PDE Teaching for Spring, 2023

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1 Second order linear PDE with constant coefficients; classification and canonical form.

We now consider the linear second order equation with **constant coefficients**, given by

$$au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y + ku = f(x, y), \quad u = u(x, y) \tag{1}$$

where $a, \ldots, k$ are all **constants** with $a^2 + b^2 + c^2 > 0$ and $f(x, y)$ is a given function defined on some open set $\Omega \subseteq \mathbb{R}^2$. We want to find a C^2 function $u(x, y)$ satisfying (1) on some open set (may be just a subset of Ω). Note that for a C^2 function $u(x, y)$, we have $u_{xy}(x, y) = u_{yx}(x, y)$ on its domain.

Remark 1.1 *Note that if $u = u(x_1, \ldots, x_n)$ depends on n variables, the discussions below are similar. For convenience, we assume that $u = u(x, y)$ depends only on 2 variables.*

We can write (1) in the matrix form as

$$Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right] + 2\begin{pmatrix} d \\ e \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix} + ku = f(x, y), \tag{2}$$

which is helpful for us to understand the effect of the change of variables. As we shall see soon, the "**type**" of the equation (1) is determined only by the **sign** of the **determinant** of the coefficient matrix.

Remark 1.2 *We shall use the notation TrA to denote the trace of a square $n\times n$ matrix $A \in M(n)$. The basic properties of the trace operator are*

$$\begin{cases} Tr(c_1A + c_2B) = c_1Tr(A) + c_2Tr(B) \\ Tr(A) = Tr(A^T), \quad Tr(P^{-1}AP) = Tr(A), \quad Tr(AB) = Tr(BA), \end{cases}$$

where $A, B, P \in M(n)$, P is invertible, and $c_1, c_2 \in \mathbb{R}$. However, unlike $\det(AB) = \det(A)\det(B)$, we do not have $Tr(AB) = Tr(A)Tr(B)$.

Lemma 1.3 *Assume $u(x, y)$ is a C^2 function defined on some domain $\Omega \subseteq \mathbb{R}^2$. If we introduce the linear change of variables given by*

$$\xi = \xi(x, y) = Ax + By, \qquad \eta = \eta(x, y) = Cx + Dy, \qquad A,\ B,\ C,\ D \text{ are all const.,} \tag{3}$$

i.e.,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = J\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \det J \neq 0, \tag{4}$$

then the equation (2) for the function $U(\xi, \eta)$ (where $U(Ax + By, Cx + Dy) = u(x, y)$) becomes

$$Tr\left[\underbrace{J\begin{pmatrix} a & b \\ b & c \end{pmatrix}J^T}\cdot\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}\right] + (\text{lower order terms}) = F(\xi, \eta), \tag{5}$$

where $Tr(\cdot)$ is the trace of a matrix.

Remark 1.4 *Denote*

$$M=\begin{pmatrix} a & b \\ b & c \end{pmatrix},\quad N=J\begin{pmatrix} a & b \\ b & c \end{pmatrix}J^T,\quad \det J\neq 0. \tag{6}$$

We see that the matrix N is also ***symmetric****. By a theorem from linear algebra,* ***all eigenvalues of both M and N are real.*** *Moreover, the* ***sign*** *of* $\det M$ *and* $\det N$ *are the* ***same*** *due to* $\det N=(\det J)^2\det M$*, where* $\det J\neq 0$*. In particular, the* ***sign*** *of the* ***eigenvalues*** $\lambda_1,\ \lambda_2$ *are unchanged under the change of variables. Finally, if J is an* ***orthogonal*** *matrix (i.e. $J^T=J^{-1}$), then both M and N are* ***similar*** *and have the* ***same*** *eigenvalues. Our goal is to* ***diagonalize*** *M (i.e. make N to be* ***diagonal****), which will reduce equation (1) into* ***canonical form****.*

Proof. We have

$$U(Ax+By,Cx+Dy)=u(x,y),$$

and by the chain rule we have

$$u_x=AU_\xi+CU_\eta,\quad u_y=BU_\xi+DU_\eta,\quad \begin{pmatrix} u_x \\ u_y \end{pmatrix}=J^T\begin{pmatrix} U_\xi \\ U_\eta \end{pmatrix},\quad \underbrace{\nabla u=J^T\nabla U}, \tag{7}$$

which is equivalent to the operator identities:

$$\frac{\partial}{\partial x}=A\frac{\partial}{\partial \xi}+C\frac{\partial}{\partial \eta},\qquad \frac{\partial}{\partial y}=B\frac{\partial}{\partial \xi}+D\frac{\partial}{\partial \eta},$$

which can be written as

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix},\qquad J=\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{8}$$

One step furthermore, we get

$$\begin{cases} \frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)=\left(A\frac{\partial}{\partial \xi}+C\frac{\partial}{\partial \eta}\right)(\cdots)=A^2\frac{\partial^2}{\partial \xi^2}+2AC\frac{\partial^2}{\partial \xi\partial\eta}+C^2\frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)=\left(A\frac{\partial}{\partial \xi}+C\frac{\partial}{\partial \eta}\right)(\cdots)=AB\frac{\partial^2}{\partial \xi^2}+(AD+BC)\frac{\partial^2}{\partial \xi\partial\eta}+CD\frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)=\left(B\frac{\partial}{\partial \xi}+D\frac{\partial}{\partial \eta}\right)(\cdots)=B^2\frac{\partial^2}{\partial \xi^2}+2BD\frac{\partial^2}{\partial \xi\partial\eta}+D^2\frac{\partial^2}{\partial \eta^2}, \end{cases}$$

i.e. we have

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix},\qquad \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} \\ \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial^2}{\partial \xi^2} & \frac{\partial^2}{\partial \xi\partial\eta} \\ \frac{\partial^2}{\partial \xi\partial\eta} & \frac{\partial^2}{\partial \eta^2} \end{pmatrix}J, \tag{9}$$

which gives the **Hessain matrix relation**:

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 u}{\partial x\partial y} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial^2 U}{\partial w^2} & \frac{\partial^2 U}{\partial w\partial z} \\ \frac{\partial^2 U}{\partial w\partial z} & \frac{\partial^2 U}{\partial z^2} \end{pmatrix}J,\quad \underbrace{\nabla^2 u=J^T\left(\nabla^2 U\right)J} \tag{10}$$

where $\nabla^2 u$ denotes the Hessian matrix of u. Thus the equation

$$Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right]+(\text{lower order terms})=f(x,y)$$

becomes (note that for any two matrices $A,\ B$, we have the identity $Trace(AB)=Trace(BA)$ in linear algebra)

$$\begin{aligned} &Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\cdot \underline{J^T\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}J}\right]+(\text{lower order terms}) \\ &=Trace\left[\underline{J\begin{pmatrix} a & b \\ b & c \end{pmatrix}J^T}\cdot\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}\right]+(\text{lower order terms})=F(\xi,\eta) \end{aligned} \tag{11}$$

The proof is done. □

Definition 1.5 *Since the matrix*

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is ***symmetric****, it has two* ***real eigenvalues*** λ_1 *and* λ_2*. If both of them are positive (or both are negative), then we say the equation (1) is* ***elliptic*** *(this is equivalent to* $\det M = ac - b^2 > 0$*). If one eigenvalue is positive and the other is negative, we say the equation is* ***hyperbolic*** *(this is equivalent to* $\det M = ac - b^2 < 0$*). If one eigenvalue is zero and the other is nonzero, we say the equation is* ***parabolic*** *(this is equivalent to* $\det M = ac - b^2 = 0$*). Note that by*

$$\det \underbrace{\left(J \begin{pmatrix} a & b \\ b & c \end{pmatrix} J^T \right)} = (\det J)^2 \det \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \textit{where } \det J \neq 0,$$

we see that ***the type of the linear equation (2) is invariant under a linear change of variables.***

Since the matrix M in (6) is **symmetric,** by linear algebra theory, we can find **orthonormal** basis $\{v_1, v_2\}$ (they are **eigenvectors** corresponding to $\lambda_1,\ \lambda_2$) such that

$$P^T M P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad P^T = P^{-1}, \tag{12}$$

where P is the **orthogonal matrix** $P = (v_1, v_2)$ (v_1, v_2 are **column vectors** of P). Assume that $v_1 = (\alpha, \beta)$ and $v_2 = (p, q)$ and let

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ p & q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad P = \begin{pmatrix} \alpha & p \\ \beta & q \end{pmatrix}, \quad P^T = \begin{pmatrix} \alpha & \beta \\ p & q \end{pmatrix}, \tag{13}$$

i.e. we choose $J = P^T$. Then

$$\begin{aligned} Trace \left[\underbrace{J \begin{pmatrix} a & b \\ b & c \end{pmatrix} J^T} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] &= Trace \left[\underbrace{P^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} P} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] \\ = Trace \left[\underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] &= \underbrace{\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta}}, \end{aligned} \tag{14}$$

which will reduce the leading terms $au_{xx} + 2bu_{xy} + cu_{yy} + \cdots$ of the PDE (1) into **canonical form** !!

By (14), we can conclude the following **classification** result for equation (1):

Lemma 1.6 *(**Canonical form.**) If the linear equation (2) is* ***elliptic****, then one can find a suitable linear change of variables (**using eigenvalues and eigenvectors**)*

$$\xi = Ax + By, \quad \eta = Cx + Dy, \quad A,\ B,\ C,\ D \textit{ are constants},$$

so that the equation for $U(\xi, \eta)$ *has the form*

$$U_{\xi\xi} + U_{\eta\eta} + (\textit{lower order terms}) = F(\xi, \eta). \tag{15}$$

For ***hyperbolic*** *case, the equation has the form*

$$U_{\xi\xi} - U_{\eta\eta} + (\textit{lower order terms}) = F(\xi, \eta), \tag{16}$$

and for ***parabolic*** *case, the equation has the form*

$$U_{\xi\xi} + (\textit{lower order terms}) = F(\xi, \eta). \tag{17}$$

Remark 1.7 *The forms in (15), (16) and (17) are said to be in* ***canonical forms****. Another canonical form of the* ***hyperbolic*** *case is*

$$U_{\xi\eta} + (\textit{lower order terms}) = F(\xi, \eta). \tag{18}$$

One can show that an equation of the form $u_{xx} - u_{yy} = 0$ *can be converted into an equation of the form* $4U_{\xi\eta} = 0$ *(by the change of variables* $\xi = x + y,\ \eta = x - y$*). Therefore, canonical form (16) and (18) are* ***equivalent****.*

Proof. For the ***elliptic*** case, by multiplying the equation by a minus sign if necessary, we may assume $\lambda_1 > 0,\ \lambda_2 > 0$ (both are eigenvalues of the coefficient matrix). By the change of variables (13), we can convert in into the form

$$\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}) = F(\xi, \eta), \quad \lambda_1 > 0,\ \lambda_2 > 0.$$

If we let

$$\tilde{\xi} = \frac{\xi}{\sqrt{\lambda_1}}, \quad \tilde{\eta} = \frac{\eta}{\sqrt{\lambda_2}}, \quad \tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = U\left(\sqrt{\lambda_1}\tilde{\xi}, \sqrt{\lambda_2}\tilde{\eta}\right), \tag{19}$$

then we have

$$\begin{aligned} &\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}), \quad \text{where} \quad \lambda_1 > 0, \quad \lambda_2 < 0 \\ &= \tilde{U}_{\tilde{\xi}\tilde{\xi}}\left(\tilde{\xi}, \tilde{\eta}\right) + \tilde{U}_{\tilde{\eta}\tilde{\eta}}\left(\tilde{\xi}, \tilde{\eta}\right) + (\text{lower order terms}) = \tilde{F}\left(\tilde{\xi}, \tilde{\eta}\right). \end{aligned}$$

Thus we have arrived at the form (15). For the **hyperbolic** case, we have $\lambda_1 > 0,\ \lambda_2 < 0$. Then we replace (19) by

$$\tilde{\xi} = \frac{\xi}{\sqrt{\lambda_1}}, \quad \tilde{\eta} = \frac{\eta}{\sqrt{-\lambda_2}}, \quad \tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = U\left(\sqrt{\lambda_1}\tilde{\xi}, \sqrt{-\lambda_2}\tilde{\eta}\right) \tag{20}$$

and get

$$\begin{aligned} &\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}), \quad \text{where} \quad \lambda_1 > 0, \quad \lambda_2 < 0 \\ &= \tilde{U}_{\tilde{\xi}\tilde{\xi}}\left(\tilde{\xi}, \tilde{\eta}\right) - \tilde{U}_{\tilde{\eta}\tilde{\eta}}\left(\tilde{\xi}, \tilde{\eta}\right) + (\text{lower order terms}) = \tilde{F}\left(\tilde{\xi}, \tilde{\eta}\right). \end{aligned}$$

For the **parabolic** case, by multiplying the equation by a minus sign if necessary, we may assume $\lambda_1 > 0,\ \lambda_2 = 0$. Then we replace (19) by

$$\tilde{\xi} = \frac{\xi}{\sqrt{\lambda_1}}, \quad \tilde{\eta} = \eta, \quad \tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = U\left(\sqrt{\lambda_1}\tilde{\xi}, \tilde{\eta}\right) \tag{21}$$

and get

$$\begin{aligned} &\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}), \quad \text{where} \quad \lambda_1 > 0, \quad \lambda_2 = 0 \\ &= \tilde{U}_{\tilde{\xi}\tilde{\xi}}\left(\tilde{\xi}, \tilde{\eta}\right) + (\text{lower order terms}) = \tilde{F}\left(\tilde{\xi}, \tilde{\eta}\right) \end{aligned}$$

The proof is done. □

Definition 1.8 *In case equation (1) is parabolic with canonical form*

$$U_{\xi\xi} + (\textit{lower order terms}) = F(\xi, \eta), \tag{22}$$

and there is ***no*** U_η *term in (lower order terms) of (22), we say the equation is* ***degenerate****. Otherwise, we say it is* ***nondegenerate.*** *A degenerate parabolic equation is just a second order* ***ODE*** *in* ξ *of the form (view* η *as a parameter):*

$$U_{\xi\xi} + aU_\xi + bU = F(\xi, \eta), \quad a,\ b \textit{ are constants.} \tag{23}$$

We ***will not*** *study a degenerate parabolic equation.* ***From now on, if we study a parabolic equation, we always assume that it is nondegenerate.***

1.1 Refined canonical form; getting rid of the first derivative terms.

One can go further to **get rid of the first derivative terms** in (15), (16) and (17) of Lemma 1.6. For simplicity, we can just look at two examples.

Example 1.9 *(**For elliptic and hyperbolic equations.**) Assume we have an **elliptic equation** in canonical form:*

$$U_{\xi\xi} + U_{\eta\eta} + 3U_\xi + 4U_\eta + 5U = F(\xi, \eta), \tag{24}$$

where we can write is as

$$\Big(\underbrace{U_{\xi\xi} + 3U_\xi}\Big) + \Big(\underbrace{U_{\eta\eta} + 4U_\eta}\Big) + 5U = F(\xi, \eta)$$

We let $v(\xi, \eta)$ be the new function given by

$$v(\xi, \eta) = e^{a\xi + b\eta} U(\xi, \eta)$$

for some constants a, b and we choose $a = 3/2$, $b = 4/2$ to get

$$v(\xi, \eta) = e^{\frac{3}{2}\xi + \frac{4}{2}\eta} U(\xi, \eta) \tag{25}$$

and compute

$$v_\xi = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\frac{3}{2}U + U_\xi\right), \quad v_\eta = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\frac{4}{2}U + U_\eta\right) \tag{26}$$

and

$$v_{\xi\xi} = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\frac{9}{4}U + \underbrace{3U_\xi + U_{\xi\xi}}\right), \quad v_{\eta\eta} = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(4U + \underbrace{4U_\eta + U_{\eta\eta}}\right). \tag{27}$$

Hence we obtain

$$v_{\xi\xi} + v_{\eta\eta} = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left[\Big(\underbrace{U_{\xi\xi} + 3U_\xi}\Big) + \Big(\underbrace{U_{\eta\eta} + 4U_\eta}\Big) + \left(\frac{9}{4} + 4\right)U\right]$$

and conclude

$$\begin{aligned} v_{\xi\xi} + v_{\eta\eta} &= e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\underbrace{U_{\xi\xi} + 3U_\xi + U_{\eta\eta} + 4U_\eta + 5U} + \overbrace{\left(\frac{9}{4} + 4\right)U - 5U}\right) \\ &= e^{\frac{3}{2}\xi + \frac{4}{2}\eta} F(\xi, \eta) + \frac{5}{4}v, \end{aligned}$$

i.e.

$$v_{\xi\xi} + v_{\eta\eta} - \frac{5}{4}v = \phi(\xi, \eta), \quad \textit{where} \ \ \phi(\xi, \eta) = e^{\frac{3}{2}\xi + \frac{4}{2}\eta} F(\xi, \eta). \tag{28}$$

*The new equation for v **has no first derivatives terms.** Note that, in general, one **cannot** choose **two** constants a and b to get rid of the **three** terms $3U_\xi + 4U_\eta + 5U$. Therefore, the term $-(5/4)v$ in (28) **cannot** be removed in general. The same method applies to the **hyperbolic equation**. We omit it.*

Example 1.10 *(**For nondegenerate parabolic equations.**) Assume we have the **nondegenerate parabolic equation** in canonical form:*

$$U_{\xi\xi} + 3U_\xi + 4U_\eta + 5U = F(\xi, \eta). \tag{29}$$

where we can write is as

$$\Big(\underbrace{U_{\xi\xi} + 3U_\xi}\Big) + \Big(\underbrace{4U_\eta + 5U}\Big) = F(\xi, \eta). \tag{30}$$

Now we let $v(\xi,\eta)$ be the new function given by

$$v(\xi,\eta) = e^{\frac{3}{2}\xi+\lambda\eta}U(\xi,\eta), \quad \lambda \text{ is a constant to be determined} \tag{31}$$

and compute

$$\begin{cases} v_\xi = e^{\frac{3}{2}\xi+\lambda\eta}\cdot\left(\frac{3}{2}U+U_\xi\right), \quad v_\eta = e^{\frac{3}{2}\xi+\lambda\eta}\cdot\left(\underbrace{\lambda U+U_\eta}\right), \\ v_{\xi\xi} = e^{\frac{3}{2}\xi+\lambda\eta}\left(\frac{9}{4}U+\underbrace{3U_\xi+U_{\xi\xi}}\right), \end{cases} \tag{32}$$

where, since there is no $U_{\eta\eta}$ term in the original equation, we ***do not have to*** *compute $v_{\eta\eta}$ (otherwise, we will get $U_{\eta\eta}$ and this does not make sense). Now, unlike the elliptic case in which we can compute $v_{\eta\eta}$ to produce the term U_η (see (27)), here to produce the term $4U_\eta$ in (30),* ***the only method*** *is to look at $4v_\eta$ and get*

$$4v_\eta = e^{\frac{3}{2}\xi+\lambda\eta}\cdot\left(\underbrace{4\lambda U+4U_\eta}\right). \tag{33}$$

Now we conclude

$$\begin{aligned} v_{\xi\xi}+4v_\eta &= e^{\frac{3}{2}\xi+\lambda\eta}\left(\frac{9}{4}U+\underbrace{3U_\xi+U_{\xi\xi}}+\underbrace{4\lambda U+4U_\eta}\right) \\ &= e^{\frac{3}{2}\xi+\lambda\eta}\left(U_{\xi\xi}+3U_\xi+4U_\eta+\underbrace{\left(\frac{9}{4}+4\lambda\right)U}\right). \end{aligned} \tag{34}$$

*By (**??**), if we choose $\lambda = 11/16$, we will have $(9/4+4\lambda)U = 5U$ and (34) becomes*

$$v_{\xi\xi}+4v_\eta = \phi(\xi,\eta), \quad \text{where} \quad v(\xi,\eta) = e^{\frac{3}{2}\xi+\frac{11}{16}\eta}U(\xi,\eta), \quad \phi(\xi,\eta) = e^{\frac{3}{2}\xi+\frac{11}{16}\eta}F(\xi,\eta). \tag{35}$$

As a comparison, we see that we have got rid of the terms $3U_\xi$ and $5U$ in (35). From the above computation, we also see that it is ***impossible*** *to reduce the* ***nondegenerate parabolic equation*** *(29) (the coefficient of U_η is not zero) into the form*

$$v_{\xi\xi}+cv = \phi(\xi,\eta) \tag{36}$$

for some constant c.

We summarize the above in the following lemma:

Lemma 1.11 *For elliptic or hyperbolic equation*

$$U_{\xi\xi} \pm U_{\eta\eta}+\underbrace{aU_\xi+bU_\eta}+cU = F(\xi,\eta),$$

where $a,\ b,\ c$ are all nonzero constants, in general we can at most get rid of the two terms $aU_\xi + bU_\eta$ only. For ***nondegenerate parabolic equation***

$$U_{\xi\xi}+\underbrace{aU_\xi}+bU_\eta+\underbrace{cU} = F(\xi,\eta),$$

where $a,\ b,\ c$ are all nonzero constants, in general we can at most get rid of the two terms $aU_\xi + cU$ only.

By the above two examples, we can improve Lemma 1.6 as:

Theorem 1.12 *(**Refined canonical form.**) If the linear equation (2) is **elliptic**, then one can find a suitable linear change of variables (**using eigenvalues, eigenvectors and scalings**) and **multiply the solution** by some suitable **exponential function** so that, eventually, the equation has the form*

$$v_{\xi\xi} + v_{\eta\eta} = cv + \phi(\xi, \eta), \quad v = v(\xi, \eta), \tag{37}$$

*for some constant $c \in (-\infty, \infty)$ and some function $\phi(\xi, \eta)$. If the equation (2) is **hyperbolic**, the equation has the form*

$$v_{\xi\xi} - v_{\eta\eta} = cv + \phi(\xi, \eta), \quad v = v(\xi, \eta), \tag{38}$$

*for some constant $c \in (-\infty, \infty)$ and some function $\phi(\xi, \eta)$. If the equation (2) is **parabolic** and **nondegenerate**, the equation has the form*

$$v_{\xi\xi} = cv_\eta + \phi(\xi, \eta), \quad v = v(\xi, \eta), \tag{39}$$

for some constant $c \in (-\infty, \infty)$, $c \neq 0$, and some function $\phi(\xi, \eta)$.

Proof. The proof is now obvious. We omit it. □

Remark 1.13 *(**Important.**) The constant c in the **elliptic case** can be $c > 0$ or $c = 0$ or $c < 0$. For $c > 0$, we can make it equal to 1 by doing the change of variables*

$$\tilde{\xi} = \sqrt{c}\xi, \quad \tilde{\eta} = \sqrt{c}\eta, \quad \tilde{v}\left(\tilde{\xi}, \tilde{\eta}\right) = v\left(\frac{\tilde{\xi}}{\sqrt{c}}, \frac{\tilde{\eta}}{\sqrt{c}}\right)$$

and for $c < 0$, we can make it equal to -1 by doing the change of variables

$$\tilde{\xi} = \sqrt{-c}\xi, \quad \tilde{\eta} = \sqrt{-c}\eta, \quad \tilde{v}\left(\tilde{\xi}, \tilde{\eta}\right) = v\left(\frac{\tilde{\xi}}{\sqrt{-c}}, \frac{\tilde{\eta}}{\sqrt{-c}}\right).$$

*Thus in the **elliptic case**, we may simply assume $c = 1$ or 0 or -1. The same for the **hyperbolic case.** Finally, for the **parabolic case**, the constant c can be $c > 0$ or $c < 0$. So eventually we can simply assume $c = 1$ or -1. However, since most parabolic equations come from physical phenomenon involving the behavior of some quantity $v(\xi, \eta)$ depending on space and time. So ξ will represent **space variable** (we rewrite it as x) and η will represent **time variable** (we rewrite it as t). In that case a **nondegenerate parabolic equation** in its **refined canonical form** looks like (assume $\phi(\xi, \eta) = 0$ for simplicity)*

$$(1).\ v_t = v_{xx} \quad \text{or} \quad (2).\ v_t = -v_{xx}, \tag{40}$$

*where, physically, the quantity v_{xx} describes the process due to **diffusion** (say, from high temperature to low temperature, or from high concentration to low concentration, ... etc). We call (1) the **"forward heat equation" (or just heat equation)** and (2) the **"backward heat equation".** Since in reality, time cannot go backwards, so in a parabolic equation, we always fucus on the behavior of a solution $v(x, t)$ **as time goes forwards**, i.e., as t is **increasing**. One can use simple examples to see that, **as time goes forwards**, the heat equation (1) will **make solution better**, while the backward heat equation (2) will **make solution worse** (look at $e^{-t}\sin x$ and $e^t \sin x$ respectively). Thus, as time goes forwards, equation (1) is **well-posed**, while (2) is **ill-posed**. Hence, we will focus only on (1).*

Finally, by the above remark, we conclude the following **final canonical form**:

Theorem 1.14 *(**Final canonical form.**) If the linear equation (2) is **elliptic**, then one can find a suitable linear change of variables (**using eigenvalues, eigenvectors and scalings**) and multiply the solution by some suitable **exponential function** so that, eventually, the equation has the form*

$$v_{\xi\xi}+v_{\eta\eta}=\begin{cases} v+\phi(\xi,\eta), \\ \phi(\xi,\eta), \\ -v+\phi(\xi,\eta), \end{cases} \tag{41}$$

*where $v=v(\xi,\eta)$. If the equation (2) is **hyperbolic**, the equation has the form*

$$v_{\xi\xi}-v_{\eta\eta}=\begin{cases} v+\phi(\xi,\eta), \\ \phi(\xi,\eta), \\ -v+\phi(\xi,\eta), \end{cases} \tag{42}$$

*where $v=v(\xi,\eta)$. If the equation (2) is **parabolic, nondegenerate** and **forward**, the equation has the form*

$$v_{\xi\xi}=v_{\eta}+\phi(\xi,\eta), \quad v=v(\xi,\eta). \tag{43}$$

Proof. The proof is now obvious. We omit it. □

Definition 1.15 *Let $v=v(\xi,\eta)$. The equations $v_{\xi\xi}+v_{\eta\eta}=0$, $v_{\xi\xi}-v_{\eta\eta}=0$ (view η as time), $v_{\eta}=v_{\xi\xi}$ (view η as time), are called **Laplace equation (elliptic equation), wave equation (hyperbolic equation), and heat equation (nondegenerate forward parabolic equation),** respectively.*

Remark 1.16 *In this elementary course we will focus only on **Laplace equation, wave equation** and **heat equation**, or focus only on equation (44) below.*

2 General solutions of hyperbolic equations without lower order terms.

In this section, we look at equations of the following form with **no** lower order terms and $f(x,y)\equiv 0$, i.e.

$$au_{xx}+2bu_{xy}+cu_{yy}=0, \quad u=u(x,y), \quad a,\ b,\ c \text{ are const.}, \tag{44}$$

where $a,\ b,\ c$ are constants with $a^2+b^2+c^2>0$. Note that (44) can be written as

$$Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right]=0, \quad \det\begin{pmatrix} a & b \\ b & c \end{pmatrix}=ac-b^2.$$

We want to find the general C^2 solution $u=u(x,y)$ of (44) defined on $\mathbb{R}^2$.

The **canonical form** of (44) is given by

$$\begin{cases} U_{\xi\xi}+U_{\eta\eta}=0 \quad (\det M>0,\ \text{elliptic}) \\ U_{\xi\xi}-U_{\eta\eta}=0 \quad \text{or} \quad U_{\xi\eta}=0 \quad (\det M<0,\ \text{hyperbolic}) \\ U_{\xi\xi}=0 \quad (\det M=0,\ \text{parabolic but } \textbf{degenerate}), \end{cases} \tag{45}$$

where M is the coefficient matrix of (44). The result is that, **for hyperbolic and parabolic cases** in (45), **we can solve them easily (but not for elliptic equation)**. The method is either by a **change of variables** (**diagonalization method**) or by a **factorization method**.

Lemma 2.1 *Let $A,\ B,\ C,\ D$ be constants with $AD-BC\neq 0$. Consider the first order equation*

$$Au_x+Bu_y=g\left(Dx-Cy\right),\quad u=u\left(x,y\right) \tag{46}$$

where $g\left(\cdot\right)$ is a given continuous function defined on $\mathbb{R}$. Then the ***general solution*** *of (46) is given by*

$$u\left(x,y\right)=F\left(Bx-Ay\right)+G\left(Dx-Cy\right), \tag{47}$$

where $F\left(\cdot\right)$ is an arbitrary C^1 function defined on $\mathbb{R}$ and the C^1 function $G\left(\theta\right)$ satisfies

$$G'\left(\theta\right)=\frac{g\left(\theta\right)}{AD-BC},\quad \forall\ \theta\in\left(-\infty,\infty\right). \tag{48}$$

In particular, if the function $g\left(\cdot\right)$ on $\mathbb{R}$ is ***arbitrary****, then the function $G\left(\cdot\right)$ on $\mathbb{R}$ is also* ***arbitrary****.*

Remark 2.2 *The condition $AD-BC\neq 0$ is necessary. The case $AD-BC=0$ will be discussed later on. See (77).*

Remark 2.3 *Also note that $AD-BC\neq 0$ implies that the two families of lines $Bx-Ay=\lambda,\ Dx-Cy=\eta$ are* ***not*** *parallel. As a consequence of this, the two terms $F\left(Bx-Ay\right),\ G\left(Dx-Cy\right)$ in (55) are essentially different.*

Proof. We do the linear change of variables

$$w=Bx-Ay,\qquad z=Dx-Cy,\quad \text{Jacobian is }\begin{vmatrix} B & -A\\ D & -C\end{vmatrix}=AD-BC\neq 0, \tag{49}$$

which is a global linear change of variables from xy-space to wz-space. Now the function $u\left(x,y\right)$ becomes $U\left(w,z\right)$ and we have

$$Au_x+Bu_y=A\left[U_wB+U_zD\right]+B\left[U_w\left(-A\right)+U_z\left(-C\right)\right]=\left(AD-BC\right)U_z=g\left(z\right),$$

which gives

$$U_z=\frac{g\left(z\right)}{AD-BC},\quad AD-BC\neq 0$$

and so

$$U\left(w,z\right)=F\left(w\right)+G\left(z\right)=F\left(Bx-Ay\right)+G\left(Dx-Cy\right),$$

where $G'\left(z\right)=\frac{g(z)}{AD-BC}$. The proof is done. $\square$

Remark 2.4 *In the above change of variables (49), we prefer not choose $z=y$ (for $B\neq 0$) or $z=x$ (for $A\neq 0$). If you choose $z=y$ or $z=x$, the method is still correct but the computation will be more involved.*

2.1 Solving hyperbolic equations; factorization method.

In this section, we will use factorization method to solve a hyperbolic equation

$$au_{xx}+2bu_{xy}+cu_{yy}=0,\quad u=u\left(x,y\right),\quad a,\ b,\ c\text{ are const.},$$

where $a,\ b,\ c$ are constants with $a^2+b^2+c^2>0$ and $ac-b^2<0$, which means the determinant of the coefficient matrix is negative. **The idea is to decompose a second order PDE into two first order PDEs.**

Lemma 2.5 *Let $a,\ b,\ c$ be three given constants with $ac-b^2<0$ (same as $b^2-ac>0$). Then one can find constants $A,\ B,\ C,\ D$ satisfying*

$$AC=a, \qquad AD+BC=2b, \qquad BD=c. \tag{50}$$

In particular, we have

$$(AD-BC)^2=4\left(b^2-ac\right)>0, \quad AD-BC\neq 0. \tag{51}$$

This means that, if we have $ac-b^2<0$ (same as $b^2-ac>0$, we can factor the second order homogeneous polynomial $ax^2+2bxy+cy^2$ as

$$ax^2+2bxy+cy^2=(Ax+By)(Cx+Dy), \quad AD-BC\neq 0, \tag{52}$$

where the two lines $Ax+By=0,\ Cx+Dy=0$ on $\mathbb{R}^2$ are not parallel (same as $Bx-Ay=0,\ Dx-Cy=0$ are ***not*** *parallel).*

Remark 2.6 *If $ac>b^2$, then (50) and (52) cannot be satisfied (check it yourself). Therefore, the method in this section cannot be used to elliptic equations.*

Remark 2.7 *If $a=1$, we can choose $A=C=1$, and then solve*

$$B+D=2b, \qquad BD=c$$

to get

$$B=b\pm\sqrt{b^2-c}, \quad D=\frac{c}{b\pm\sqrt{b^2-c}}, \quad \text{for} \quad c\neq 0$$

and

$$B=2b, \quad D=0, \quad \text{for} \quad c=0.$$

Remark 2.8 *If $a>0,\ b=0,\ c<0$, we can choose $A=C=\sqrt{a}$ and $B=\sqrt{-c},\ D=-\sqrt{-c}$.*

Proof. (**Read it yourself**. **We omit it.**) If $a=0$, then by $ac=0<b^2$, we must have $b\neq 0$. The numbers

$$A=1, \quad B=\frac{c}{2b}, \quad C=0, \quad D=2b,$$

satisfy (50). If $c=0$, then we still have $b\neq 0$. The numbers

$$A=\frac{a}{2b}, \quad B=1, \quad C=2b, \quad D=0,$$

satisfy (50). If $ac\neq 0$, then $b+\sqrt{b^2-ac}\neq 0$ and $b-\sqrt{b^2-ac}\neq 0$. The numbers

$$A=1, \quad B=\frac{b+\sqrt{b^2-ac}}{a}, \quad C=a, \quad D=\frac{ac}{b+\sqrt{b^2-ac}}$$

satisfy (50). Finally, in each case we can see that $AD-BC\neq 0$. □

We now focus on the **hyperbolic** ($ac-b^2<0$) case in (44). That is, the eigenvalues $\lambda_1,\ \lambda_2$ of the coefficient matrix have different sign and we may assume $\lambda_1>0,\ \lambda_2<0$.

Lemma 2.9 ***(Factorization method for hyperbolic equation.)*** *Assume (44) is hyperbolic, i.e., $ac<b^2$. Then one can decompose it as*

$$au_{xx}+2bu_{xy}+cu_{yy}=\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)\left[\left(C\frac{\partial}{\partial x}+D\frac{\partial}{\partial y}\right)u\right]=0 \tag{53}$$

for some constants $A,\ B,\ C,\ D$ satisfying

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0. \tag{54}$$

In particular, the ***general solution*** *of (44) is given by*

$$u(x,y) = F(Bx - Ay) + G(Dx - Cy), \quad (x,y) \in \mathbb{R}^2 \tag{55}$$

for arbitrary C^2 ***functions*** *$F(\cdot),\ G(\cdot)$ defined on $\mathbb{R}$.*

Proof. For a C^2 function u, we have

$$\begin{aligned}&\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u\right]\\ &= A(Cu_x + Du_y)_x + B(Cu_x + Du_y)_y = ACu_{xx} + (AD + BC)u_{xy} + BDu_{yy}.\end{aligned}$$

Now by Lemma 2.5, there are numbers $A,\ B,\ C,\ D$ satisfying

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0. \tag{56a}$$

Hence, for a C^2 function u, it satisfies $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ if and only if it satisfies

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u\right] = 0, \tag{57}$$

where $A,\ B,\ C,\ D$ satisfy (56a). We can find solutions of (57) by solving two first order PDE. Let

$$v = \left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u.$$

It satisfies $\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)v = 0$. Hence $v(x,y) = f(Bx - Ay)$ for **arbitrary** C^1 function f and the equation for u becomes

$$Cu_x + Du_y = f(Bx - Ay), \quad f \in C^1.$$

By Lemma 2.1, the general solution for $u(x,y)$ is

$$u(x,y) = G(Dx - Cy) + F(Bx - Ay), \quad (x,y) \in \mathbb{R}^2,$$

where $F,\ G$ are two arbitrary C^2 functions defined on $(-\infty, \infty)$ (since we want $u(x,y)$ to be a C^2 solution, we must require $F,\ G$ to be C^2 functions). The proof is done. □

Remark 2.10 ***(Important.)*** *Lemma 2.9 says that to solve the second order hyperbolic equation, it suffices to solve* ***two first order equations.***

Definition 2.11 *We call the* ***2-parameter family of lines***

$$Bx - Ay = \lambda, \quad Dx - Cy = \eta, \quad AD - BC \neq 0$$

where $\lambda,\ \eta$ are arbitrary constants, the ***characteristic lines*** *of the hyperbolic equation (53).*

Example 2.12 *Consider the second order linear equation in two variables:*

$$u_{xx} - 4u_{xy} - 2u_{yy} = 0, \quad (x,y) \in \mathbb{R}^2, \quad u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}.$$

What is the type (elliptic, hyperbolic, or parabolic) of this equation ? Use ***factorization method*** *to find the general solution of the equation.*

Solution:

The equation has the form $au_{xx} + 2bu_{xy} + cu_{yy} = 0$, where $a = 1$, $b = -2$, $c = -2$ and $ac - b^2 = -6 < 0$. Therefore the equation is **hyperbolic**. We know that one can decompose it into two **first order equations** of the form (by Remark 2.7, we can choose $A = C = 1$)

$$u_{xx} - 4u_{xy} - 2u_{yy} = (\partial_x + B\partial_y)(\partial_x + D\partial_y)u = u_{xx} + (B + D)u_{xy} + BDu_{yy}.$$

Thus we solve $B,\ D$ to satisfy the equation (note that now we have $A = C = 1$)

$$B + D = -4, \quad BD = -2.$$

We obtain $(B, D) = \left(-2 + \sqrt{6}, -2 - \sqrt{6}\right)$ or $(B, D) = \left(-2 - \sqrt{6}, -2 + \sqrt{6}\right)$. Thus we choose $(B, D) = \left(-2 + \sqrt{6}, -2 - \sqrt{6}\right)$ and get

$$u_{xx} - 4u_{xy} - 2u_{yy} = \left[\partial_x + \left(-2 + \sqrt{6}\right)\partial_y\right]\left[\partial_x + \left(-2 - \sqrt{6}\right)\partial_y\right]u \tag{58}$$

and the general solution is

$$\begin{aligned} u(x, y) &= F(Bx - Ay) + G(Dx - Cy) = F\left(\left(-2 + \sqrt{6}\right)x - y\right) + G\left(\left(-2 - \sqrt{6}\right)x - y\right) \\ &= F\left(\left(\sqrt{6} - 2\right)x - y\right) + G\left(\left(\sqrt{6} + 2\right)x + y\right), \end{aligned} \tag{59}$$

where $F(z) : \mathbb{R} \to \mathbb{R}$ and $G(z) : \mathbb{R} \to \mathbb{R}$ are two arbitrary C^2 functions. □

Remark 2.13 *In terms of polynomial, the decomposition (58) is the same as*

$$x^2 - 4xy - 2y^2 = \left(x + \left(-2 + \sqrt{6}\right)y\right)\left(x + \left(-2 - \sqrt{6}\right)y\right). \tag{60}$$

2.2 Solving hyperbolic equations; change of variables method.

Remark 2.14 *By (63) below, we see that this method is essentially the same as the factorization method. Computationally, you can just use the factorization method. Note that solving the equation $U_{\xi\eta} = 0$ is the same as solving **two first order equations**. See equation (57) also.*

We can also use a change of variables method to solve the equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0, \quad \text{where} \quad ac < b^2. \tag{61}$$

Introduce the change of variables

$$\xi = Bx - Ay, \quad \eta = Dx - Cy, \quad \text{Jacobian} = \begin{vmatrix} B & -A \\ D & -C \end{vmatrix} = AD - BC, \tag{62}$$

where $A,\ B,\ C,\ D$ are constants satisfying

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0, \tag{63}$$

where we note that the existence of $A,\ B,\ C,\ D$ satisfying (63) is guaranteed by Lemma 2.5. Since $AD - BC \neq 0$, this change of variables is good on all $\mathbb{R}^2$. Let $U(\xi, \eta)$ be the function $u(x, y)$ in (ξ, η) variables. We have:

Lemma 2.15 *(**Change of variables method for hyperbolic equation.**) Under the change of variables (62), (63), the equation for $U(\xi,\eta)$ is given by*

$$-(AD-BC)^2 U_{\xi\eta}=0 \quad (\textit{same as } U_{\xi\eta}=0 \textit{ since } AD-BC\neq 0), \tag{64}$$

which has general solution given by $U(\xi,\eta)=F(\xi)+G(\eta)$ for arbitrary C^2 functions (we want $U(\xi,\eta)$ to be a C^2 function) $F(\xi)$ and $G(\eta)$ defined on $\mathbb{R}$. As a consequence, the general solution $u(x,y)$ of (61) is

$$u(x,y)=F(Bx-Ay)+G(Dx-Cy), \quad (x,y)\in\mathbb{R}^2,$$

for arbitrary C^2 functions $F(\xi)$ and $G(\eta)$ defined on $\mathbb{R}$.

Proof. We have

$$\begin{cases} u_x=BU_\xi+DU_\eta, \qquad u_y=-AU_\xi-CU_\eta, \\ u_{xx}=B^2U_{\xi\xi}+2BDU_{\xi\eta}+D^2U_{\eta\eta}, \\ u_{xy}=-ABU_{\xi\xi}-(BC+AD)U_{\xi\eta}-CDU_{\eta\eta}, \\ u_{yy}=A^2U_{\xi\xi}+2ACU_{\xi\eta}+C^2U_{\eta\eta} \end{cases}$$

and so

$$\begin{aligned} &au_{xx}+2bu_{xy}+cu_{yy} \\ &=\begin{cases} a\left[B^2U_{\xi\xi}+2BDU_{\xi\eta}+D^2U_{\eta\eta}\right] \\ +2b\left[-ABU_{\xi\xi}-(BC+AD)U_{\xi\eta}-CDU_{\eta\eta}\right]+c\left[A^2U_{\xi\xi}+2ACU_{\xi\eta}+C^2U_{\eta\eta}\right] \end{cases} \\ &=\begin{cases} \left(\underline{aB^2-2bAB+cA^2}\right)U_{\xi\xi} \\ +\left(\underbrace{a2BD-2bBC-2bAD+c2AC}\right)U_{\xi\eta}+\left(\overbrace{aD^2-2bCD+cC^2}\right)U_{\eta\eta}. \end{cases} \end{aligned}$$

By

$$AC=a, \qquad AD+BC=2b, \qquad BD=c, \qquad AD-BC\neq 0,$$

we have

$$\begin{cases} \underline{aB^2-2bAB+cA^2}=ACB^2-(AD+BC)AB+BDA^2=0 \\ \overbrace{aD^2-2bCD+cC^2}=ACD^2-(AD+BC)CD+BDC^2=0 \end{cases}$$

and

$$\begin{aligned} &\underbrace{a2BD-2bBC-2bAD+c2AC} \\ &=2ACBD-(AD+BC)BC-(AD+BC)AD+BD2AC \\ &=2ABCD-B^2C^2-A^2D^2=-(AD-BC)^2\neq 0. \end{aligned}$$

Hence the equation for $U(\xi,\eta)$ is

$$-(AD-BC)^2U_{\xi\eta}=0 \quad (\text{same as } U_{\xi\eta}=0).$$

Its general solution is $U(\xi,\eta)=F(\xi)+G(\eta)$ for arbitrary C^2 functions $F(\xi)$ and $G(\eta)$ defined on $\mathbb{R}$. □

Example 2.16 *Consider the second order linear equation in two variables:*

$$u_{xx}-4u_{xy}-2u_{yy}=0, \quad (x,y)\in\mathbb{R}^2, \quad u=u(x,y):\mathbb{R}^2\to\mathbb{R}.$$

*Use **change of variables method** to reduce it to canonical form ($U_{\xi\xi}-U_{\eta\eta}=0$ or $U_{\xi\eta}=0$) and then solve it.*

Solution:

Recall that the numbers $A,\ B,\ C,\ D$ satisfying

$$AC = a = 1, \qquad AD + BC = 2b = -4, \qquad BD = c = -2, \tag{65}$$

are given by $A = C = 1,\ B = -2 + \sqrt{6},\ D = -2 - \sqrt{6}$. By Lemma 2.15, if we do the change of variables

$$\begin{cases} \xi = Bx - Ay = \left(-2 + \sqrt{6}\right)x - y \\ \eta = Dx - Cy = \left(-2 - \sqrt{6}\right)x - y, \end{cases} \tag{66}$$

the new for $U(\xi, \eta)$ is given by $U_{\xi\eta}(\xi, \eta) = 0$. The general solution for $U(\xi, \eta)$ is $U(\xi, \eta) = F(\xi) + G(\eta)$. Hence the general solution for $u(x, y)$ is

$$\begin{aligned} u(x, y) &= F(Bx - Ay) + G(Dx - Cy) = F\left(\left(-2 + \sqrt{6}\right)x - y\right) + G\left(\left(-2 - \sqrt{6}\right)x - y\right) \\ &= F\left(\left(\sqrt{6} - 2\right)x - y\right) + G\left(\left(\sqrt{6} + 2\right)x + y\right), \end{aligned} \tag{67}$$

where $F(z) : \mathbb{R} \to \mathbb{R}$ and $G(z) : \mathbb{R} \to \mathbb{R}$ are two arbitrary C^2 functions. □

2.3 Solving hyperbolic equations; diagonalization method (eigenvalue-eigenvector method).

Remark 2.17 *Interesting question:* ***Can you find the relation between this method and the factorization method ?***

Example 2.18 *Consider the second order linear equation in two variables:*

$$u_{xx} - 4u_{xy} - 2u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad u = u(x, y) : \mathbb{R}^2 \to \mathbb{R}.$$

Use ***diagonalization method*** *to reduce it to canonical form* ($U_{\xi\xi} - U_{\eta\eta} = 0$) *and then solve it.*

Solution:

One can write the equation as

$$Trace\left[\begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right] = 0.$$

The eigenvalues of the coefficient matrix are $\lambda_1 = 2,\ \lambda_2 = -3$ with corresponding **orthonormal** eigenvectors

$$v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{68}$$

By (13) and (14), we introduce the change of variables (recall that the change of variables matrix J is chosen as $J = P^T$, see (13))

$$\xi = -\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y, \quad \eta = \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y.$$

Then, in terms of the variables (ξ, η), we have

$$2U_{\xi\xi} - 3U_{\eta\eta} = 0.$$

Finally, we let $\tilde{\xi} = \frac{1}{\sqrt{2}}\xi,\ \tilde{\eta} = \frac{1}{\sqrt{3}}\eta$, we have the final **canonical form:**

$$\tilde{U}_{\tilde{\xi}\tilde{\xi}} - \tilde{U}_{\tilde{\eta}\tilde{\eta}} = 0 \text{ (which can be decomposed as } \left(\frac{\partial}{\partial\tilde{\xi}} + \frac{\partial}{\partial\tilde{\eta}}\right)\left[\left(\frac{\partial}{\partial\tilde{\xi}} - \frac{\partial}{\partial\tilde{\eta}}\right)\tilde{U}\right] = 0 \text{)}$$

and its general solution is

$$\tilde{U}\left(\tilde{\xi},\tilde{\eta}\right) = F\left(\tilde{\xi} - \tilde{\eta}\right) + G\left(-\tilde{\xi} - \tilde{\eta}\right) \text{ (same as } G\left(\tilde{\xi} + \tilde{\eta}\right) \text{)},$$

where $F(z)$ and $G(z)$ are two arbitrary C^2 functions. Hence the general solution for $u(x,y)$ is

$$u(x,y) = \begin{cases} F\left(\frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)\right) \\ +G\left(\frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)\right) \end{cases} \tag{69}$$

$$= \begin{cases} F\underbrace{\left(\left(-\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{15}}\right)y\right)} \\ +G\underline{\left(\left(-\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{15}}\right)y\right)}. \end{cases}$$

By the identity

$$\frac{-\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{15}}}{\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{15}}} = \frac{2\sqrt{15} + \sqrt{10}}{-\sqrt{15} + 2\sqrt{10}} = \sqrt{6} + 2$$

we can write $F\underbrace{\left(\left(-\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{15}}\right)y\right)}$ as $F\underbrace{\left(\left(\sqrt{6}+2\right)x + y\right)}$ and by the identity

$$\frac{-\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{15}}}{\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{15}}} = \frac{-2\sqrt{15} + \sqrt{10}}{\sqrt{15} + 2\sqrt{10}} = -\sqrt{6} + 2$$

we can write $G\underline{\left(\left(-\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{15}}\right)y\right)}$ as $G\underbrace{\left(\left(-\sqrt{6}+2\right)x + y\right)}$. Thus the general solution can also be expressed as

$$u(x,y) = F\left(\left(\sqrt{6}+2\right)x + y\right) + G\left(\left(-\sqrt{6}+2\right)x + y\right), \tag{70}$$

which is the same as (59) and (67). □

3 General solutions of parabolic equations without lower order terms.

3.1 Solving parabolic equations; factorization method.

We now come to the **parabolic** case for the equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0, \quad \text{where} \quad ac = b^2. \tag{71}$$

Here we may assume $a,\ b,\ c$ are **all nonzero** (otherwise we will get into a trivial case). By the identity $ac = b^2 > 0$, we know that $a,\ c$ have the same sign. By multiplying the equation by a minus sign if necessary, we may assume that $a > 0,\ c > 0$. However, b can be either $b > 0$ or $b < 0$.

As there is no lower order terms in (71), the parabolic equation is **degenerate**. Hence it is **essentially an ODE**. We have the following:

Lemma 3.1 ***(Factorization method for parabolic equation.)*** *Assume the equation (71) is parabolic, i.e.* $ac = b^2$, *with* $a > 0,\ b \neq 0,\ c > 0$ *(if* $b = 0$*, then we are in the trivial case; hence we assume* $b \neq 0$ *and multiply the equation by* -1 *if necessary, we can assume* $a > 0$ *and* $c > 0$*). Then one can decompose it as*

$$au_{xx} + 2bu_{xy} + cu_{yy} = \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left[\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)u\right] = 0 \tag{72}$$

for some constants $A > 0,\ B \neq 0$. *More precisely, we have*

$$\begin{cases} A = \sqrt{a}, & B = \sqrt{c}, \quad \text{if} \quad b > 0 \\ A = \sqrt{a}, & B = -\sqrt{c}, \quad \text{if} \quad b < 0. \end{cases} \tag{73}$$

The ***general solution*** *of (71) can be expressed as either one of the following two forms*

$$(1).\ \ u(x,y) = F(Bx - Ay) + \frac{x}{A}G(Bx - Ay) \tag{74}$$

or

$$(2).\ \ u(x,y) = F(Bx - Ay) + \frac{y}{B}G(Bx - Ay) \tag{75}$$

for arbitrary C^2 *functions* $F(z),\ G(z)$ *defined on* $\mathbb{R}$.

Remark 3.2 *If* $a = 0$ *(then* $b = 0,\ c \neq 0$*) or* $c = 0$ *(then* $b = 0,\ a \neq 0$*), then we are in a trivial case. The equation now has the form* $cu_{yy} = 0$ *or* $au_{xx} = 0$. *We have not much to discuss.*

Remark 3.3 *See Remark 3.6 below for a third form of the solution.*

Remark 3.4 *The two forms in (74) and (75) are the same due to the identity*

$$\begin{aligned} &\frac{x}{A}G(Bx - Ay) \\ &= \frac{(Bx - Ay) + Ay}{AB}G(Bx - Ay) \\ &= \frac{(Bx - Ay)}{AB}G(Bx - Ay)\ \textit{(absorb this term into } F(Bx - Ay)\,\textit{)} + \frac{y}{B}G(Bx - Ay). \end{aligned} \tag{76}$$

Proof. For a C^2 function u, (72) is the same as

$$A(Au_x + Bu_y)_x + B(Au_x + Bu_y)_y = A^2u_{xx} + 2ABu_{xy} + B^2u_{yy} = 0.$$

Hence we need to solve

$$A^2 = a, \quad AB = b, \quad B^2 = c,$$

which is **solvable** due to $ac - b^2 = 0$. If $a > 0,\ b > 0,\ c > 0$, then we can choose $A = \sqrt{a},\ B = \sqrt{c}$. If $a > 0,\ b < 0,\ c > 0$, then we can choose $A = \sqrt{a},\ B = -\sqrt{c}$.

Let $w = \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)u$. Then by $\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)w = 0$, we see that $w = G(Bx - Ay)$ for some arbitrary C^1 function $G(z)$ defined on $\mathbb{R}$. Next we solve

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)u(x,y) = G(Bx - Ay). \tag{77}$$

Since $A > 0$ and $B \neq 0$, we can do the change of variables $w = Bx - Ay,\ z = x$, to get

$$Au_x + Bu_y = A[U_wB + U_z] + BU_w(-A) = \underbrace{AU_z = G(w)}$$

and obtain the general solution for $U(w,z)$:

$$U(w,z) = F(w) + \frac{z}{A} G(w),$$

which gives

$$u(x,y) = F(Bx - Ay) + \frac{x}{A} G(Bx - Ay), \quad \text{if } A \neq 0,$$

where now $F(z)$, $G(z)$ are two arbitrary C^2 functions defined on $\mathbb{R}$. This is the form (74).

Similarly, we can also do the change of variables $w = Bx - Ay,\ z = y$, to get

$$Au_x + Bu_y = A(U_w B) + B[U_w(-A) + U_z] = \underbrace{BU_z = G(w)}$$

which gives the general solution of the form (75). □

Remark 3.5 *Note that, in solving (77), Lemma 2.1 is* ***not*** *applicable here.*

Remark 3.6 *(**Important.**) We can also do the change of variables $w = Bx - Ay,\ z = Ax + By$ in (77) and get*

$$Au_x + Bu_y = A[U_w B + U_z A] + B[U_w(-A) + U_z B] = \underbrace{(A^2 + B^2) U_z = G(w)}.$$

We now have

$$U(w,z) = F(w) + \frac{z}{A^2 + B^2} G(w), \tag{78}$$

which gives the ***symmetric*** *form*

$$u(x,y) = F(Bx - Ay) + \frac{Ax + By}{A^2 + B^2} G(Bx - Ay). \tag{79}$$

The ***three forms*** *(74), (75) and (79) are all* ***equivalent*** *due to the following identities:*

$$\frac{Ax + By}{A^2 + B^2} = \frac{1}{A^2 + B^2}\left[-\frac{B(Bx - Ay)}{A} + \frac{(A^2 + B^2)x}{A}\right] \tag{80}$$

and

$$\frac{Ax + By}{A^2 + B^2} = \frac{1}{A^2 + B^2}\left[\frac{A(Bx - Ay)}{B} + \frac{(A^2 + B^2)y}{B}\right]. \tag{81}$$

Definition 3.7 *Unlike the hyperbolic case, we have only* ***1-parameter family of characteristic lines***

$$Bx - Ay = \lambda \tag{82}$$

for the parabolic equation (71), where λ is an arbitrary constant.

3.2 Solving parabolic equations; diagonalization method (eigenvalue-eigenvector method).

What happens if we use **diagonalization method** to solve a parabolic equation ($ac = b^2$) ? One can check that the coefficient matrix of equation (71) has two eigenvalues $\lambda_1 = a + c,\ \lambda_2 = 0$. The corresponding orthonormal eigenvectors are

$$v_1 = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a \\ b \end{pmatrix}, \qquad v_2 = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} b \\ -a \end{pmatrix}$$

and we get the matrix

$$P=\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad P^T=P=P^{-1}$$

By (14), under the change of variables (recall that the change of variables matrix J is chosen as $J=P^T$, see (13))

$$\xi=\frac{1}{\sqrt{a^2+b^2}}(ax+by), \qquad \eta=\frac{1}{\sqrt{a^2+b^2}}(bx-ay), \tag{83}$$

equation (71) can be reduced to

$$Trace\left[\begin{pmatrix} a+c & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}\right]=0.$$

Since $a+c\neq 0$, the above is same as

$$U_{\xi\xi}=0, \tag{84}$$

which implies

$$U(\xi,\eta)=\xi g(\eta)+f(\eta) \tag{85}$$

for arbitrary C^2 functions $f(\eta)$ and $g(\eta)$. The corresponding $u(x,y)$ is given by

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{a^2+b^2}}(ax+by)\,g\left(\frac{1}{\sqrt{a^2+b^2}}(bx-ay)\right)+f\left(\frac{1}{\sqrt{a^2+b^2}}(bx-ay)\right) \\ &= H(bx-ay)+\frac{ax+by}{\sqrt{a^2+b^2}}K(bx-ay) \end{aligned} \tag{86}$$

for arbitrary C^2 functions $H(\eta)$ and $K(\eta)$. We can compare (86) with (79). Assume we are in the case $a>0,\ b>0,\ c>0$, then we have $A=\sqrt{a},\ B=\sqrt{c}$ and so $b=\sqrt{ac}=AB$. Hence

$$\begin{aligned} & H(bx-ay)+\frac{ax+by}{\sqrt{a^2+b^2}}K(bx-ay) \\ &= H\left(ABx-A^2y\right)+\frac{A^2x+ABy}{\sqrt{A^4+A^2B^2}}K\left(ABx-A^2y\right) \\ &= H\left(ABx-A^2y\right)+\frac{Ax+By}{\sqrt{A^2+B^2}}K\left(ABx-A^2y\right)=F(Bx-Ay)+\frac{Ax+By}{A^2+B^2}G(Bx-Ay). \end{aligned}$$

for another two arbitrary C^2 functions $F(\eta)$ and $G(\eta)$. **Therefore, (86) is the same as (79).** The check for the case $a>0,\ b<0,\ c>0$ is similar. Therefore, both methods are actually equivalent.

4 Hyperbolic and parabolic equations of the form $au_{xx}+2bu_{xy}+cu_{yy}=0$ with initial conditions.

If we write the equation $au_{xx}+2bu_{xy}+cu_{yy}=0$ as $au_{xx}+2bu_{xt}+cu_{tt}=0$ and view x as space variable, t as time variable, then a pair of initial conditions (at time $t=0$) of the form

$$u(x,0)=\phi(x), \quad u_t(x,0)=\psi(x), \quad x\in(-\infty,\infty) \tag{87}$$

can determine the solution **uniquely**, i.e. the solution exists and is unique. Here $\phi(x)$ and $\psi(x)$ are two given functions defined on $(-\infty,\infty)$. We will demonstrate this by direct computations.

Remark 4.1 *(**Useful motivation**.) Roughly speaking, a hyperbolic equation (imagine it is a **wave equation**) comes from **Newtonian mechanics**, hence as long as the **initial position** and **initial velocity** are known, the whole process of motion is **uniquely** determined.*

Lemma 4.2 ***(Hyperbolic equation with initial conditions.)*** *Let $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$ be two given functions. Assume $ac < b^2$ in equation $au_{xx} + 2bu_{xt} + cu_{tt} = 0$. Consider the* ***hyperbolic equation with initial conditions:***

$$\begin{cases} \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial t}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}\right)u\right] = 0, \quad u = u(x,t) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty,\infty), \end{cases} \tag{88}$$

where A, B, C, D are constants satisfying $AD - BC \neq 0$ and ***here we also assume that*** *$B \neq 0$* ***and*** *$D \neq 0$ (they are the coefficients of the operator $\frac{\partial}{\partial t}$). Then the initial value problem (88) has a unique solution $u(x,t) \in C^2(\mathbb{R}^2)$. Moreover, the solution is given by*

$$u(x,t) = \frac{1}{AD-BC}\left[AD\cdot\phi\left(x - \frac{C}{D}t\right) - BC\cdot\phi\left(x - \frac{A}{B}t\right)\right] + \frac{BD}{AD-BC}\int_{x-\frac{A}{B}t}^{x-\frac{C}{D}t}\psi(s)\,ds, \tag{89}$$

where $(x,t) \in \mathbb{R}^2$. Note: since we assume $B \neq 0$ and $D \neq 0$, the line $t = 0$ is ***not a characteristic line****, which is good. See Remark 4.3 below.*

Remark 4.3 ***(Important.)*** *The assumption $B \neq 0$ and $D \neq 0$ is* ***essential*** *in the formula (89). There are* ***2-parameter families of characteristic lines*** *for the hyperbolic equation*

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial t}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}\right)u\right] = 0, \tag{90}$$

namely the lines

$$Bx - At = const., \quad \text{and} \quad Dx - Ct = const.. \tag{91}$$

If we have $B = 0$, then by $AD - BC \neq 0$, we must have $A \neq 0$ and $D \neq 0$. Hence the line $t = const.$ is a characteristic line. Similarly, if we have $D = 0$, then $B \neq 0$ and $C \neq 0$, and again the line $t = const.$ is a characteristic line. By this, the initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty,\infty)$$

happen to occur on the ***characteristic line*** *$t = const.$ (i.e. $t = 0$). In general, for $B = 0$, we have either* ***no*** *solution to the initial value problem or* ***infinitely many solutions*** *to the initial value problem. The same conclusion holds for the case $D = 0$.* ***We will leave this as a homework problem for you to verify****. Also see Remark 4.10.*

Proof. The general solution of the equation is

$$u(x,t) = F(Bx - At) + G(Dx - Ct) \tag{92}$$

for arbitrary C^2 functions $F(\cdot)$, $G(\cdot)$ defined on $\mathbb{R}$ and we need to solve

$$\begin{cases} F(Bx) + G(Dx) = \phi(x) \\ -AF'(Bx) - CG'(Dx) = \psi(x). \end{cases} \tag{93}$$

Differentiate the first equation with respect to x to get the system of equations:

$$\begin{cases} BF'(Bx) + DG'(Dx) = \phi'(x) \\ -AF'(Bx) - CG'(Dx) = \psi(x), \end{cases}$$

i.e.

$$\begin{pmatrix} B & D \\ -A & -C \end{pmatrix}\begin{pmatrix} F'(Bx) \\ G'(Dx) \end{pmatrix} = \begin{pmatrix} \phi'(x) \\ \psi(x) \end{pmatrix}.$$

Hence we get

$$\begin{pmatrix} F'(Bx) \\ G'(Dx) \end{pmatrix} = \frac{1}{AD-BC} \begin{pmatrix} -C & -D \\ A & B \end{pmatrix} \begin{pmatrix} \phi'(x) \\ \psi(x) \end{pmatrix}$$

and so

$$F'(Bx) = \frac{-1}{AD-BC}\left(C\phi'(x) + D\psi(x)\right), \qquad G'(Dx) = \frac{1}{AD-BC}\left(A\phi'(x) + B\psi(x)\right),$$

which are the same as (note that we assume $B \neq 0$ and $D \neq 0$)

$$\begin{cases} \frac{d}{dx}F(Bx) = BF'(Bx) = \frac{-B}{AD-BC}\left(C\phi'(x) + D\psi(x)\right), \\ \frac{d}{dx}G(Dx) = DG'(Dx) = \frac{D}{AD-BC}\left(A\phi'(x) + B\psi(x)\right). \end{cases} \tag{94}$$

If we let $\tilde{\psi}(x)$ be an **antiderivative** of $\psi(x)$ (it is not unique), we get

$$\begin{cases} F(Bx) = \frac{-B}{AD-BC}\left(C\phi(x) + D\tilde{\psi}(x)\right) + C_1 \\ G(Dx) = \frac{D}{AD-BC}\left(A\phi(x) + B\tilde{\psi}(x)\right) + C_2 \end{cases} \tag{95}$$

for some integration constants $C_1,\ C_2$. Now by the first equation of (93), we must have $C_1 + C_2 = 0$. Therefore, we conclude (note that we assume $B \neq 0$ and $D \neq 0$)

$$\begin{aligned} &u(x,t) \\ &= F(Bx - At) + G(Dx - Ct) = F\left(B\left(x - \frac{A}{B}t\right)\right) + G\left(D\left(x - \frac{C}{D}t\right)\right) \\ &= \begin{cases} \frac{-B}{AD-BC}\left[C\phi\left(x - \frac{A}{B}t\right) + D\tilde{\psi}\left(x - \frac{A}{B}t\right)\right] \\ +\frac{D}{AD-BC}\left[A\phi\left(x - \frac{C}{D}t\right) + B\tilde{\psi}\left(x - \frac{C}{D}t\right)\right] \end{cases} \\ &= \frac{AD}{AD-BC}\phi\left(x - \frac{C}{D}t\right) - \frac{BC}{AD-BC}\phi\left(x - \frac{A}{B}t\right) + \frac{BD}{AD-BC}\left[\underbrace{\tilde{\psi}\left(x - \frac{C}{D}t\right) - \tilde{\psi}\left(x - \frac{A}{B}t\right)}\right]. \end{aligned}$$

Note that the antiderivative $\tilde{\psi}(x)$ is **not unique**. However, the quantity

$$\tilde{\psi}\left(x - \frac{C}{D}t\right) - \tilde{\psi}\left(x - \frac{A}{B}t\right) \tag{96}$$

is **unique** and is **independent of** the choice of the antiderivative $\tilde{\psi}(x)$. For convenience, we can choose $\tilde{\psi}(x) = \int_0^x \psi(s)\,ds$ and obtain

$$\tilde{\psi}\left(x - \frac{C}{D}t\right) - \tilde{\psi}\left(x - \frac{A}{B}t\right) = \int_{x-\frac{A}{B}t}^{x-\frac{C}{D}t} \psi(s)\,ds. \tag{97}$$

Thus the solution formula for $u(x,t)$ is given by

$$u(x,t) = \frac{AD}{AD-BC}\phi\left(x - \frac{C}{D}t\right) - \frac{BC}{AD-BC}\phi\left(x - \frac{A}{B}t\right) + \frac{BD}{AD-BC}\int_{x-\frac{A}{B}t}^{x-\frac{C}{D}t} \psi(s)\,ds. \tag{98}$$

The solution is defined on all $(x,t) \in \mathbb{R}^2$. Since $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$, we have $u(x,t) \in C^2(\mathbb{R} \times \mathbb{R})$. The solution (98) satisfies the initial value problem (88).

To see uniqueness of the solution (98), we may look at the case $\phi(x) = \psi(x) \equiv 0$ for all $x \in (-\infty, \infty)$ and from the above derivation, we must have $u(x,t) \equiv 0$ for all $(x,t) \in \mathbb{R}^2$. Hence the uniqueness follows. □

Remark 4.4 *By direct check, one can see that (98) is indeed a solution of (88). First, the solution $u(x,t)$ has the form $F(Bx-At)+G(Dx-Ct)$. Hence it must be a solution. Second, for any $x\in\mathbb{R}$, we have*

$$u(x,0)=\frac{AD}{AD-BC}\phi(x)-\frac{BC}{AD-BC}\phi(x)+\frac{BD}{AD-BC}\int_x^x\psi(s)\,ds=\phi(x)$$

and

$$\begin{aligned}u_t(x,0)&=\begin{cases}\frac{AD}{AD-BC}\left(-\frac{C}{D}\right)\phi'(x)-\frac{BC}{AD-BC}\left(-\frac{A}{B}\right)\phi'(x)\\+\frac{BD}{AD-BC}\left(-\frac{C}{D}\right)\psi(x)-\frac{BD}{AD-BC}\left(-\frac{A}{B}\right)\psi(x)\end{cases}\\&=\frac{BD}{AD-BC}\left(-\frac{C}{D}\right)\psi(x)-\frac{BD}{AD-BC}\left(-\frac{A}{B}\right)\psi(x)=\psi(x).\end{aligned}$$

Thus $u(x,t)$ given by (98) is indeed a solution of the equation satisfying the initial conditions.

Remark 4.5 *(**Domain of dependence for hyperbolic equations.**) By (98), the **domain of dependence interval** of the point (x_0,t_0) is the interval $\left[x_0-\frac{A}{B}t_0,x_0-\frac{C}{D}t_0\right]$ (or the interval $\left[x_0-\frac{C}{D}t_0,x_0-\frac{A}{B}t_0\right]$ if $x_0-\frac{C}{D}t_0$ is smaller) lying on the x-axis. Only the values of $\phi(x)$ and $\psi(x)$ on the interval will determine the value of u at the point (x_0,t_0). Draw a picture for the two characteristic lines L_1, L_2 passing through (x_0,t_0).*

In case equation (88) is parabolic (i.e. $A=C,\ B=D$) and has the same initial conditions, we can still solve it (note that now the equation is **degenerate parabolic**).

Lemma 4.6 *(**Parabolic equation with initial conditions.**) Let $\phi\in C^3(\mathbb{R})$ and $\psi\in C^2(\mathbb{R})$ be two given functions. Assume $ac=b^2$ in equation (44) (we view y as time and here we denote it as t). Consider the **parabolic equation with initial conditions (at time** $t=0$**):***

$$\begin{cases}\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial t}\right)\left[\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial t}\right)u\right]=0, & u=u(x,t)\\ u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x), & x\in(-\infty,\infty),\end{cases}\tag{99}$$

*where $A,\ B$ are constants with $B\neq0$ (this is same as in Lemma 4.2; B is the coefficient of the operator $\frac{\partial}{\partial t}$). Then the initial value problem (99) has a **unique** solution $u(x,t)\in C^2(\mathbb{R}^2)$. Moreover, the solution is given by*

$$u(x,t)=\phi\left(x-\frac{A}{B}t\right)+\frac{t}{B}\left[A\phi'\left(x-\frac{A}{B}t\right)+B\psi\left(x-\frac{A}{B}t\right)\right],\tag{100}$$

*where $(x,t)\in\mathbb{R}^2$. Note: there is only **1-parameter family of characteristic lines** $Bx-At=$ const. for the parabolic equation in (99). Since we assume $B\neq0$, the line $t=0$ is **not a characteristic line**, which is good.*

Remark 4.7 *Assume $B\neq0$. Note that (100) is still correct even if we have $A=0$. In such a case, we have*

$$u_{tt}(x,t)=0,\quad u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x),\quad x\in(-\infty,\infty),$$

and the solution is unique, given by

$$u(x,t)=\phi(x)+t\psi(x),\quad x\in(-\infty,\infty).\tag{101}$$

However, if we have $B=0,\ A\neq0$, then the solution either does not exist or is not unique. See Remark 4.10 below.

Proof. Since $B \neq 0$, we choose the general solution to have the form (see (75))

$$u(x,t) = F(Bx - At) + \frac{t}{B} G(Bx - At), \quad B \neq 0, \tag{102}$$

then we have

$$\begin{cases} F(Bx) = \phi(x) \\ -AF'(Bx) + \frac{1}{B} G(Bx) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

which gives

$$\begin{cases} \underbrace{BF'(Bx)} = \phi'(x) \\ -A \underbrace{BF'(Bx)} + G(Bx) = B\psi(x), \quad x \in \mathbb{R}, \end{cases}$$

and we conclude

$$\begin{cases} F(Bx) = \phi(x) \\ G(Bx) = B\psi(x) + A\phi'(x). \end{cases} \tag{103}$$

By the above we will get

$$\begin{aligned} u(x,t) &= F(Bx - At) + \frac{t}{B} G(Bx - At) = F\left(B\left(x - \frac{A}{B}t\right)\right) + \frac{t}{B} G\left(B\left(x - \frac{A}{B}t\right)\right) \\ &= \phi\left(x - \frac{A}{B}t\right) + \frac{t}{B}\left[B\psi\left(x - \frac{A}{B}t\right) + A\phi'\left(x - \frac{A}{B}t\right)\right], \end{aligned} \tag{104}$$

Since we assume $\phi \in C^3(\mathbb{R})$ and $\psi \in C^2(\mathbb{R})$, the solution $u(x,t)$ given by (104) satisfies $u(x,t) \in C^2(\mathbb{R} \times \mathbb{R})$. □

Remark 4.8 *By direct check, one can see that (104) is indeed a solution of (99).*

Remark 4.9 *(**Important.**) Note that the value of the solution u at (x_0, t_0) depends only on the initial data ϕ and ψ at the point $x_0 - \frac{A}{B}t_0$. Another way to see this is that the two characteristic lines L_1, L_2 passing through (x_0, t_0) **degenerates into one characteristic line** only.*

Remark 4.10 *(**Important.**) (**This is a continuation of Remark 4.3.**) In case $B = 0$ (of course, we must have $A \neq 0$), equation becomes*

$$\begin{cases} u_{xx}(x,t) = 0, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty, \infty), \end{cases} \tag{105}$$

and the general solution for $u(x,t)$ is $u(x,t) = h(t)x + g(t)$ for arbitrary functions of $h(t)$ and $g(t)$, $t \in (-\infty, \infty)$. Now we need to require

$$\begin{cases} u(x,0) = h(0)x + g(0) = \phi(x), \quad x \in (-\infty, \infty), \\ u_t(x,0) = h'(0)x + g'(0) = \psi(x), \quad x \in (-\infty, \infty). \end{cases} \tag{106}$$

*Clearly, in general, (106) has **no solution**. But if $\phi(x)$ and $\psi(x)$ are of the form $ax+b$, $cx+d$, then (106) has **infinitely many solutions**. Again, the reason for this is that **the data is prescribed on the line** $t = 0$, **which is a characteristic line.** If we change*

Example 4.11 *Find the solution $u(x,t)$ satisfying:*

$$u_{xx} + 2u_{xt} + u_{tt} = 0, \quad u(x,0) = x^2, \quad u_t(x,0) = e^x, \quad x \in \mathbb{R}.$$

Solution:

By (74) in Lemma 3.1 and the identity

$$u_{xx} + 2u_{xt} + u_{tt} = (\partial_x + \partial_t)\left[(\partial_x + \partial_t) u\right] = 0, \quad A = B = 1,$$

we see that the general solution of this degenerate parabolic equation has the form

$$u(x,t) = F(x-t) + xG(x-t)$$

for arbitrary C^2 functions $F(\cdot)$, $G(\cdot)$ defined on $\mathbb{R}$. Hence we need to solve

$$\begin{cases} F(x) + xG(x) = x^2 \\ -F'(x) - xG'(x) = e^x \end{cases}$$

and obtain

$$\begin{cases} F'(x) + G(x) + xG'(x) = 2x \\ -F'(x) - xG'(x) = e^x \end{cases}$$

and then

$$G(x) = 2x + e^x.$$

Next, by the identity $F(x) + xG(x) = x^2$ we have

$$F(x) = x^2 - xG(x) = x^2 - x(2x + e^x) = -x^2 - xe^x.$$

The answer for $u(x,t)$ is

$$\begin{aligned} u(x,t) &= F(x-t) + xG(x-t) \\ &= -(x-t)^2 - (x-t)e^{x-t} + x\left[2(x-t) + e^{x-t}\right] = (x-t)(x+t) + te^{x-t} \end{aligned}$$

□

5 The wave equation with initial conditions.

Remark 5.1 *One can see W. A. Strauss PDE book (second edition), p. 11-13, for a brief explanation of how to derive wave equation from a flexible, elastic homogeneous string which undergoes relatively small transverse vibrations. However, it is difficult to understand his explanation.*

In this section, we look at **one-dimensional** wave equation for the function $u(x,t)$, given by

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{107}$$

Under some assumptions (small amplitude, etc ...), the equation describes **the motion of a vibrating string**, where $u(x,t)$ represents the position of the string (I will not derive this in class). Here $c > 0$ is a constant given by

$$c = \sqrt{\frac{T}{\rho}}, \tag{108}$$

where T is the **tension** of the string and ρ is the **density** of the string (both are assumed to be constants, not very realistic at all).

Remark 5.2 *By a change of variable in time:*

$$x = x, \quad \tilde{t} = ct,$$

the function $v\left(x, \tilde{t}\right) = u\left(x, t\right) = u\left(x, \frac{\tilde{t}}{c}\right)$ *will satisfy (107) with* $c = 1$. *Hence, the two equations* $u_{tt} = c^2 u_{xx}$ *and* $u_{tt} = u_{xx}$ *are equivalent. Some books discuss wave equation in the form* $u_{tt} = u_{xx}$ *only. However, we prefer to keep the constant* $c > 0$ *in the equation to reveal some of its physical implications.*

As (107) is a physical equation, it has initial conditions. They are the **initial position** and **initial velocity** of the string, given by

$$u\left(x, 0\right) = \phi\left(x\right), \quad u_t\left(x, 0\right) = \psi\left(x\right), \quad x \in \mathbb{R}. \tag{109}$$

Remark 5.3 *(**Useful motivation**.) Roughly speaking, the wave equation comes from **Newtonian mechanics**, hence as long as the initial position and initial velocity are known, the whole process of motion is uniquely determined.*

With the above conditions, the solution $u\left(x, t\right)$ satisfying (107) and (109) **exists** and is **unique** (which can be seen from (98)). Moreover, if we change $\phi\left(x\right)$ and $\psi\left(x\right)$ a little bit, then the corresponding solution will also change a little bit (we shall see this soon). In this sense, we say that the problem (107) and (109) is **well-posed**.

Since one can factorize the equation (107) as

$$\left(c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left[\left(c\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u\right] = 0, \quad u = u\left(x, t\right), \tag{110}$$

by previous discussion we know the general solution of (107), defined on the whole space $\mathbb{R}^2$, is given by

$$u\left(x, t\right) = F\left(x + ct\right) + G\left(x - ct\right), \quad \left(x, t\right) \in \left(-\infty, \infty\right) \times \left(-\infty, \infty\right) \tag{111}$$

for arbitrary C^2 functions $F,\ G$ defined on $\mathbb{R}$.

Definition 5.4 *Any line of the form* $x + ct = const.$ *or* $x - ct = const.$ *is called a **characteristic line** of the wave equation (107). A wave equation has **2-parameter family** of characteristic lines.*

Remark 5.5 *(**The geometric meaning of the wave equation.**) The solution* $u\left(x, t\right)$ *given by (111) consists of two **traveling waves** moving in opposite directions (positive and negative* x*-direction) with the same speed* c *(the graph of* $F\left(x + ct\right)$ *moves to the left and the graph of* $G\left(x - ct\right)$ *moves to the right; draw a picture on blackboard). Moreover, since* $c = \sqrt{T/\rho}$, *if the tension* T *is large and the density* ρ *is small, then the traveling wave speed is large. This matches with physical observation.*

Without remembering the formula in Lemma 4.2, one can easily derive the solution formula satisfying the conditions (109). We need to require

$$\begin{cases} u\left(x, 0\right) = F\left(x\right) + G\left(x\right) = \phi\left(x\right) \\ u_t\left(x, 0\right) = cF'\left(x\right) - cG'\left(x\right) = \psi\left(x\right). \end{cases} \tag{112}$$

By this we obtain

$$F'\left(x\right) = \frac{c\phi'\left(x\right) + \psi\left(x\right)}{2c}, \qquad G'\left(x\right) = \frac{c\phi'\left(x\right) - \psi\left(x\right)}{2c}, \tag{113}$$

and so

$$F\left(x\right) = \frac{\phi\left(x\right)}{2} + \frac{1}{2c}\int_0^x \psi\left(s\right)ds + \delta, \qquad G\left(x\right) = \frac{\phi\left(x\right)}{2} - \frac{1}{2c}\int_0^x \psi\left(s\right)ds + \varepsilon \tag{114}$$

with suitable constants $\delta,\ \varepsilon$. Here $\delta+\varepsilon=0$ by (112). Hence we get the **unique solution** given by

$$u(x,t)=F(x+ct)+G(x-ct)=\underbrace{\frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds}. \tag{115}$$

We can conclude the following:

Lemma 5.6 *Assume ϕ and ψ in the initial conditions (109) satisfy $\phi\in C^2(\mathbb{R})$ and $\psi\in C^1(\mathbb{R})$. Then the function $u(x,t)$ given by*

$$u(x,t)=\frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds. \tag{116}$$

is the ***unique*** *C^2 solution of the initial value problem*

$$\begin{cases} u_{tt}(x,t)=c^2u_{xx}(x,t) \\ u(x,0)=\phi(x), \quad u_t(x,0)=\psi(x), \quad x\in(-\infty,\infty) \end{cases} \tag{117}$$

defined on the domain $(x,t)\in\mathbb{R}^2$.

Proof. (**Due to Lemma 4.2, we can omit the proof of this lemma.**) Given $\phi\in C^2(\mathbb{R})$ and $\psi\in C^1(\mathbb{R})$, we define $u(x,t)$ as in (116) for $(x,t)\in\mathbb{R}^2$. It clearly satisfies $u(x,0)=\phi(x)$ for all $x\in(-\infty,\infty)$. We also have

$$u_t(x,t)=\frac{c}{2}\left[\phi'(x+ct)-\phi'(x-ct)\right]+\frac{1}{2}\left[\psi(x+ct)+\psi(x-ct)\right]$$

for all $(x,t)\in\mathbb{R}^2$, and so $u(x,t)$ satisfies $u_t(x,0)=\psi(x)$ for all $x\in(-\infty,\infty)$. Finally, note that

$$u_{tt}(x,t)=\frac{c^2}{2}\left[\phi''(x+ct)+\phi''(x-ct)\right]+\frac{c}{2}\left[\psi'(x+ct)-\psi'(x-ct)\right]$$

and

$$u_{xx}(x,t)=\frac{1}{2}\left[\phi''(x+ct)+\phi''(x-ct)\right]+\frac{1}{2c}\left[\psi'(x+ct)-\psi'(x-ct)\right],$$

which implies $u_{tt}(x,t)=c^2u_{xx}(x,t)$ for all $(x,t)\in\mathbb{R}^2$. Therefore, $u(x,t)$ given by (116) is indeed a C^2 solution of (117) on the domain $(x,t)\in\mathbb{R}^2$.

To check **uniqueness**, assume we have two C^2 solutions $u_1(x,t)$ and $u_2(x,t)$ of (117) on $\mathbb{R}^2$. Then the function $u(x,t)=u_1(x,t)-u_2(x,t)$ is a C^2 solution of the problem

$$\begin{cases} u_{tt}(x,t)=c^2u_{xx}(x,t) \\ u(x,0)=0, \quad u_t(x,0)=0, \quad x\in(-\infty,\infty) \end{cases}$$

on $\mathbb{R}^2$. Since we know the general solution of $u_{tt}(x,t)=c^2u_{xx}(x,t)$ on $\mathbb{R}^2$ has the form $u(x,t)=F(x+ct)+G(x-ct)$ for some C^2 functions $F(\cdot),\ G(\cdot)$ defined on $\mathbb{R}$, we have

$$\begin{cases} u(x,0)=F(x)+G(x)=0 \\ u_t(x,0)=cF'(x)-cG'(x)=0, \quad x\in(-\infty,\infty) \end{cases} \tag{118}$$

which implies $F'(x)=G'(x)=0$ on $\mathbb{R}$. Therefore, both $F(\cdot),\ G(\cdot)$ are **constant** functions and by $u(x,t)=F(x+ct)+G(x-ct)$, $u(x,t)$ is a constant function on $\mathbb{R}^2$. Since we have $u(x,0)\equiv0$ for all x, we must have $u(x,t)\equiv0$ on $\mathbb{R}^2$. The uniqueness property is verified. $\square$

Remark 5.7 ***(Important.)*** *We call (116)* ***d'Alembert solution****. It was due to him in 1746. Note that the function*

$$\frac{1}{2}\left[\phi\left(x+ct\right)+\phi\left(x-ct\right)\right] \tag{119}$$

is an ***even function*** *in t and the function*

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\psi\left(s\right)ds \tag{120}$$

is an ***odd function*** *in t.*

Remark 5.8 ***(Important.)*** *One can also express the solution as*

$$u\left(x,t\right)=\frac{1}{2}\left[\phi\left(x+ct\right)+\phi\left(x-ct\right)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi\left(s\right)ds=F\left(x+ct\right)+G\left(x-ct\right)$$

for some suitable function $F\left(z\right)$ and $G\left(z\right)$. To see this, let

$$\tilde{\psi}\left(\theta\right)=\frac{1}{2c}\int_{0}^{\theta}\psi\left(s\right)ds,\quad \theta\in\left(-\infty,\infty\right),\quad \tilde{\psi}\left(0\right)=0.$$

We have

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\psi\left(s\right)ds=\frac{1}{2c}\int_{0}^{x+ct}\psi\left(s\right)ds-\frac{1}{2c}\int_{0}^{x-ct}\psi\left(s\right)ds=\tilde{\psi}\left(x+ct\right)-\tilde{\psi}\left(x-ct\right)$$

and then

$$\begin{aligned}u\left(x,t\right)&=\frac{1}{2}\left[\phi\left(x+ct\right)+\phi\left(x-ct\right)\right]+\tilde{\psi}\left(x+ct\right)-\tilde{\psi}\left(x-ct\right)\\&=\left(\frac{1}{2}\phi\left(x+ct\right)+\tilde{\psi}\left(x+ct\right)\right)+\left(\frac{1}{2}\phi\left(x-ct\right)-\tilde{\psi}\left(x-ct\right)\right)=F\left(x+ct\right)+G\left(x-ct\right),\end{aligned}$$

where $F\left(z\right)=\frac{1}{2}\phi\left(z\right)+\tilde{\psi}\left(z\right)$ and $G\left(z\right)=\frac{1}{2}\phi\left(z\right)-\tilde{\psi}\left(z\right),\ z\in\mathbb{R}$.

Remark 5.9 ***(Omit this in class)*** *If $\phi\in C^{k}\left(\mathbb{R}\right)$ and $\psi\in C^{k-1}\left(\mathbb{R}\right)$, then u given by (116) represents a classical solution $u\in C^{k}\left(\mathbb{R}\times[0,\infty)\right)$ of the initial value problem, but* ***its regularity is not smoother (or worse) in general. Thus the wave equation does not produce instantaneous smoothing of the initial data as the heat equation does.***

5.1 Domain of dependence and influence of the initial conditions for wave equation.

For convenience of discussion, we confine to **nonnegative** time $t\geq 0$ (this is not really essential). Recall that the solution of the initial value problem (117) is

$$u\left(x,t\right)=\frac{1}{2}\left[\phi\left(x+ct\right)+\phi\left(x-ct\right)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi\left(s\right)ds,\quad (x,t)\in(-\infty,\infty)\times[0,\infty). \tag{121}$$

For each fixed time $t=t_0\in[0,\infty)$ and fixed $x_0\in\left(-\infty,\infty\right)$, we get an interval of the form $[x_0-ct_0,x_0+ct_0]$. The value of $u\left(x_0,t_0\right)$ depends only on ϕ at $x_0-ct_0,\ x_0+ct_0$, and ψ on the interval $[x_0-ct_0,x_0+ct_0]$. The values of ϕ and ψ **outside** the interval $[x_0-ct_0,x_0+ct_0]$ **will not** affect the value of $u\left(x_0,t_0\right)$.

Definition 5.10 *The interval $[x_0-ct_0,x_0+ct_0]$ lying on the x-axis is called the* ***domain of dependence interval*** *of the point $\left(x_0,t_0\right)$.*

Remark 5.11 ***The above also says that for our wave equation "disturbances" or "signals" only travel with speed*** *c. To understand this more clearly, see the definition of* ***"domain of influence".***

In view of this, for each fixed $x_0 \in (-\infty, \infty)$, there is a region $R \subset xt$-plane (**unbounded closed set**, lying on the upper half xt-plane) so that the values of u on this region can be **affected** by the values of $\phi(x_0)$ and $\psi(x_0)$. This region $R \subset xt$-plane is called the **domain of influence** of the point $(x_0, 0)$ (equivalently, a point $p \in R$ if and only if its domain of dependence interval **contains** the point x_0). The value of u at any point (x, t) lying **out of** of the region R is **not** affected by the values of $\phi(x_0)$ and $\psi(x_0)$. The **domain of influence** region R can be described as

$$R = \{(x,t) \in \mathbb{R} \times [0,\infty) : x - ct \leq x_0 \text{ and } x + ct \geq x_0\}, \tag{122}$$

where the two half-lines $x - ct = x_0,\ t \geq 0$, and $x + ct = x_0,\ t \geq 0$, intersect at the point $(x_0, 0)$.

Remark 5.12 *Draw a picture for the region R (or see Figure 1 in p. 39 of Strauss's undergraduate PDE book).*

Outside the **domain of influence** of the point $(x_0, 0)$, the value of $u(x,t)$ is not affected by the values of $\phi(x_0)$ and $\psi(x_0)$. In view of this, we have the following obvious fact:

Lemma 5.13 *For any given $\phi(x)$ and $\psi(x)$, the* ***domain of influence*** *of the interval (lying on the x-axis) $|x| \leq \sigma$ is the region (lying on $\mathbb{R} \times [0,\infty)$) $|x| \leq \sigma + ct$. In particular, if $\phi(x) \equiv \psi(x) \equiv 0$ for $|x| > \sigma$ (i.e. both $\phi(x)$ and $\psi(x)$ have* ***compact support*** *), then $u(x,t) \equiv 0$ on the region $|x| \geq \sigma + ct$.*

Remark 5.14 *Draw a picture for the above lemma. Note that the wave speed is $c > 0$ and the wave is propagating on the x-axis (the space dimension). From this observation, it is easy to see that if $\phi(x) \equiv \psi(x) \equiv 0$ for $|x| > \sigma$, then $u(x,t) \equiv 0$ on the region $|x| \geq \sigma + ct$.*

Corollary 5.15 *Assume both $\phi(x)$ and $\psi(x)$ have* ***compact support****. Then $u(x,t)$ given by (116) has compact support in x for each fixed t (however, the support of $u(x,t)$ will become larger if t gets larger).*

Example 5.16 *Consider the wave equation with initial conditions:*

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad c > 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty, \infty). \end{cases}$$

Assume that $\phi = \psi \equiv 0$ on the interval $[-1-c, 1+c]$. Determine the region $R \subset \mathbb{R} \times [0,\infty)$ such that $u(x,t) \equiv 0$ on R.

<u>**Solution**</u>:

The region R is the trianglular-shaped region given by

$$R = \{(x,t) : x - ct \geq -1 - c\} \bigcap \{(x,t) : x + ct \leq 1 + c\} \bigcap \{(x,t) : t \geq 0\}.$$

$\square$

Remark 5.17 *Draw a picture for the above answer.*

5.2 Space-time separable solutions of the wave equation.

We look for certain special solutions of the wave equation $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ with the **separable form:**

$$u(x,t) = f(x)\, g(t),$$

where $f(x)$ and $g(t)$ are C^2 functions defined on $(-\infty, \infty)$. Plug it into the equation to get

$$f(x)\, g''(t) = c^2 f''(x)\, g(t)$$

and then we get the identity (assume the denominators are nonzero)

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)}.$$

The above identity cannot hold unless there is some constant K such that

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = K, \quad \forall\, x \in \text{domain of } f, \quad \forall\, t \in \text{domain of } g.$$

There are three possibilities for the constant $K : K = \lambda^2 > 0,\ K = 0,\ K = -\lambda^2 < 0$. For the first case, we have

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda^2, \quad \lambda > 0,$$

which gives

$$g(t) = c_1 e^{\lambda ct} + c_2 e^{-\lambda ct}, \quad f(x) = d_1 e^{\lambda x} + d_2 e^{-\lambda x}, \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty), \tag{123}$$

where $c_1,\ c_2,\ d_1,\ d_2$ are integration constants. For the second case, we have

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = 0,$$

which gives

$$g(t) = c_1 t + c_2, \quad f(x) = d_1 x + d_2, \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty). \tag{124}$$

For the third case, we have

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = -\lambda^2, \quad \lambda > 0,$$

which gives

$$g(t) = c_1 \sin(\lambda ct) + c_2 \cos(\lambda ct), \quad f(x) = d_1 \sin(\lambda x) + d_2 \cos(\lambda x), \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty). \tag{125}$$

Thus we have:

Lemma 5.18 *(Classification of space-time separable solutions of the wave equation.) The following are the **space-time separable** solutions of the wave equation*

$$u(x,t) = \begin{cases} \left(c_1 e^{\lambda ct} + c_2 e^{-\lambda ct}\right)\left(d_1 e^{\lambda x} + d_2 e^{-\lambda x}\right) \\ (c_1 t + c_2)(d_1 x + d_2) \\ \left[c_1 \sin(\lambda ct) + c_2 \cos(\lambda ct)\right]\left[d_1 \sin(\lambda x) + d_2 \cos(\lambda x)\right], \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty) \end{cases} \tag{126}$$

and there are no others. Here λ $(\lambda > 0)$, $c_1,\ c_2,\ d_1,\ d_2$ are all arbitrary constants.

Remark 5.19 ***(Important.)*** *The space-time separable solutions are important if we want to use* ***Fourier series*** *to express solutions of the wave equation, especially using the third solution in (126).*

Example 5.20 *For each $n \in \mathbb{N}$ and $L > 0$, there is a solution, defined on $\mathbb{R}^2$, of the wave equation of the form (choose $\lambda = n\pi/L$ in the above lemma)*

$$u_n(x,t) = \left[A_n \sin\left(\frac{n\pi c}{L}t\right) + B_n \cos\left(\frac{n\pi c}{L}t\right)\right] \sin\left(\frac{n\pi x}{L}\right), \quad A_n,\ B_n \text{ are const.,} \tag{127}$$

which satisfies the ***initial-boundary conditions (fixed-end condition)*** *on the domain $(x,t) \in [0,L] \times [0,\infty)$:*

$$\begin{cases} (u_n)_{tt}(x,t) = c^2 (u_n)_{xx}(x,t), & (x,t) \in [0,L] \times [0,\infty) \\ u_n(0,t) = u_n(L,t) = 0, \quad \forall\, t \in [0,\infty) & \text{(boundary cond.)} \\ u_n(x,0) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad (u_n)_t(x,0) = A_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right), \quad \forall\, x \in [0,L] & \text{(initial cond.).} \end{cases} \tag{128}$$

For each $n \in \mathbb{N}$, we call the above $u_n(x,t)$ a ***harmonic*** *of the wave equation $u_{tt} = c^2 u_{xx}$ with* ***fixed ends*** *at $x = 0$ and $x = L$ (by Fourier series expansion applying to solutions of the wave equation with fixed ends, any such solution can be expressed as an* ***infinite sum*** *of* ***different harmonics****). For each $n \in \mathbb{N}$, the initial data $u_n(x,0) = B_n \sin\left(\frac{n\pi x}{L}\right)$ in (127) has n zeros on the interval $[0,L]$. Note that for a wave equation with* ***fixed-end*** *boundary condition, we also need the initial physical conditions*

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,L], \tag{129}$$

to guarantee a unique solution (imagine the vibrations of a guitar string). One can rewrite $u_n(x,t)$ as

$$\begin{aligned} u_n(x,t) &= R_n \left[\cos\theta_n \sin\left(\frac{n\pi c}{L}t\right) + \sin\theta_n \cos\left(\frac{n\pi c}{L}t\right)\right] \sin\left(\frac{n\pi x}{L}\right) \\ &= R_n \sin\left(\frac{n\pi c}{L}t + \theta_n\right) \sin\left(\frac{n\pi x}{L}\right), \end{aligned} \tag{130}$$

where $R_n = \sqrt{A_n^2 + B_n^2}$ and the angle θ_n satisfies $\cos\theta_n = \frac{A_n}{R_n}$, $\sin\theta_n = \frac{B_n}{R_n}$. Note that the value of $u_n(x,t)$ lies between $-R_n$ and R_n. We call R_n the ***amplitude*** *of $u_n(x,t)$ and call θ_n the* ***phase*** *of $u_n(x,t)$. The solution (130), defined on $[0,L]$, is* ***periodic*** *in time with*

$$u_n\left(x, t + \frac{2L}{nc}\right) = u_n(x,t), \quad \forall\, x \in [0,L], \quad t \in \mathbb{R}.$$

The time $\frac{2L}{nc}$ is called the ***period*** *of the solution $u_n(x,t)$.*

5.3 Conservation of the total energy for wave equation.

Lemma 5.21 ***(Equipartition of energy for wave equations with compact support initial data.)*** *Assume both $\phi(x)$ and $\psi(x)$ have* ***compact support*** *(this assumption is essential) in (117) and $u \in C^2(\mathbb{R} \times (-\infty,\infty))$ solves the initial value problem (117). Define the* ***kinetic energy*** *and the* ***potential energy*** *for the solution $u(x,t)$ as*

$$k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx \quad \text{and} \quad p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx, \quad t \in (-\infty,\infty). \tag{131}$$

Then we have

$$k(t) + c^2 p(t) \quad \text{is a } \textbf{constant}, \quad \forall\, t \in (-\infty,\infty). \tag{132}$$

Moreover, for $t \in (-\infty, \infty)$ such that $|t|$ is ***large enough****, we have*

$$k(t) = c^2 p(t), \quad \forall\, t \in (-\infty, \infty). \tag{133}$$

In particular, we have

$$k(t) = c^2 p(t) = const. \tag{134}$$

when $|t|$ is large enough.

Remark 5.22 *Be careful that there is a coefficient c^2 before $p(t)$.*

Remark 5.23 *By (132), we see that the energy is equal to*

$$\begin{aligned} & k(t) + c^2 p(t) = k(0) + c^2 p(0) \\ & = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,0)\,dx + \frac{c^2}{2}\int_{-\infty}^{\infty} u_x^2(x,0)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \psi^2(x)\,dx + \frac{c^2}{2}\int_{-\infty}^{\infty} (\varphi'(x))^2\,dx, \end{aligned} \tag{135}$$

where we see that the improper integrals are actually proper integrals since $\phi(x)$ and $\psi(x)$ have ***compact support*** *...*

Proof. To prove (132), by Corollary 5.15, we know that $u(x,t)$ given by (115) have compact support in x **for each fixed** $t \in (-\infty, \infty)$. Computing

$$\begin{cases} u_t(x,t) = \frac{c}{2}\left[\phi'(x+ct) - \phi'(x-ct)\right] + \frac{1}{2}\left[\psi(x+ct) + \psi(x-ct)\right] \\ u_x(x,t) = \frac{1}{2}\left[\phi'(x+ct) + \phi'(x-ct)\right] + \frac{1}{2c}\left[\psi(x+ct) - \psi(x-ct)\right], \end{cases} \tag{136}$$

we can see that both $u_t(x,t)$ and $u_x(x,t)$ also have **compact support** in x for each fixed $t \in (-\infty, \infty)$. Hence the two **improper** integrals in (131) both **converge** (both are actually ***proper*** integrals). **Hence, the differentiation with respect to time can commute with the integral.** That is, we have

$$\begin{aligned} & \frac{d}{dt}\left[k(t) + c^2 p(t)\right] = \frac{d}{dt}\left[\frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx + \frac{c^2}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx\right] \\ & = \int_{-\infty}^{\infty} u_t(x,t)\,u_{tt}(x,t)\,dx + c^2\int_{-\infty}^{\infty} u_x(x,t)\,u_{xt}(x,t)\,dx \\ & = c^2\int_{-\infty}^{\infty} \left[u_t(x,t)\,u_{xx}(x,t) + u_x(x,t)\,u_{xt}(x,t)\right]dx \\ & = c^2\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[u_t(x,t)\,u_x(x,t)\right]dx = c^2\left[u_x(x,t)\,u_t(x,t)\right]\Big|_{x=-\infty}^{x=\infty} = 0, \quad \forall\, t \in (-\infty, \infty), \end{aligned}$$

where we have used the fact that both $u_t(x,t)$ and $u_x(x,t)$ also have compact support in x for each fixed $t \in (-\infty, \infty)$. The proof of (132) is done.

For (133), by (136) we have

$$u_t^2(x,t) = \begin{cases} \frac{c^2}{4}\left[\underbrace{(\phi'(x+ct))^2 + (\phi'(x-ct))^2} - 2\phi'(x+ct)\,\phi'(x-ct)\right] \\ +\frac{1}{4}\left[\underbrace{\psi^2(x+ct) + \psi^2(x-ct)} + 2\psi(x+ct)\,\psi(x-ct)\right] \\ +\frac{c}{2}\left(\begin{array}{c} \underbrace{\phi'(x+ct)\,\psi(x+ct)} + \phi'(x+ct)\,\psi(x-ct) \\ -\phi'(x-ct)\,\psi(x+ct) - \underbrace{\phi'(x-ct)\,\psi(x-ct)} \end{array}\right) \end{cases}$$

and

$$c^2u_x^2(x,t)=\begin{cases}\frac{c^2}{4}\left[\underbrace{(\phi'(x+ct))^2+(\phi'(x-ct))^2}+2\phi'(x+ct)\,\phi'(x-ct)\right]\\ +\frac{1}{4}\left[\underbrace{\psi^2(x+ct)+\psi^2(x-ct)}-2\psi(x+ct)\,\psi(x-ct)\right]\\ +\frac{c}{2}\left(\begin{array}{c}\underbrace{\phi'(x+ct)\,\psi(x+ct)}-\phi'(x+ct)\,\psi(x-ct)\\ +\phi'(x-ct)\,\psi(x+ct)-\underbrace{\phi'(x-ct)\,\psi(x-ct)}\end{array}\right).\end{cases}$$

By above, it suffices to show that

$$\int_{-\infty}^{\infty}\left(\begin{array}{c}\frac{c^2}{4}\left[-2\phi'(x+ct)\,\phi'(x-ct)\right]+\frac{1}{4}\left[2\psi(x+ct)\,\psi(x-ct)\right]\\ +\frac{c}{2}\left[\phi'(x+ct)\,\psi(x-ct)-\phi'(x-ct)\,\psi(x+ct)\right]\end{array}\right)dx$$
$$=\int_{-\infty}^{\infty}\left(\begin{array}{c}\frac{c^2}{4}\left[2\phi'(x+ct)\,\phi'(x-ct)\right]+\frac{1}{4}\left[-2\psi(x+ct)\,\psi(x-ct)\right]\\ +\frac{c}{2}\left[-\phi'(x+ct)\,\psi(x-ct)+\phi'(x-ct)\,\psi(x+ct)\right]\end{array}\right)dx.$$

which is equivalent to showing that

$$\int_{-\infty}^{\infty}\left(\begin{array}{c}-c^2\phi'(x+ct)\,\phi'(x-ct)+\psi(x+ct)\,\psi(x-ct)\\ +c\phi'(x+ct)\,\psi(x-ct)-c\phi'(x-ct)\,\psi(x+ct)\end{array}\right)dx=0. \tag{137}$$

Since, for any $x\in\mathbb{R}$, the two points $p=x+ct,\ q=x-ct$ have distance $2c\,|t|$. They **both** can not stay in the support of ϕ and ψ for large $|t|$ for **any** $x\in\mathbb{R}$. Hence if $|t|\geq 0$ is large enough, we must have

$$\begin{aligned}&\phi'(x+ct)\,\phi'(x-ct)=\psi(x+ct)\,\psi(x-ct)\\ &=\phi'(x+ct)\,\psi(x-ct)=\phi'(x-ct)\,\psi(x+ct)=0\quad\textbf{for all}\quad x\in(-\infty,\infty).\end{aligned}$$

The proof is done. □

Remark 5.24 *(**Read the following by yourself.**) If you have difficulty assuring the identity*

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx=\int_{-\infty}^{\infty}\left(\frac{d}{dt}u_t^2(x,t)\right)dx=2\int_{-\infty}^{\infty}u_t(x,t)\,u_{tt}(x,t)\,dx,\quad\forall\ t\in(-\infty,\infty),$$

you can use definition and mean value theorem to see that for fixed $t_0\in(-\infty,\infty)$ (let $F(t)=\int_{-\infty}^{\infty}u_t^2(x,t)\,dx,\ t\in(-\infty,\infty)$) we have

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=t_0}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx=\lim_{h\to0}\frac{F(t_0+h)-F(t_0)}{h}\\ &=\lim_{h\to0}\int_{-\infty}^{\infty}\frac{u_t^2(x,t_0+h)-u_t^2(x,t_0)}{h}dx=\lim_{h\to0}\underbrace{\int_{-\infty}^{\infty}2u_t(x,t_0+\theta(x,h))\,u_{tt}(x,t_0+\theta(x,h))\,dx},\end{aligned}$$

where the number $\theta(x,h)$ lies between 0 and h and so $t_0+\theta(x,h)$ lies between t_0 and t_0+h (note that h may be positive or negative). As we will let $h\to0$, we may assume that $h\in[-1,1]$ and so $t_0+\theta(h)\in[t_0-1,t_0+1]$ for all $h\in[-1,1]$. One can find a large number $M>0$ so that

$$2u_t(x,t_0+\theta(h))\,u_{tt}(x,t_0+\theta(h))\equiv0\ \text{for all}\ |x|\geq M\ \text{and all}\ h\in[-1,1], \tag{138}$$

which gives

$$\int_{-\infty}^{\infty}2u_t(x,t_0+\theta(h))\,u_{tt}(x,t_0+\theta(h))\,dx=\int_{-M}^{M}2u_t(x,t_0+\theta(h))\,u_{tt}(x,t_0+\theta(h))\,dx,\ \forall\ h\in[-1,1].$$

Since the integrand $2u_t(x,s)\,u_{tt}(x,s)$ *is a* ***continuous function*** *in* $(x,s)\in[-M,M]\times[t_0-1,t_0+1]$, *we have*

$$\lim_{s\to t_0}\int_{-M}^{M}2u_t(x,s)\,u_{tt}(x,s)\,dx=\int_{-M}^{M}2u_t(x,t_0)\,u_{tt}(x,t_0)\,dx,$$

which is an elementary fact in Advanced Calculus (note that the situation here is easier because we are doing integral on a compact interval $[-M,M]$. *The above implies*

$$\begin{aligned}&\underbrace{\lim_{h\to 0}\int_{-\infty}^{\infty}2u_t(x,t_0+\theta(h))\,u_{tt}(x,t_0+\theta(h))\,dx}\\&=\lim_{s\to t_0}\int_{-M}^{M}2u_t(x,s)\,u_{tt}(x,s)\,dx=\int_{-M}^{M}2u_t(x,t_0)\,u_{tt}(x,t_0)\,dx=\int_{-\infty}^{\infty}2u_t(x,t_0)\,u_{tt}(x,t_0)\,dx\end{aligned}$$

and we have

$$\left.\frac{d}{dt}\right|_{t=t_0}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx=\int_{-\infty}^{\infty}2u_t(x,t_0)\,u_{tt}(x,t_0)\,dx,$$

where $t_0\in(-\infty,\infty)$ *is fixed but can be arbitrary. The proof is done.*

Another energy property for the wave equation is the following:

Lemma 5.25 *(**Conservation of energy for wave equations with fixed-end condition.**) Assume* $u\in C^2(\mathbb{R}^2)$ *is a solution of the wave equation* $u_{tt}(x,t)=c^2u_{xx}(x,t)$ *and there is some* $L>0$ *such that*

$$u(0,t)=u(L,t)=0,\quad\forall\ t\in(-\infty,\infty),\tag{139}$$

i.e. $u(x,t)$ *satisfies "**fixed-end**" condition, then the **total energy over the interval*** $[0,L]$:

$$\frac{1}{2}\left(\int_0^L u_t^2(x,t)\,dx+c^2\int_0^L u_x^2(x,t)\,dx\right),\quad t\in(-\infty,\infty)\tag{140}$$

*is **independent** of time.*

Proof. We first note that by (139) we have $u_t(0,t)=u_t(L,t)=0$ for all $t\in(-\infty,\infty)$. Compute

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left(\int_0^L u_t^2(x,t)\,dx+c^2\int_0^L u_x^2(x,t)\,dx\right)\right]\\&=\int_0^L u_tu_{tt}dx+c^2\int_0^L u_xu_{xt}dx=c^2\int_0^L(u_tu_{xx}+u_xu_{xt})\,dx\\&=c^2\int_0^L\frac{\partial}{\partial x}(u_tu_x)\,dx=c^2\underbrace{u_t(L,t)}u_x(L,t)-c^2\underbrace{u_t(0,t)}u_x(0,t)=0,\quad\forall t\in(-\infty,\infty).\end{aligned}$$

Hence the total energy over the interval $[0,L]$ is independent of time. □

The above will be the coverage of the midterm exam on 2023-4-13.

6 Nonhomogeneous wave equation.

We now consider the nonhomogeneous wave equation

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), \quad (x,t) \in \mathbb{R}^2 \tag{141}$$

with initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty, \infty). \tag{142}$$

Here the functions $f(x,t) \in C^1(\mathbb{R}^2)$, $\phi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$ are all given.

Remark 6.1 *(**Important.**) Note that here we assume $f(x,t) \in C^1(\mathbb{R}^2)$ instead of $f(x,t) \in C^0(\mathbb{R}^2)$. We will explain the reason later on.*

By linearity, it suffices to look at the case when $\phi(x) = \psi(x) = 0$ due to the following observation:

Lemma 6.2 *If $w(x,t) \in C^2(\mathbb{R}^2)$ solves the problem*

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ w(x,0) = 0, \quad w_t(x,0) = 0, & x \in \mathbb{R} \end{cases} \tag{143}$$

and $v(x,t) \in C^2(\mathbb{R}^2)$ solves the problem

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = 0, & (x,t) \in \mathbb{R}^2 \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{144}$$

then $u(x,t) = w(x,t) + v(x,t) \in C^2(\mathbb{R}^2)$ solves the problem

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{145}$$

Since we already know how to solve (144), it suffices to focus on (143). We want to derive a **solution formula** for $u(x,t) \in C^2(\mathbb{R}^2)$ satisfying (143).

We shall use the **change of variables method (characteristic coordinates method)** to solve (143). This method is quite **straightforward** and **natural**. We first note the following simple fact:

Lemma 6.3 *Let $p(x,y)$ be a C^1 function defined on $\mathbb{R}^2$. Then the following two conditions are equivalent: satisfying*

$$\begin{cases} (1).\ p(\lambda,\lambda) = 0 \quad \text{and} \quad p_x(\lambda,\lambda) = p_y(\lambda,\lambda), \quad \forall\, \lambda \in \mathbb{R} \\ (2).\ p(\lambda,\lambda) = 0 \quad \text{and} \quad p_x(\lambda,\lambda) = 0, \quad \forall\, \lambda \in \mathbb{R}. \end{cases} \tag{146}$$

Proof. Assume $p(\lambda,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. By the chain rule we have

$$0 = \frac{d}{d\lambda} p(\lambda,\lambda) = \frac{\partial p}{\partial x}(\lambda,\lambda)\frac{d\lambda}{d\lambda} + \frac{\partial p}{\partial y}(\lambda,\lambda)\frac{d\lambda}{d\lambda} = p_x(\lambda,\lambda) + p_y(\lambda,\lambda). \tag{147}$$

By (147), we see that the two conditions in (146) are equivalent. Moreover, both conditions imply

$$p_x(\lambda,\lambda) = p_y(\lambda,\lambda) = 0, \quad \forall\, \lambda \in \mathbb{R}. \tag{148}$$

The proof is done. $\square$

Recall that (see Section 2.2) when we solve the homogeneous wave equation $u_{tt} - c^2 u_{xx} = 0$, we can use the change of variables (by use of the two characteristic lines)

$$\xi = x + ct, \qquad \eta = x - ct \tag{149}$$

to reduce the equation into a simple form (let $\tilde{u}(\xi, \eta) = u(x, t)$). By

$$\begin{cases} u_t = c\tilde{u}_\xi - c\tilde{u}_\eta, \quad u_x = \tilde{u}_\xi + \tilde{u}_\eta \\ u_{tt} = c^2 \tilde{u}_{\xi\xi} - 2c^2 \tilde{u}_{\xi\eta} + c^2 \tilde{u}_{\eta\eta}, \quad u_{xx} = \tilde{u}_{\xi\xi} + 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}, \end{cases} \tag{150}$$

the simple new equation for $\tilde{u}(\xi, \eta)$ is

$$-4c^2 \tilde{u}_{\xi\eta}(\xi, \eta) = 0, \tag{151}$$

where, in (151), we have used the identity $\tilde{u}_{\xi\eta} = \tilde{u}_{\eta\xi}$ for a C^2 solution. Hence we obtain the general solution for $\tilde{u}(\xi, \eta)$:

$$\tilde{u}(\xi, \eta) = F(\xi) + G(\eta) \tag{152}$$

and then the general C^2 solution of $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x, t) = F(x + ct) + G(x - ct) \tag{153}$$

for arbitrary C^2 functions $F(z)$ and $G(z)$ defined on $(-\infty, \infty)$.

For the nonhomogeneous equation $u_{tt}(x, t) - c^2 u_{xx}(x, t) = f(x, t)$ with initial conditions in (143), we can **do the same change of variables** and the equation for $\tilde{u}(\xi, \eta)$ becomes

$$-4c^2 \tilde{u}_{\xi\eta}(\xi, \eta) = f\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right). \tag{154}$$

To solve (154), **we also need to convert the initial conditions for $u(x, t)$ into the initial conditions for $\tilde{u}(\xi, \eta)$.** Since we have $u(x, t) = \tilde{u}(x + ct, x - ct)$, the condition $u(x, 0) = 0$ for all $x \in (-\infty, \infty)$ becomes

$$\tilde{u}(\lambda, \lambda) = 0, \quad \forall\, \lambda \in (-\infty, \infty). \tag{155}$$

Next, by

$$u_t(x, t) = c\tilde{u}_\xi(x + ct, x - ct) - c\tilde{u}_\eta(x + ct, x - ct), \tag{156}$$

the condition $u_t(x, 0) = 0$ for all $x \in (-\infty, \infty)$ becomes

$$\tilde{u}_\xi(\lambda, \lambda) = \tilde{u}_\eta(\lambda, \lambda), \quad \forall\, \lambda \in (-\infty, \infty). \tag{157}$$

Now by Lemma 6.3, the conditions in (155) and (157) are equivalent to the following:

$$(1).\ \tilde{u}(\lambda, \lambda) = 0, \qquad (2).\ \tilde{u}_\xi(\lambda, \lambda) = 0, \qquad \forall\, \lambda \in (-\infty, \infty). \tag{158}$$

Thus the initial value problem (143) for the new function $\tilde{u}(\xi, \eta)$ becomes the following equation **with condition on the diagonal line**:

$$\begin{cases} \tilde{u}_{\xi\eta}(\xi, \eta) = \dfrac{\partial}{\partial \eta}(\tilde{u}_\xi(\xi, \eta)) = -\dfrac{1}{4c^2} f\left(\dfrac{\xi + \eta}{2}, \dfrac{\xi - \eta}{2c}\right), \quad \forall\ (\xi, \eta) \in \mathbb{R}^2 \\ \tilde{u}(\xi, \xi) = 0, \quad \tilde{u}_\xi(\xi, \xi) = 0, \quad \forall\, \xi \in \mathbb{R}. \end{cases} \tag{159}$$

We first integrate the equation with respect to η to get $\tilde{u}_\xi(\xi,\eta)$. We need to take the condition $\tilde{u}_\xi(\xi,\xi)=0$ into consideration. In calculus, if we want to find a one-variable function $H(\eta)$ satisfying $H'(\eta)=g(\eta),\ H(\xi)=0$, then the answer is **unique** and is given by

$$H(\eta)=\int_\xi^\eta g(\sigma)\,d\sigma. \tag{160}$$

Motivated by the above, we write the equation in (159) as

$$\frac{\partial}{\partial\theta}\left(\tilde{u}_\xi(\xi,\theta)\right)=-\frac{1}{4c^2}f\left(\frac{\xi+\theta}{2},\ \frac{\xi-\theta}{2c}\right)$$

and apply the integral $\int_\xi^\eta d\sigma$ onto it to get

$$\tilde{u}_\xi(\xi,\eta)=\frac{\partial}{\partial\xi}\tilde{u}(\xi,\eta)=-\frac{1}{4c^2}\int_\xi^\eta f\left(\frac{\xi+\theta}{2},\ \frac{\xi-\theta}{2c}\right)d\theta,\quad \tilde{u}_\xi(\xi,\xi)=0. \tag{161}$$

Next, we integrate the above with respect to ξ to get $\tilde{u}(\xi,\eta)$ and we need to take the condition $\tilde{u}(\xi,\xi)=0$ into consideration. By the same trick as in (160), if we want to find $G(\xi)$ satisfying $G'(\xi)=p(\xi),\ G(\eta)=0$, then the answer is **unique** and is given by

$$G(\xi)=\int_\eta^\xi p(r)\,dr. \tag{162}$$

Therefore, by similar trick, we write the equation in (161) as

$$\frac{\partial}{\partial r}\tilde{u}(r,\eta)=-\frac{1}{4c^2}\int_r^\eta f\left(\frac{r+\theta}{2},\ \frac{r-\theta}{2c}\right)d\theta$$

and apply the integral $\int_\eta^\xi dr$ onto it to get

$$\tilde{u}(\xi,\eta)=\int_\eta^\xi\left[-\frac{1}{4c^2}\int_r^\eta f\left(\frac{r+\theta}{2},\ \frac{r-\theta}{2c}\right)d\theta\right]dr, \tag{163}$$

which is the same as

$$\tilde{u}(\xi,\eta)=\frac{1}{4c^2}\int_\eta^\xi\left[\int_\eta^r f\left(\frac{r+\theta}{2},\ \frac{r-\theta}{2c}\right)d\theta\right]dr,\quad \forall\ (\xi,\eta)\in\mathbb{R}^2. \tag{164}$$

It satisfies

$$\tilde{u}(\xi,\xi)=0,\quad \tilde{u}_\xi(\xi,\xi)=0,\quad \forall\ \xi\in\mathbb{R}. \tag{165}$$

Therefore, we conclude that the unique solution of problem (159) is given by the formula (164). To get a better picture for the domain of integration in (ξ,η)-plane, we write (164) as

$$\tilde{u}(\xi_0,\eta_0)=\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\left[\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\eta\right]d\xi,\quad \forall\ (\xi_0,\eta_0)\in\mathbb{R}^2 \tag{166}$$

or as the double integral

$$\tilde{u}(\xi_0,\eta_0)=\frac{1}{4c^2}\iint_{\tilde{\triangle}} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\xi d\eta. \tag{167}$$

The domain of integration $\tilde{\triangle}$ in the (ξ,η)-plane for the above double integral is given by (for convenience, in the picture below we assume $\eta_0<\xi_0$)

$\tilde{\triangle}$: DRAW A PICTURE HERE IN (ξ,η)-PLANE !!!.

We note that $\tilde{\triangle}$ is the region inside a **right triangle** bounded by the three lines $L_1 : \xi = \xi_0,\ L_2 : \eta = \eta_0$, and $L_3 : \xi = \eta$ (the point (ξ_0, η_0) is on $L_1 \bigcap L_2$). Going back to (x, t)-plane, by the relation

$$\xi = x + ct, \quad \eta = x - ct \qquad \Longleftrightarrow \qquad x = \frac{\xi + \eta}{2}, \quad t = \frac{\xi - \eta}{2c},$$

we have the following correspondence (denote the point corresponding to (ξ_0, η_0) as (x_0, t_0)):

$$\begin{cases} (1).\ (\xi_0, \eta_0) = (x_0 + ct_0, x_0 - ct_0) \Longleftrightarrow (x_0, t_0) = \left(\frac{\xi_0+\eta_0}{2}, \frac{\xi_0-\eta_0}{2c}\right) \text{ (top point of the triangle)}, \\ (2).\ L_1 : \xi = \xi_0 \Longleftrightarrow \text{ line } \ x + ct = x_0 + ct_0 \text{ (characteristic line with negative slope)}, \\ (3).\ L_2 : \eta = \eta_0 \Longleftrightarrow \text{ line } \ x - ct = x_0 - ct_0 \text{ (characteristic line with positive slope)}, \\ (4).\ L_3 : \xi = \eta \Longleftrightarrow \text{ line } t = 0 \text{ (the } x\text{-axis)}. \end{cases} \tag{168}$$

Denote the region of (168) as $\triangle \subset (x, t)$ plane, which is a triangle with top point at (x_0, t_0). Its picture is given by

$\triangle$: DRAW A PICTURE HERE IN (x, t)-PLANE !!!.

As a consequence of the above, from Advanced Calculus, we have the change of variables identity for **double integrals** (the **absolute value** of the Jacobian $J = \partial(\xi, \eta)/\partial(x, t)$ is $2c,\ c > 0$) :

$$\tilde{u}(\xi_0, \eta_0) = \frac{1}{4c^2} \iint_{\tilde{\triangle}} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\xi d\eta = \frac{1}{2c} \iint_{\triangle} f(x, t)\, dxdt = u(x_0, t_0). \tag{169}$$

The last step is to express the double integral $\iint_\triangle f(x, t)\, dxdt$ as an **iterated integral**. Based on the shape of $\triangle$, it is easier to integrate **with respect to** x **first and then with respect to** t. We have

$$u(x_0, t_0) = \frac{1}{2c} \iint_{\triangle} f(x, t)\, dxdt = \frac{1}{2c} \int_0^{t_0} \left(\int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(x, t)\, dx \right) dt, \quad (x_0, t_0) \in \mathbb{R}^2. \tag{170}$$

Back to general $u(x, t)$, we can express the solution as

$$u(x, t) = \frac{1}{2c} \int_0^t \left(\int_{x - c(t-s)}^{x + c(t-s)} f(\theta, s)\, d\theta \right) ds, \quad (x, t) \in \mathbb{R}^2. \tag{171}$$

At this moment, we are ready to state **the main theorem in this section**, which is:

Theorem 6.4 ***(Solution for nonhomogeneous wave equation with zero initial data.)*** *Assume $f \in C^1(\mathbb{R}^2)$. Then the function $u(x, t)$ given by*

$$u(x, t) = \frac{1}{2c} \int_0^t \left(\int_{x - c(t-s)}^{x + c(t-s)} f(\theta, s)\, d\theta \right) ds, \quad (x, t) \in \mathbb{R}^2 \tag{172}$$

lies in the space $C^2(\mathbb{R}^2)$ and is a C^2 ***solution*** *of the initial value problem*

$$\begin{cases} u_{tt}(x, t) - c^2 u_{xx}(x, t) = f(x, t), \quad (x, t) \in \mathbb{R}^2 \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad x \in \mathbb{R} \end{cases} \tag{173}$$

on the domain $(x, t) \in \mathbb{R}^2$. Moreover, the $C^2(\mathbb{R}^2)$ solution of the problem (173) is uniquely given by (172).

Proof. One first give a direct proof that, for $f \in C^1$, the function $u(x,t)$ given by (172) is indeed a C^2 **solution** of problem (173) defined on $\mathbb{R}^2$. For this purpose, we recall the following **derivative formula** from Calculus: Assume $\alpha(x),\ \beta(x) : [a,b] \to [c,d]$ are **differentiable** with respect to $x \in [a,b]$ and $h(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ is a C^1 function. Then we have the identity (also see Remark 6.5 below)

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} h(x,y)\,dy = h(x,\beta(x))\,\beta'(x) - h(x,\alpha(x))\,\alpha'(x) + \int_{\alpha(x)}^{\beta(x)} h_x(x,y)\,dy. \tag{174}$$

Now let $u(x,t)$ be the function given by (172). We first note that

$$\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta \quad \text{is} \quad C^1 \quad \text{in} \quad s \in (-\infty,\infty), \qquad f \in C^1\left(\mathbb{R}^2\right) \tag{175}$$

and

$$\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta \quad \text{is} \quad C^2 \quad \text{in} \quad (x,t) \in \mathbb{R}^2, \quad f \in C^1\left(\mathbb{R}^2\right). \tag{176}$$

We clearly have

$$u(x,0) = \frac{1}{2c}\int_0^0 (\cdots)\,ds = 0, \quad \forall\, x \in \mathbb{R} \tag{177}$$

and

$$\begin{aligned} u_x(x,t) &= \frac{1}{2c}\frac{d}{dx}\left[\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta\right) ds\right] = \frac{1}{2c}\int_0^t \frac{\partial}{\partial x}\left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta\right) ds \\ &= \frac{1}{2c}\int_0^t \left[f(x+c(t-s),s) - f(x-c(t-s),s)\right] ds \end{aligned} \tag{178}$$

and

$$u_{xx}(x,t) = \frac{1}{2c}\int_0^t \underbrace{\frac{\partial}{\partial x}\left[f(x+c(t-s),s) - f(x-c(t-s),s)\right]}\,ds. \tag{179}$$

On the other hand, by the identity (174), we have

$$\begin{aligned} u_t(x,t) &= \frac{1}{2c}\frac{d}{dt}\int_0^t (h(x,t,s))\,ds, \quad \text{where} \quad h(x,t,s) = \int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta \\ &= h(x,t,t)\ \text{(this is zero)} + \frac{1}{2c}\int_0^t \frac{\partial}{\partial t}(h(x,t,s))\,ds \\ &= \frac{1}{2c}\int_0^t \frac{\partial}{\partial t}\left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta\right) ds \\ &= \frac{1}{2c}\int_0^t \left[f(x+c(t-s),s)\,c - f(x-c(t-s),s)\,(-c)\right] ds \\ &= \frac{1}{2}\int_0^t \left[f(x+c(t-s),s) + f(x-c(t-s),s)\right] ds, \end{aligned} \tag{180}$$

which gives

$$u_t(x,0) = \frac{1}{2c}\int_0^0 (\cdots)\,ds = 0, \quad \forall\, x \in \mathbb{R} \tag{181}$$

and also

$$u_{tt}(x,t) = f(x,t) + \frac{1}{2}\int_0^t \underbrace{\frac{\partial}{\partial t}\left[f(x+c(t-s),s) + f(x-c(t-s),s)\right]}\,ds. \tag{182}$$

By the identity

$$\underbrace{\frac{\partial}{\partial t}\left[f\left(x+c\left(t-s\right),s\right)+f\left(x-c\left(t-s\right),s\right)\right]}=c\underbrace{\frac{\partial}{\partial x}\left[f\left(x+c\left(t-s\right),s\right)-f\left(x-c\left(t-s\right),s\right)\right]},$$

we conclude

$$u_{tt}\left(x,t\right)=f\left(x,t\right)+c^{2}u_{xx}\left(x,t\right),\quad \left(x,t\right)\in\mathbb{R}^{2}. \tag{183}$$

The check is done.

Now we check uniqueness. If there is another C^2 solution $w(x,t)$ of the problem (173) on $\mathbb{R}^2$, we can look at the difference

$$v\left(x,t\right)=\frac{1}{2c}\int_{0}^{t}\left(\int_{x-c(t-s)}^{x+c(t-s)}f\left(\theta,s\right)d\theta\right)ds-w\left(x,t\right),\quad \left(x,t\right)\in\mathbb{R}^{2}.$$

Then $v\in C^{2}\left(\mathbb{R}^{2}\right)$ and it satisfies

$$\begin{cases} v_{tt}\left(x,t\right)-c^{2}v_{xx}\left(x,t\right)=0, & \left(x,t\right)\in\mathbb{R}^{2}\\ v\left(x,0\right)=0,\quad v_{t}\left(x,0\right)=0, & x\in\mathbb{R}.\end{cases}$$

By Lemma 5.6, we must have $v(x,t)\equiv 0$ on $\mathbb{R}^2$. The proof is done. □

Remark 6.5 ***(Change the order of differentiation and integration.)*** *If $h(x,y)$ and $\frac{\partial h}{\partial x}(x,y)$ are both in $C^{0}\left([a,b]\times[c,d]\right)$, then the function*

$$H\left(x\right):=\int_{c}^{d}h\left(x,y\right)dy,\quad x\in\left[a,b\right]$$

is differentiable with respect to $x\in[a,b]$ and satisfies

$$H'\left(x\right)=\frac{d}{dx}\left(\int_{c}^{d}h\left(x,y\right)dy\right)=\int_{c}^{d}\frac{\partial h}{\partial x}\left(x,y\right)dy,\quad\forall\ x\in\left[a,b\right]. \tag{184}$$

In particular, we note that $H'(x)$ is continuous on $[a,b]$ due to $\frac{\partial h}{\partial x}\in C^{0}\left([a,b]\times[c,d]\right)$.

Remark 6.6 ***(Important observation.)*** *In calculus, if $f(s)\in C^{0}(\mathbb{R})$ is a* ***continuous*** *function on $\mathbb{R}$, and we* ***integrate it twice over*** *$\mathbb{R}$, then the result will be a C^2 function on $\mathbb{R}$, i.e.*

$$u\left(t\right)=\int_{0}^{t}\left(\int_{0}^{\rho}f\left(s\right)ds\right)d\rho\in C^{2}\left(\mathbb{R}\right). \tag{185}$$

Note that here $\dim\mathbb{R}=1$ and the times of integration over the domain space $\mathbb{R}$ is 2. However, if $f\in C^{0}\left(\mathbb{R}^{2}\right)$ and we look at

$$u\left(x,t\right)=\frac{1}{2c}\int_{0}^{t}\left(\int_{x-c(t-s)}^{x+c(t-s)}f\left(\theta,s\right)d\theta\right)ds=\frac{1}{2c}\iint_{\triangle}f\left(\theta,s\right)d\theta ds\ \textit{(top point of } \triangle \textit{ is } (x,t)\,), \tag{186}$$

then we have $\dim\mathbb{R}^{2}=2$ and the times of integration over the domain space $\mathbb{R}^2$ is 1 (2 iterated integrals over $\mathbb{R}$ is equal to 1 ***double integral*** *over the domain space $\mathbb{R}^2$).* ***Therefore, we expect*** *$u(x,t)$* ***given by (186) to lie in the space*** *$C^{1}\left(\mathbb{R}^{2}\right)$* ***only (and this is so in general !!!).*** *Finally, if we assume $f\in C^{1}\left(\mathbb{R}^{2}\right)$, then $u(x,t)$ given by (186) will lie in the space $C^{2}\left(\mathbb{R}^{2}\right)$. In Evans PDE book, p. 81, Theorem 4, the author also assumes that $f\in C^{1}\left(\mathbb{R}^{2}\right)$ in order for $u(x,t)$ to be a C^2 solution of the problem (173) on $\mathbb{R}^2$.*

Remark 6.7 *(**Important observation.**) Another way to see that if $f \in C^0(\mathbb{R}^2)$, then the function $u(x,t)$ in (186) will, in general, lie in the space $C^1(\mathbb{R}^2)$ only. This is because $f(\theta,s)$ is continuous in θ and we only do integral with respect to θ **once**. Similarly, $f(\theta,s)$ is continuous in s and we only do integral with respect to s **once**.*

Example 6.8 *For simplicity of computation, we assume $f \in C^0(\mathbb{R}^2)$ and look at the double integral over a **rectangle**:*

$$u(x,t) = \iint_{\Box} f(\theta,s)\, d\theta ds \ \textit{(top right point of } \Box \textit{ is } (x,t)\textit{)},$$

where $\Box$ is the domain $[0,x] \times [0,t] \subset \mathbb{R}^2$. In terms of iterated integrals, it is equal to

$$u(x,t) = \iint_{\Box} f(\theta,s)\, d\theta ds = \int_0^t \int_0^x f(\theta,s)\, d\theta ds$$

and if we choose $f(\theta,s) = |\theta s| = |\theta|\,|s| \in C^0(\mathbb{R}^2)$, we get

$$u(x,t) = \iint_{\Box} |\theta|\,|s|\, d\theta ds = \int_0^t \int_0^x |\theta|\,|s|\, d\theta ds = \int_0^t |s|\, ds \cdot \int_0^x |\theta|\, d\theta,$$

which satisfies

$$u_x(x,t) = |x| \int_0^t |s|\, ds, \quad u_t(x,t) = |t| \int_0^x |\theta|\, d\theta, \quad u_{xt}(x,t) = u_{tx}(x,t) = |x|\,|t|.$$

*However, $u_{xx}(x,t)$ does not exist at $x = 0$ and $u_{tt}(x,t)$ does not exist at $t = 0$. $u(x,t)$ is a C^1 function on $\mathbb{R}^2$, but **not a C^2 function on** $\mathbb{R}^2$.*

By Theorem 6.4, we can state the following:

Theorem 6.9 *Consider the following initial value problem for nonhomogeneous wave equation:*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{187}$$

*where $f \in C^1(\mathbb{R}^2)$, $\phi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$, are given functions. Then the solution $u(x,t) \in C^2(\mathbb{R}^2)$ of (187) is **unique**, defined on $\mathbb{R}^2$, and is given by the formula*

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds \\ +\frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\, d\theta\right) ds, \quad (x,t) \in \mathbb{R}^2. \end{cases} \tag{188}$$

Remark 6.10 *(**Important.**) By the representation formula (188), we can conclude the following: The **domain of dependence** of the solution u at the point (x_0,t_0), $t_0 > 0$, is given by a **compact triangular set** $\triangle$ in the upper (x,t) plane, enclosed by the three lines:*

$$\textit{line } t = 0 \textit{ (x-axis)}; \quad \textit{line } x + ct = x_0 + ct_0; \quad \textit{line } x - ct = x_0 - ct_0. \tag{189}$$

*Note that in case $t_0 < 0$, the description (189) is still correct. Now the **compact triangular set** ∇ is in the lower (x,t) plane.*

In the above lemma we assume $f \in C^1(\mathbb{R}^2)$ so that $u \in C^2(\mathbb{R}^2)$. **However, if $f(x,t)$ depends only on x (or only on t), then it suffices to assume that $f \in C^0(\mathbb{R})$. We have:**

Corollary 6.11 *Assume $f(x,t)$ in (173) is a* ***continuous*** *function depending only on x, i.e., $f(x,t) = f(x) \in C^0(\mathbb{R})$. Let $G(x) \in C^2(\mathbb{R})$ be a function satisfying $G''(x) = f(x)$ for all $x \in \mathbb{R}$, then the function $u(x,t)$ in (172) has the form*

$$u(x,t) = \frac{1}{2c^2}\left[G(x+ct) + G(x-ct) - 2G(x)\right], \qquad (x,t) \in \mathbb{R}^2, \tag{190}$$

where $u \in C^2(\mathbb{R}^2)$. Note that the solution $u(x,t)$ depends on both space and time.

Remark 6.12 *One can check that the solution (190) does not depend on the function $G(x)$ as long as it satisfies $G''(x) = f(x)$ for all $x \in \mathbb{R}$. That is, if we replace $G(x)$ by $\tilde{G}(x) = G(x) + ax + b$ for some constants a, b, then we still have $\tilde{G}''(x) = f(x)$. Using this $\tilde{G}(x)$ in (190) will give the same answer $u(x,t)$.*

Proof. The proof is straightforward. Let $F(x)$ be such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$ and $G(x)$ be such that $G'(x) = F(x)$ for all $x \in \mathbb{R}$ (and so $G''(x) = f(x)$). We have

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta)\, d\theta\right) ds = \frac{1}{2c}\int_0^t \left[F(x+c(t-s)) - F(x-c(t-s))\right] ds.$$

By

$$\frac{d}{ds}\left(\frac{-1}{c}G(x+c(t-s))\right) = F(x+c(t-s)), \quad \frac{d}{ds}\left(\frac{1}{c}G(x-c(t-s))\right) = F(x-c(t-s)),$$

we get

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left\{\left(\frac{-1}{c}G(x+c(t-s))\right)\Big|_{s=0}^{s=t} - \left(\frac{1}{c}G(x-c(t-s))\right)\Big|_{s=0}^{s=t}\right\} \\ &= -\frac{1}{2c^2}G(x) + \frac{1}{2c^2}G(x+ct) - \frac{1}{2c^2}G(x) + \frac{1}{2c^2}G(x-ct) \\ &= \frac{1}{2c^2}\left[G(x+ct) + G(x-ct) - 2G(x)\right], \qquad (x,t) \in \mathbb{R}^2. \end{aligned}$$

The proof is done. □

Corollary 6.13 *Assume $f(x,t)$ in (173) is a* ***continuous*** *function depending only on t, i.e., $f(x,t) = f(t) \in C^0(\mathbb{R})$. Then the function $u(x,t)$ in (172) has the form*

$$u(x,t) = \int_0^t f(s)(t-s)\, ds = \int_0^t \left(\int_0^\rho f(s)\, ds\right) d\rho \in C^2(\mathbb{R}^2), \qquad \forall\ (x,t) \in \mathbb{R}^2. \tag{191}$$

Note that $u(x,t)$ depends only on time t and is a C^2 function of $t \in \mathbb{R}$.

Proof. By (172), we have

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\, d\theta\right) ds = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(s)\, d\theta\right) ds = \underbrace{\int_0^t f(s)(t-s)\, ds}.$$

We have

$$\begin{cases} u_t(x,t) = f(t)(t-t) + \int_0^t \frac{\partial}{\partial t}\left(f(s)(t-s)\right) ds = \int_0^t f(s)\, ds, \quad u_{tt}(x,t) = f(t), \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u_x(x,t) = u_{xx}(x,t) = 0, \quad u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(t). \end{cases}$$

□

Remark 6.14 ***(Interesting observation !!)*** *A remarkable thing is that for a* ***continuous*** *function* $f(t) \in C^0(\mathbb{R})$, *the function* $u(t) := \int_0^t f(s)(t-s)\,ds$ *is a* C^2 *function of* $t \in (-\infty, \infty)$ *(not just a* C^1 *function). Note that* $\int_0^t f(s)(t-s)\,ds$ *is a* ***convolution-type integral*** *and we have*

$$\frac{d}{dt}\int_0^t f(s)(t-s)\,ds = \int_0^t f(s)\,ds \quad \text{and} \quad \frac{d^2}{dt^2}\int_0^t f(s)(t-s)\,ds = f(t), \quad \forall\, t \in \mathbb{R}. \tag{192}$$

On the other hand, since $u(t)$ *satisfies* $u''(t) = f(t)$, $u(0) = 0$, $u'(0) = 0$, *we can also express* $u(t)$ *as the* ***double integral***

$$u(t) = \int_0^t \left(\int_0^\rho f(s)\,ds\right) d\rho, \quad \forall\, t \in \mathbb{R}. \tag{193}$$

Therefore, we have

$$\int_0^t f(s)(t-s)\,ds = \int_0^t \left(\int_0^\rho f(s)\,ds\right) d\rho, \tag{194}$$

which can be verified using the ***change of order of integration in the*** ρs***-plane****, i.e.*

$$\int_0^t \left(\int_0^\rho f(s)\,ds\right) d\rho = \int_0^t \left(\int_s^t f(s)\,d\rho\right) ds = \int_0^t f(s)(t-s)\,ds. \tag{195}$$

From (194), we see that $\int_0^t f(s)(t-s)\,ds$ *is a* C^2 *function of* $t \in (-\infty, \infty)$.

This is the end of second-order equations