial with degree n and $\lambda,\ \alpha,\ \beta \in \mathbb{R}$ with $\beta > 0$. Note that the case $\lambda = 0$ and the case $\alpha = 0$ are allowed (in case $\lambda = 0$, $P_n(t)\, e^{\lambda t} = P_n(t)$ is just a polynomial in t).

We know that the general solution $y(t)$ of (235) is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \quad t \in (-\infty, \infty),$$

where $y_p(t)$ is a **particular solution** of the nonhomogeneous equation (235) and $y_1(t)$, $y_2(t)$ are solutions of $ay''(t) + by'(t) + cy(t) = 0$, determined by the roots of the characteristic equation $ar^2 + br + c = 0$. Since we know how to find $y_1(t)$, $y_2(t)$, it suffices to find a particular solution $y_p(t)$ of (235).

The "**method of undetermined coefficients**" says that we can try a particular solution of the **form** given by **Table 3.5.1 in p. 182** of the book and then plug in the form into the nonhomogeneous equation (235) to **determine the coefficients**. After that, one can find a particular solution $y_p(t)$.

Remark 3.55 *Explain Table 3.5.1 in p. 182 ....*

3.8.1 Motivation of the undetermined coefficients method.

One can use simple first order equation to explain the method. Consider the simple equation

$$y'(t) - \lambda y(t) = a_0 e^{\alpha t}, \quad a_0,\ \lambda,\ \alpha \text{ are constants}, a_0 \neq 0. \tag{236}$$

The characteristic equation of the homogeneous equation $y'(t) - \lambda y(t) = 0$ is $r - \lambda = 0$, which has root $r = \lambda$ and so the general solution of $y'(t) - \lambda y(t) = 0$ is given by $y(t) = Ce^{\lambda t}$ for arbitrary constant C. To find the general solution of (236), it suffices to find a particular solution $y_p(t)$. If $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), then the function $(e^{\alpha t})' - \lambda(e^{\alpha t})$ is **not zero** and is still of the form $Ke^{\alpha t}$ for constant $K = \alpha - \lambda \neq 0$. This form matches with the function $a_0 e^{\alpha t}$ on the right hand side of the equation. Therefore, if we try $y_p(t)$ to have the form

$$y_p(t) = A_0 e^{\alpha t} \tag{237}$$

and **choose the coefficient** A_0 **suitably**, we can obtain a particular solution of the equation (236). To find A_0, we plug $y_p(t) = A_0 e^{\alpha t}$ into (236) and get the identity

$$(\alpha - \lambda) A_0 e^{\alpha t} = a_0 e^{\alpha t}, \quad a_0 \neq 0. \tag{238}$$

Hence, if we choose $A = \frac{a_0}{\alpha - \lambda}$ (denominator is not zero), we can obtain a particular solution $y_p(t) = \frac{a_0}{\alpha - \lambda} e^{\alpha t}$ of (236). Thus the general solution of (236) is

$$y(t) = Ce^{\lambda t} + \frac{a_0}{\alpha - \lambda} e^{\alpha t}, \quad t \in (-\infty, \infty), \quad C \text{ is arbitrary const..} \tag{239}$$

On the other hand, if $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), identity (238) will becomes $0 = a_0 e^{\alpha t}$, which is impossible and it suggests that we cannot try $y_p(t)$ to have the form $y_p(t) = A_0 e^{\alpha t}$. instead, if we try

$$y_p(t) = A_0 t e^{\alpha t}, \tag{240}$$

and plug it into (236), we get the identity

$$A_0 e^{\alpha t} + \alpha A_0 t e^{\alpha t} - \lambda A_0 t e^{\alpha t} = a_0 e^{\alpha t} \quad (\text{note that } \alpha = \lambda).$$

Hence if we choose $A_0 = a_0$, the function $y_p(t) = a_0 t e^{\alpha t}$ will be a particular solution of (236) and from this we can obtain general solution of (236).

One step further, now we look at the equation

$$y'(t) - \lambda y(t) = (a_0 + b_0 t)e^{\alpha t}, \quad a_0,\ b_0,\ \lambda,\ \alpha \text{ are constants}, a_0 \neq 0,\ b_0 \neq 0. \tag{241}$$

For the case $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), based on the above observation, the only way you can try is

$$y_p(t) = (A_0 + B_0 t)e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0, \tag{242}$$

and if you plug it into equation (241), you get

$$B_0 e^{\alpha t} + \alpha(A_0 + B_0 t)e^{\alpha t} - \lambda(A_0 + B_0 t)e^{\alpha t} = (a_0 + b_0 t)e^{\alpha t},$$

which is same as

$$B_0 + \alpha(A_0 + B_0 t) - \lambda(A_0 + B_0 t) = a_0 + b_0 t, \tag{243}$$

and you need to choose $A_0,\ B_0$ satisfying

$$\begin{cases} B_0 + (\alpha - \lambda) A_0 = a_0 \\ (\alpha - \lambda) B_0 = b_0 \end{cases}$$

and conclude that if we choose

$$A_0 = \frac{a_0}{\alpha - \lambda} - \frac{b_0}{(\alpha - \lambda)^2}, \qquad B_0 = \frac{b_0}{\alpha - \lambda}, \quad \alpha \neq \lambda,$$

then $y_p(t)$ in (242) will be a **particular solution** of the ODE (241).

Finally, for the case $\alpha = \lambda$, the identity (243) becomes $B_0 = a_0 + b_0 t$, **impossible to hold**. Therefore you need to modify your choice of $y_p(t)$ in (242). A natural next choice is

$$y_p(t) = (A_0 + B_0 t + C_0 t^2) e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0,\ C_0.$$

However, note that $A_0 e^{\alpha t}$ is already a solution of the homogeneous equation $y'(t) - \lambda y(t) = 0$, there is **no need** to include it. Hence we choose

$$y_p(t) = (B_0 t + C_0 t^2) e^{\alpha t} = t (B_0 + C_0 t) e^{\alpha t}$$

and for consistency of notations, we write it as

$$y_p(t) = t (A_0 + B_0 t) e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0. \tag{244}$$

If you plug the above $y_p(t)$ into (241), you get

$$(A_0 + B_0 t) e^{\alpha t} + t B_0 e^{\alpha t} = (a_0 + b_0 t) e^{\alpha t}$$

and conclude

$$A_0 = a_0, \quad B_0 = \frac{b_0}{2}.$$

Thus when $\alpha = \lambda$, the function

$$y_p(t) = t \left(a_0 + \frac{b_0}{2} t \right) e^{\alpha t}, \quad t \in (-\infty, \infty)$$

will be a **particular solution** of the equation (241).

From (237), (240), (242), and (244), you can understand the undetermined coefficients method in Table 3.5.1 in p. 182 of the book.

3.8.2 P. 183, Case 2.

This is to verify that the **method of undetermined coefficients** can be used to solve a nonhomogeneous second order linear ODE (with constant coefficients) of the form

$$a y''(t) + b y'(t) + c y(t) = P_n(t) e^{\lambda t}, \quad a \neq 0, \tag{245}$$

where

$$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$$

is a polynomial with degree n.

Remark 3.56 *Of course, one can also use reduction method to solve (245), but the* ***method of undetermined coefficients*** *will be easier for $g(t)$ of the form $P_n(t) e^{\lambda t}$.*

We let $y_p(t) = u(t) e^{\lambda t}$ be the particular solution to be found (there is no other better try than this), where $u(t)$ is to be determined. Plug $y_p(t) = u(t) e^{\lambda t}$ into (245) to get

$$a \left[u''(t) e^{\lambda t} + 2u'(t) \lambda e^{\lambda t} + u(t) \lambda^2 e^{\lambda t} \right] + b \left[u'(t) e^{\lambda t} + u(t) \lambda e^{\lambda t} \right] + c u(t) e^{\lambda t} = P_n(t) e^{\lambda t}.$$

We can cancel $e^{\lambda t}$ and the equation becomes

$$\underbrace{au''(t) + (2a\lambda + b)\, u'(t) + \left(a\lambda^2 + b\lambda + c\right) u(t)} = P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n. \qquad (246)$$

Assume first that λ is **not a root of the characteristic equation** $ar^2+br+c=0$. Hence $a\lambda^2+b\lambda+c \neq 0$. One can try

$$u(t) = A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n. \qquad (247)$$

Note that
$$\begin{cases} u'(t) = nA_0 t^{n-1} + (n-1)\, A_1 t^{n-2} + \cdots + 2A_{n-2} t + A_{n-1} \\ u''(t) = n(n-1)\, A_0 t^{n-2} + (n-1)(n-2)\, A_1 t^{n-3} + \cdots + 2A_{n-2}. \end{cases}$$

If we plug (247) into (246) and compare coefficients, we can get the following system of equations (note that $P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$):

$$\begin{cases} \left(a\lambda^2 + b\lambda + c\right) A_0 = a_0 \text{ (coefficients of } t^n), \quad \text{where} \quad a\lambda^2 + b\lambda + c \neq 0 \\ (2a\lambda + b)\, nA_0 + \left(a\lambda^2 + b\lambda + c\right) A_1 = a_1 \text{ (coefficients of } t^{n-1}) \\ an(n-1)\, A_0 + (2a\lambda + b)(n-1)\, A_1 + \left(a\lambda^2 + b\lambda + c\right) A_2 = a_2 \text{ (coefficients of } t^{n-2}) \\ \cdots \\ a2A_{n-2} + (2a\lambda + b)\, A_{n-1} + \left(a\lambda^2 + b\lambda + c\right) A_n = a_n \text{ (coefficients of } t^0). \end{cases} \qquad (248)$$

Then one can solve all A_0, ..., A_n and obtain $u(t)$, and conclude that $y(t) = u(t)\, e^{\lambda t}$ is a solution of the nonhomogeneous equation (245).

If λ is **a root with multiplicity** $s=1$, then $a\lambda^2+b\lambda+c=0$ and $2a\lambda+b \neq 0$. The above trial solution (247) **does not work out**. Instead we try

$$u(t) = t\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n\right) = A_0 t^{n+1} + A_1 t^n + \cdots + A_{n-1} t^2 + A_n t$$

Then (246) becomes

$$\underbrace{au''(t) + (2a\lambda + b)\, u'(t)} = P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n \qquad (249)$$

and (248) becomes

$$\begin{cases} (2a\lambda + b)(n+1)\, A_0 = a_0 \text{ (coefficients of } t^n), \quad \text{where} \quad 2a\lambda + b \neq 0 \\ an(n+1)\, A_0 + (2a\lambda + b)\, nA_1 = a_1 \text{ (coefficients of } t^{n-1}) \\ \cdots \\ a2A_{n-2} + (2a\lambda + b)\, A_n = a_n \text{ (coefficients of } t^0). \end{cases} \qquad (250)$$

In this case we can solve all A_0, ..., A_n and conclude that $y(t) = u(t)\, e^{\lambda t}$ is a solution of (245).

Finally if λ is **a root with multiplicity** $s=2$, then $a\lambda^2+b\lambda+c=0$ and $2a\lambda+b=0$, but $a \neq 0$. Then we try

$$u(t) = t^2\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n\right) = A_0 t^{n+2} + A_1 t^{n+1} + \cdots + A_n t^2.$$

Now (246) becomes

$$\underbrace{au''(t)} = P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n \qquad (251)$$

and (248) becomes

$$\begin{cases} a(n+2)(n+1)A_0 = a_0 \text{ (coefficients of } t^n\text{)}, \quad \text{where} \quad a \neq 0 \\ an(n+1)A_1 = a_1 \text{ (coefficients of } t^{n-1}\text{)} \\ \cdots \\ a2A_n = a_n \text{ (coefficients of } t^0\text{)}. \end{cases} \tag{252}$$

Again, we can solve all A_0, ..., A_n and obtain a particular solution of (245).

In conclusion, the method works for the case $g(t) = P_n(t)e^{\lambda t}$, $\lambda \in \mathbb{R}$. The verification is done.□

Example 3.57 $y'' + 3y = 4e^{-5t}, \quad y_p(t) = \frac{1}{7}e^{-5t}.$

Example 3.58 $y'' - 3y' - 4y = 2e^{-t}, \quad y_p(t) = -\frac{2}{5}te^{-t}.$

Example 3.59 $y'' + 2y = \sin 3t, \quad y_p(t) = -\frac{1}{7}\sin 3t.$

Example 3.60 $y'' + 9y = \sin 3t, \quad y_p(t) = -\frac{1}{6}t\cos 3t.$

Example 3.61 $y'' - 3y = t^2, \quad y_p(t) = -\frac{1}{3}t^2 - \frac{2}{9}.$

Example 3.62 *Do Example 3 in p. 179.*

Example 3.63 *Find general solution of the equation*

$$y'' + 2y' + y = te^{-t}.$$

Solution:

By the rule for $y_p(t)$, it has the form

$$y_p(t) = t^s(At + B)e^{-t} = \left(At^3 + Bt^2\right)e^{-t}, \quad \text{where } s = 2.$$

Plugging it into equation to get

$$\left\{\begin{array}{l} \left[(6At + 2B)e^{-t} - 2(3At^2 + 2Bt)e^{-t} + (At^3 + Bt^2)e^{-t}\right] \\ +2\left[(3At^2 + 2Bt)e^{-t} - (At^3 + Bt^2)e^{-t}\right] + (At^3 + Bt^2)e^{-t} \end{array}\right. = te^{-t}.$$

Hence, after simplification, we need to solve $6At + 2B = t$, which gives

$$A = \frac{1}{6}, \quad B = 0.$$

Thus $y_p(t) = \frac{1}{6}t^3e^{-t}$ is a particular solution of the equation. The general solution is

$$y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{6}t^3e^{-t}, \quad t \in (-\infty, \infty).$$

□

Remark 3.64 *If an equation has the form*

$$ay'' + by' + cy = f(t) + g(t), \tag{253}$$

where $f(t)$ *and* $g(t)$ *both have the form in the above case 1 or case 2 (say* $f(t) = t^2e^{5t}$ *and* $g(t) = (t^3 + 2t^2 - 6t - 5)e^{-t}\cos 7t$*), then use the undetermined coefficients to find* $y_p(t)$ *for the equation*

$$ay'' + by' + cy = f(t)$$

and then use the same method to find $\tilde{y}_p(t)$ *for the equation*

$$ay'' + by' + cy = g(t).$$

Then the general solution of (253) is given by

$$x(t) = y_p(t) + \tilde{y}_p(t) + c_1y_1(t) + c_2y_2(t),$$

where $c_1x_1(t) + c_2x_2(t)$ *is the general solution of the corresponding homogeneous equation.*

Example 3.65 *(**This is Exercise 30 in p. 185 with one extra term.**) Find general solution of the equation*

$$y'' + \lambda^2 y = \sum_{m=1}^{N}(a_m \sin m\pi t + b_m \cos m\pi t), \quad t \in (-\infty, \infty), \tag{254}$$

where $\lambda > 0$ *and* $\lambda \neq m\pi$ *for* $m = 1, 2, ..., N$.

<u>**Solution**</u>:

The two roots of the characteristic polynomial $r^2 + \lambda^2 = 0$ are $r = \pm\lambda i$, where $\lambda \neq m\pi$ for any $m = 1, ..., N$. Hence **for each** $m = 1, ..., N$, we try a particular solution $y_m(t)$ of the form

$$y_m(t) = A_m \sin m\pi t + B_m \cos m\pi t, \tag{255}$$

which is for the equation

$$y'' + \lambda^2 y = a_m \sin m\pi t + b_m \cos m\pi t. \tag{256}$$

We plug the above $y_m(t)$ into equation (256) to get

$$\left(\lambda^2 - m^2\pi^2\right)A_m \sin m\pi t + \left(\lambda^2 - m^2\pi^2\right)B_m \cos m\pi t = a_m \sin m\pi t + b_m \cos m\pi t$$

and obtain

$$A_m = \frac{a_m}{\lambda^2 - m^2\pi^2}, \qquad B_m = \frac{b_m}{\lambda^2 - m^2\pi^2}, \qquad m = 1, ..., N.$$

Hence, the general solution of the equation is given by (add all $y_m(t)$ together):

$$y(t) = c_1 \sin\lambda t + c_2\cos\lambda t + \sum_{m=1}^{N}\left(\frac{a_m}{\lambda^2 - m^2\pi^2}\right)\sin m\pi t + \left(\frac{b_m}{\lambda^2 - m^2\pi^2}\right)\cos m\pi t.$$

The proof is done. □

3.9 Variation of parameters method (this is Section 3.6 of the book) for constant coefficients.

In this section we first focus on the **nonhomogeneous linear equation with constant coefficients**, given by

$$ay'' + by' + cy = f(t), \quad a \neq 0,\ b,\ c \text{ are constants}, \tag{257}$$

where now $f(t)$ can be an **arbitrary** continuous function defined on some interval $I \subseteq \mathbb{R}$. The reason of requiring constant coefficients in (257) is that we can know general solution of the corresponding homogeneous equation $ay'' + by' + cy = 0$.

Choose a pair of **fundamental solutions** $\{y_1(t),\ y_2(t)\}$. To solve (257), we try a solution of the form:

$$y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t), \quad t \in I \tag{258}$$

and look for suitable $u_1(t)$ and $u_2(t)$.

Remark 3.66 *(**Useful observation.**) One can view (258) as a generalization of the **reduction method** because if we only try $y(t) = u_1(t)\, y_1(t)$, it is exactly the reduction method. See Section 3.6.1.*

We need to impose suitable conditions on $u_1(t)$ and $u_2(t)$ so that the above $y(t)$ is a solution of (257).

We first note that

$$y'(t) = \underbrace{\left[u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t)\right]} + \left[u_1(t)\, y_1'(t) + u_2(t)\, y_2'(t)\right]$$

and **impose the first condition**

$$\underbrace{u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t)} = 0, \quad t \in I. \tag{259}$$

Remark 3.67 *If we impose the condition $u_1(t)\, y_1'(t) + u_2(t)\, y_2'(t)$, then in $y''(t)$ we will encounter $u_1''(t)$ and $u_2''(t)$. With this, the method will not help us too much ...*

Then $y'(t)$ becomes

$$y'(t) = u_1(t)\, y_1'(t) + u_2(t)\, y_2'(t), \quad t \in I$$

and so

$$y''(t) = \underbrace{\left[u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t)\right]} + \left[u_1(t)\, y_1''(t) + u_2(t)\, y_2''(t)\right], \quad t \in I.$$

Then we **impose the second condition** as

$$\underbrace{u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t)} = \frac{f(t)}{a} \text{ (make sure to divide } a \text{ here).} \tag{260}$$

We now have

$$\begin{aligned}
&ay''(t) + by'(t) + cy(t) \\
&= a\left[\frac{f(t)}{a} + u_1(t)\, y_1''(t) + u_2(t)\, y_2''(t)\right] + b\left[u_1(t)\, y_1'(t) + u_2(t)\, y_2'(t)\right] + c\left[u_1(t)\, y_1(t) + u_2(t)\, y_2(t)\right] \\
&= f(t) + u_1(t)\left[\underbrace{ay_1''(t) + by_1'(t) + cy_1(t)}\right] + u_2(t)\left[\underbrace{ay_2''(t) + by_2'(t) + cy_2(t)}\right] \\
&= f(t) + u_1(t) \cdot 0 + u_2(t) \cdot 0 = f(t), \quad t \in I,
\end{aligned}$$

which is exactly what we want.

It remains to claim that (259) and (260) can be satisfied. We need to solve

$$\begin{cases} u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t) = 0 \\ u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t) = \frac{f(t)}{a} \end{cases}$$

and get

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ \frac{f(t)}{a} & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{-y_2(t)\frac{f(t)}{a}}{W(y_1, y_2)(t)}, \qquad u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & \frac{f(t)}{a} \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{y_1(t)\frac{f(t)}{a}}{W(y_1, y_2)(t)}.$$

The above gives

$$u_1(t) = -\frac{1}{a}\int \frac{y_2(t)\, f(t)}{W(y_1, y_2)(t)} dt + c_1, \qquad u_2(t) = \frac{1}{a}\int \frac{y_1(t)\, f(t)}{W(y_1, y_2)(t)} dt + c_2, \tag{261}$$

and the general solution of (257) is given by

$$\begin{aligned} y(t) &= \left(-\frac{1}{a}\int \frac{y_2(t) f(t)}{W(y_1,y_2)(t)}dt + c_1\right) y_1(t) + \left(\frac{1}{a}\int \frac{y_1(t) f(t)}{W(y_1,y_2)(t)}dt + c_2\right) y_2(t) \\ &= c_1 y_1(t) + c_2 y_2(t) + y_p(t), \end{aligned}$$

where

$$y_p(t) = -\frac{1}{a}\left(\int \frac{y_2(t) f(t)}{W(y_1,y_2)(t)}dt\right) y_1(t) + \frac{1}{a}\left(\int \frac{y_1(t) f(t)}{W(y_1,y_2)(t)}dt\right) y_2(t) \tag{262}$$

is a **particular solution** of (257). The above method is called "**variation of parameters**" method. It is a powerful method.

Remark 3.68 ***(Important.)*** *If the equation (257) has initial conditions $y(t_0) = y_0$, $y'(t_0) = z_0$, $t_0 \in I$, then the unique solution $y(t)$ can be expressed as*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \left[-\frac{1}{a}\left(\int_{t_0}^{t} \frac{y_2(s) f(s)}{W(y_1,y_2)(s)}ds\right) y_1(t) + \frac{1}{a}\left(\int_{t_0}^{t} \frac{y_1(s) f(s)}{W(y_1,y_2)(s)}ds\right) y_2(t)\right], \tag{263}$$

where c_1, c_2 satisfy the following

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = z_0. \end{cases}$$

This is due to the fact that the ***particular solution***

$$y_p(t) = -\frac{1}{a}\left(\int_{t_0}^{t} \frac{y_2(s) f(s)}{W(y_1,y_2)(s)}ds\right) y_1(t) + \frac{1}{a}\left(\int_{t_0}^{t} \frac{y_1(s) f(s)}{W(y_1,y_2)(s)}ds\right) y_2(t) \tag{264}$$

satisfies $y(t_0) = y'(t_0) = 0$. To see this, we clearly have $y_p(t_0) = 0$. As for $y_p'(t_0) = 0$, we note that

$$y_p'(t_0) = \begin{cases} -\frac{1}{a}\left(\frac{y_2(t_0)f(t_0)}{W(y_1,y_2)(t_0)}\right) y_1(t_0) + 0 \cdot y_1'(t_0) \\ +\frac{1}{a}\left(\frac{y_1(t_0)f(t_0)}{W(y_1,y_2)(t_0)}\right) y_2(t_0) + 0 \cdot y_2'(t_0) \end{cases} = 0. \tag{265}$$

Example 3.69 ***(This is Exercise 5 in p. 190.)*** *Solve the equation*

$$y''(t) + y(t) = 2\tan t, \quad 0 < t < \frac{\pi}{2}.$$

Solution:

Since we know two independent solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of $y''(t) + y(t) = 0$, we can use variation of parameters method. We first compute

$$W(y_1,y_2)(t) = \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} = 1$$

and by (262) we conclude

$$y(t) = c_1 \cos t + c_2 \sin t + \left(-\frac{1}{a}\int \frac{y_2(t) f(t)}{W(y_1,y_2)(t)}dt\right) y_1(t) + \left(\frac{1}{a}\int \frac{y_1(t) f(t)}{W(y_1,y_2)(t)}dt\right) y_2(t),$$

where

$$\begin{cases} -\frac{1}{a}\int \frac{y_2(t)f(t)}{W(y_1,y_2)(t)}dt = -\int (\sin t)(2\tan t)\,dt = -2\int \frac{(1-\cos^2 t)}{\cos}dt = -2\int (\sec t - \cos t)\,dt \\ \frac{1}{a}\int \frac{y_1(t)f(t)}{W(y_1,y_2)(t)}dt = \int (\cos t)(2\tan t)\,dt = 2\int \sin t dt. \end{cases}$$

We conclude

$$\begin{aligned}
y(t) &= c_1\cos t + c_2\sin t + \left(-2\int(\sec t - \cos t)\,dt\right)\cos t + \left(2\int \sin t dt\right)\sin t \\
&= c_1\cos t + c_2\sin t + \left(-2\int \sec t dt + 2\sin t\right)\cos t + (-2\cos t)\sin t \\
&= c_1\cos t + c_2\sin t + (-2\log|\sec t + \tan t|)\cos t. \qquad (266)
\end{aligned}$$

□

Remark 3.70 ***(Compare with the reduction method.)*** *If we use reduction method, we can let* $y(t) = v(t)\sin t$ *(*$\sin t$ *is a solution of* $y''(t) + y(t) = 0$*) and get*

$$v''(t)\sin t + 2v'(t)\cos t - v(t)\sin t + v(t)\sin t = 2\tan t,$$

which gives (let $w(t) = v'(t)$*)*

$$w'(t) + 2\frac{\cos t}{\sin t}\cdot w(t) = \frac{2}{\cos t}, \qquad 0 < t < \frac{\pi}{2}$$

and then

$$\begin{aligned}
w(t) = v'(t) &= e^{-\int \frac{2\cos}{\sin t}dt}\left(\int e^{\int \frac{2\cos}{\sin t}dt}\frac{2}{\cos t}dt + C\right) = \frac{1}{\sin^2 t}\left(2\int \sin t\tan t dt + C\right) \\
&= \frac{1}{\sin^2 t}\left(2\int(\sec t - \cos t)\,dt + C\right) = \frac{C}{\sin^2 t} + \frac{1}{\sin^2 t}(2\log|\sec t + \tan t| - 2\sin t).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
v(t) &= \int \frac{C}{\sin^2 t}dt + \int \frac{1}{\sin^2 t}(2\log|\sec t + \tan t| - 2\sin t)\,dt + K \\
&= -C\cot t + K + \underbrace{\int \frac{1}{\sin^2 t}(2\log|\sec t + \tan t|)\,dt - 2\int \frac{1}{\sin t}dt},
\end{aligned}$$

where, by the integration by parts, we find

$$\begin{aligned}
&\underbrace{\int \frac{1}{\sin^2 t}(2\log|\sec t + \tan t|)\,dt} = -\int(2\log|\sec t + \tan t|)\,d(\cot t) \\
&= \underbrace{-(2\log|\sec t + \tan t|)(\cot t) + 2\int \frac{1}{\sin t}dt}
\end{aligned}$$

and conclude

$$\begin{aligned}
y(t) = v(t)\sin t = y(t) &= [-C\cot t + K - (2\log|\sec t + \tan t|)(\cot t)]\sin t \\
&= c_1\cos t + c_2\sin t - (2\log|\sec t + \tan t|)\cos t. \qquad (267)
\end{aligned}$$

We see that (267) is the same as (266). This method clearly involves more computations. This is because we only make use of one solution $\sin t$*.*

Example 3.71 ***(This is Exercise 10 in p. 190.)*** *Solve the equation*

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2}, \qquad t \in (-\infty, \infty).$$

Solution:

We have $y_1(t) = e^t,\ y_2(t) = te^t,$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & e^t \\ te^t & e^t + te^t \end{vmatrix} = e^{2t}$$

and so

$$\begin{aligned} y(t) &= \left(-\int \frac{te^t \cdot \frac{e^t}{1+t^2}}{e^{2t}} dt + c_1 \right) e^t + \left(\int \frac{e^t \cdot \frac{e^t}{1+t^2}}{e^{2t}} dt + c_2 \right) te^t \\ &= \left(-\int \frac{t}{1+t^2} dt + c_1 \right) e^t + \left(\int \frac{1}{1+t^2} dt + c_2 \right) te^t \\ &= \left(-\frac{1}{2} \log\left(1+t^2\right) + c_1 \right) e^t + \left(\tan^{-1} t + c_2 \right) te^t, \end{aligned}$$

which is the general solution. $\square$

3.9.1 Nonhomogeneous Euler equation.

One can combine the variation of parameters method and change of variables to solve a **nonhomogeneous Euler equation**, given by

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = f(t), \quad t \in (0, \infty), \quad \alpha,\ \beta \text{ constants}, \tag{268}$$

where $f(t)$ can be any arbitrary continuous function defined on $t \in (0, \infty)$. By the change of variables $x = \log t,\ x \in (-\infty, \infty)$, the above equation becomes

$$\frac{d^2 \tilde{y}}{dx^2} + (\alpha - 1) \frac{d\tilde{y}}{dx} + \beta \tilde{y}(x) = F(x), \quad x \in (-\infty, \infty), \tag{269}$$

where $\tilde{y}(x) = y(e^x)$ and $F(x) = f(e^x)$. We can know a pair of **fundamental solutions** $\{y_1(t),\ y_2(t)\}$ for $\tilde{y}''(x) + (\alpha - 1)\tilde{y}'(x) + \beta \tilde{y}(x) = 0$ and then use the variation of parameters method to find the general solution $\tilde{y}(x)$ of (269) and then change back to get $y(t)$. It will be the general solution of (268).

Remark 3.72 *In case the Euler equation has the form*

$$At^2 y''(t) + Bty'(t) + Cy(t) = f(t), \quad t \in (0, \infty), \quad A \neq 0,\ B,\ C \text{ are constants,}$$

then equation (269) becomes

$$A \frac{d^2 \tilde{y}}{dx^2} + (B - A) \frac{d\tilde{y}}{dx} + C\tilde{y}(x) = F(x). \tag{270}$$

3.10 Variation of parameters method (this is Section 3.6 of the book) for variable coefficients.

The above variation of parameters method can also be applied to the equation (which has **leading coefficient** 1):

$$y'' + p(t) y' + q(t) y = g(t), \quad t \in I, \tag{271}$$

as long as we know two fundamental solutions $y_1(t)$ **and** $y_2(t)$ **of the equation** $y'' + p(t) y' + q(t) y = 0$ (here $p(t),\ q(t),\ g(t)$ can be arbitrary continuous functions). The method is exactly the same as the method for the case $ay'' + by' + cy = f(t)$. Assume we are given two independent solutions $y_1(t)$ and $y_2(t)$ of the equation $y''+p(t)y'+q(t)y=0$ on I and assume that $u_1(t)$ and $u_2(t)$ satisfy the equation

$$\begin{cases} u_1'(t)\,y_1(t)+u_2'(t)\,y_2(t)=0 \\ u_1'(t)\,y_1'(t)+u_2'(t)\,y_2'(t)=g(t) \end{cases}, \quad t\in I. \tag{272}$$

Then we will have (here $y(t)=u_1(t)\,y_1(t)+u_2(t)\,y_2(t)$)

$$\begin{aligned} & y''(t)+p(t)\,y'(t)+q(t)\,y(t) \\ &= \begin{cases} [g(t)+u_1(t)\,y_1''(t)+u_2(t)\,y_2''(t)]+p(t)\,[u_1(t)\,y_1'(t)+u_2(t)\,y_2'(t)] \\ +q(t)\,[u_1(t)\,y_1(t)+u_2(t)\,y_2(t)] \end{cases} \\ &= \begin{cases} g(t)+u_1(t)\left[\underbrace{y_1''(t)+p(t)\,y_1'(t)+q(t)\,y_1(t)}\right] \\ +u_2(t)\left[\underbrace{y_2''(t)+p(t)\,y_2'(t)+q(t)\,y_2(t)}\right] \end{cases} = g(t), \quad t\in I, \end{aligned}$$

which means that $y(t)=u_1(t)\,y_1(t)+u_2(t)\,y_2(t)$, $t\in I$, does satisfy the equation (271). Similar to (262), we get a particular solution for (271):

$$y_p(t)=\left(-\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}dt\right)y_1(t)+\left(\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}dt\right)y_2(t), \quad t\in I \tag{273}$$

and if we want to find the solution $y(t)$ of (271) with $y(t_0)=y_0$, $y'(t_0)=z_0$, $t_0\in I$, then the solution is given by

$$y(t)=c_1y_1(t)+c_2y_2(t)+y_p(t), \quad t\in I, \quad c_1,\ c_2 \text{ constants}, \tag{274}$$

where now we choose $y_p(t)$ as

$$y_p(t)=\left(-\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(y_1,y_2)(s)}ds\right)y_1(t)+\left(\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(y_1,y_2)(s)}ds\right)y_2(t), \quad t\in I \tag{275}$$

(the above $y_p(t)$ satisfies $y_p(t_0)=y_p'(t_0)=0$) and choose $c_1,\ c_2$ satisfying

$$\begin{cases} c_1y_1(t_0)+c_2y_2(t_0)=y_0 \\ c_1y_1'(t_0)+c_2y_2'(t_0)=z_0. \end{cases} \tag{276}$$

Example 3.73 *Find the general solution of the equation*

$$ty''(t)-(1+t)\,y'(t)+y(t)=t^2e^{2t}, \quad t\in(0,\infty), \tag{277}$$

given that $y_1(t)=1+t$ and $y_2(t)=e^t$ is a pair of fundamental solutions for the corresponding homogeneous equation.

Solution:

To apply the variation of parameters method, we need to rewrite the equation to have leading coefficient of $y''(t)$ equal to 1. We have

$$y''(t)-\left(\frac{1+t}{t}\right)y'(t)+\frac{1}{t}y(t)=te^{2t}, \quad t\in(0,\infty),$$

and obtain $g(t) = te^{2t}$. By the variation of parameters method, we have

$$y_p(t) = \left(-\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}dt\right) y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}dt\right) y_2(t), \quad t \in (0,\infty),$$

where

$$W(y_1,y_2)(t) = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = te^t.$$

Hence

$$\begin{aligned} y_p(t) &= \left(-\int \frac{e^t \cdot te^{2t}}{te^t}dt\right)(1+t) + \left(\int \frac{(1+t)\cdot te^{2t}}{te^t}dt\right)e^t \\ &= \left(-\int e^{2t}dt\right)(1+t) + \left(\int (1+t)\,e^t dt\right)e^t \\ &= \left(-\frac{1}{2}e^{2t}\right)(1+t) + \left(te^t\right)e^t = \frac{1}{2}(t-1)\,e^{2t}. \end{aligned}$$

and conclude the general solution for equation (277):

$$y(t) = C_1(1+t) + C_2e^t + \frac{1}{2}(t-1)\,e^{2t}, \quad t \in (0,\infty).$$

□

To end this section, we look at one more interesting example arising from **"mechanics of vibrations".**

Example 3.74 *Consider the equation*

$$y''(t) + y(t) = g(t), \quad g(t) \text{ is continuous on } I$$

with initial condition

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \quad t_0 \in I.$$

This equation appears frequently in ***mechanics of vibrations****. If we choose* $y_1(t) = \cos t$, $y_2(t) = \sin t$, *then by Remark 3.68 the particular solution* $y_p(t)$ *(with* $y_p(t_0) = y_p'(t_0) = 0$*) in (264) is given by*

$$\begin{aligned} y_p(t) &= -y_1(t)\int_{t_0}^t \frac{y_2(s)\,g(s)}{W(y_1,y_2)(s)}ds + y_2(t)\int_{t_0}^t \frac{y_1(s)\,g(s)}{W(y_1,y_2)(s)}ds, \quad W(y_1,y_2)(s) \equiv 1 \\ &= -(\cos t)\int_{t_0}^t g(s)\sin s ds + (\sin t)\int_{t_0}^t g(s)\cos s ds \\ &= \int_{t_0}^t g(s)\sin(t-s)\,ds, \quad t \in I \end{aligned} \tag{278}$$

Then the solution satisfying the initial condition is given by the nice formula:

$$y(t) = y_0\cos(t-t_0) + z_0\sin(t-t_0) + \int_{t_0}^t g(s)\sin(t-s)\,ds, \quad t \in I. \tag{279}$$

Now assume that $I = (-\infty,\infty)$ *and* $g(t)$ *is a* 2π***-periodic function*** *defined on* $(-\infty,\infty)$ *(*$g(t)$ *usually comes from the* ***external force*** *acting on the mechanical system, say string vibration).* ***The***

particular solution $y_p(t)$ ***in (278) may not be*** 2π***-periodic in general*** *(but the homogeneous part* $y_0\cos(t-t_0)+z_0\sin(t-t_0)$ *is clearly* 2π*-periodic). Note that we have*

$$\begin{aligned}
&y_p(t+2\pi)-y_p(t)\\
&=\begin{cases}\left[-(\cos(t+2\pi))\int_{t_0}^{t+2\pi}g(s)\sin s ds+(\sin(t+2\pi))\int_{t_0}^{t+2\pi}g(s)\cos s ds\right]\\ -\left[-(\cos t)\int_{t_0}^{t}g(s)\sin s ds+(\sin t)\int_{t_0}^{t}g(s)\cos s ds\right]\end{cases}\\
&=-(\cos t)\underbrace{\int_t^{t+2\pi}g(s)\sin s ds}+(\sin t)\underbrace{\int_t^{t+2\pi}g(s)\cos s ds}\\
&=-(\cos t)\int_0^{2\pi}g(s)\sin s ds+(\sin t)\int_0^{2\pi}g(s)\cos s ds
\end{aligned}$$

and so if the 2π*-periodic function* $g(s)$ *satisfies*

$$\int_0^{2\pi}g(s)\sin s ds=\int_0^{2\pi}g(s)\cos s ds=0, \tag{280}$$

we would have $y_p(t+2\pi)=y_p(t)$ *for all* $t\in(-\infty,\infty)$*. If we take* $g(t)=\cos t$ *((280) is not satisfied), then* $y_p(t)=\frac{1}{2}t\sin t$ *is a particular solution (with* $y_p(0)=y_p'(0)=0$*), but it is **not*** 2π*-periodic even that* $g(t)=\cos t$ *is* 2π*-periodic. In fact, one can see that* $y_p(t)$ ***in (278) is*** 2π***-periodic if and only if*** $g(t)$ ***is*** 2π***-periodic and satisfies (280),*** *for example, say* $g(t)=\cos 2t$*.*

3.11 Summary of solution methods.

Remark 3.75 *(**Useful.**) This is a summary for solving the nonhomogeneous equation:* $ay''+by'+cy=g(t)$*,* $t\in I$*, where* a*,* b*,* c *are constants with* $a\neq 0$ *and* $g(t)$ *is a continuous **nonzero** function on* I*. Here you can easily find two independent solutions* $y_1(t)$ *and* $y_2(t)$ *of* $ay''+by'+cy=0$*.*

1. *In case* $g(t)$ *is of the form* $P_n(t)e^{\lambda t}$*,* $P_n(t)e^{\alpha t}\cos\beta t$*,* $P_n(t)e^{\alpha t}\sin\beta t$*, where* $P_n(t)$ *is a polynomial with degree* n *and* λ*,* α*,* $\beta\in\mathbb{R}$ *with* $\beta>0$*, use the **method of undetermined coefficients** (the easiest way).*
2. *In case* $g(t)$ *is not of the form in* (1)*, you can use **decomposition method**, or **reduction method**, or **variation of parameters method**. **Variation of parameters method** seems to be the best one.*

Remark 3.76 *(**Useful.**) This is a summary for solving the nonhomogeneous equation:* $y''+p(t)y'+q(t)y=g(t)$*,* $t\in I$*, where* $p(t)$*,* $q(t)$*,* $g(t)$ *are a continuous functions on* I*. Here we are given one nonzero solution* $y_1(t)$ *of the homogeneous equation* $y''+p(t)y'+q(t)y=0$ *on* I*.*

1. *In case* $g(t)\equiv 0$ *on* I*, use **reduction method** or **Wronskian method**.*
2. *In case* $g(t)$ *is nonzero on* I*, use **reduction method.***
3. *In case we know two independent solutions* $y_1(t)$*,* $y_2(t)$ *(**fundamental set** of solutions) of* $y''+p(t)y'+q(t)y=0$ *on* I*, use **variation of parameters method** (to be explained below).*

Remark 3.77 *(**Useful.**) Finally, if an equation has the **Euler** form*

$$At^2y''(t)+Bty'(t)+Cy(t)=f(t),\quad t\in(0,\infty),\quad A\neq 0, \tag{281}$$

then use the change of variables $x=\log t$ *to convert it into the form*

$$A\tilde{y}''(x)+(B-A)\tilde{y}'(x)+C\tilde{y}(x)=F(x),\quad x\in(-\infty,\infty), \tag{282}$$

and solve $\tilde{y}_p(x)$ *using either* ***the method of undetermined coefficients*** *(if* $F(s)$ *has the form in Table 3.5.1 in p. 182) or* ***variation of parameters*** *(if* $F(s)$ *is arbitrary) to find* $\tilde{y}_p(x)$ *and then go back to* $y_p(t)$*. On the other hand, once we know* $y_1(t)$ *and* $y_2(t)$ *for* $At^2y''(t) + Bty'(t) + Cy(t) = 0$*, one can also use* ***variation of parameters*** *method directly to the original equation (281).*

END OF PART I, 2020-11-19