PDE Teaching for Spring, 2022

Revised on 2022-4-14

Remark 0.1 *This notes is based on "Lecture-notes-on-PDE-first-part-2022.tex".*

1 Second order linear PDE with constant coefficients; classification and canonical form.

We now consider the linear second order equation with **constant coefficients**, given by

$$au_{xx}+2bu_{xy}+cu_{yy}+2du_x+2eu_y+ku=f(x,y), \quad u=u(x,y) \tag{1}$$

where $a,\ \ldots,\ k$ are all **constants** with $a^2+b^2+c^2>0$ and $f(x,y)$ is a given function defined on some open set $\Omega\subseteq\mathbb{R}^2$. We want to find a C^2 function $u(x,y)$ satisfying (1) on some open set (may be just a subset of Ω). Note that for a C^2 function $u(x,y)$, we have $u_{xy}(x,y)=u_{yx}(x,y)$ on its domain.

Remark 1.1 *Note that if $u=u(x_1,\ \ldots\ ,x_n)$ depends on n variables, the discussions below are similar. For convenience, we assume that $u=u(x,y)$ depends only on 2 variables.*

We can write (1) in the matrix form as

$$Trace\left[\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\right]+2\begin{pmatrix} d\\ e\end{pmatrix}\cdot\begin{pmatrix} u_x\\ u_y\end{pmatrix}+ku=f(x,y), \tag{2}$$

which is helpful for us to understand the effect of the change of variables. As we shall see soon, the "**type**" of the equation (1) is determined by the **sign** of the **determinant** of the coefficient matrix.

Remark 1.2 *We shall use the notation TrA to denote the trace of a square $n\times n$ matrix $A\in M(n)$. The basic properties of the trace operator are*

$$\begin{cases} Tr(c_1A+c_2B)=c_1Tr(A)+c_2Tr(B)\\ Tr(A)=Tr\left(A^T\right), \quad Tr\left(P^{-1}AP\right)=Tr(A), \quad Tr(AB)=Tr(BA), \end{cases}$$

where $A,\ B,\ P\in M(n)$, P is invertible, and $c_1,\ c_2\in\mathbb{R}$. However, unlike $\det(AB)=\det(A)\det(B)$, *we do not have $Tr(AB)=Tr(A)Tr(B)$.*

Lemma 1.3 *Assume $u(x,y)$ is a C^2 function defined on some domain $\Omega\subseteq\mathbb{R}^2$. If we introduce the linear change of variables given by*

$$\xi=\xi(x,y)=Ax+By, \qquad \eta=\eta(x,y)=Cx+Dy, \qquad A,\ B,\ C,\ D \text{ are all const.,} \tag{3}$$

i.e.,

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}=J\begin{pmatrix}x\\ y\end{pmatrix}, \quad \text{where} \quad J=\begin{pmatrix}A & B\\ C & D\end{pmatrix}, \quad \det J\neq 0, \tag{4}$$

then the equation (2) for the function $U(\xi,\eta)$ (where $U(Ax+By,Cx+Dy)=u(x,y)$) becomes

$$Tr\left[\underbrace{J\begin{pmatrix} a & b\\ b & c\end{pmatrix}J^T}\cdot\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta}\\ U_{\xi\eta} & U_{\eta\eta}\end{pmatrix}\right]+(\text{lower order terms})=F(\xi,\eta), \tag{5}$$

where $Tr(\cdot)$ is the trace of a matrix.

Remark 1.4 *Denote*

$$M=\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad N=J\begin{pmatrix} a & b \\ b & c \end{pmatrix}J^T, \quad \det J\neq 0. \tag{6}$$

*We see that the matrix N is also **symmetric**. By a theorem from linear algebra, **all eigenvalues of both** M **and** N **are real.** Moreover, the **sign** of $\det M$ and $\det N$ are the **same** due to $\det N = (\det J)^2 \det M$, where $\det J\neq 0$. In particular, the **sign** of the **eigenvalues** $\lambda_1,\ \lambda_2$ are unchanged under the change of variables. Finally, if J is an **orthogonal** matrix (i.e. $J^T=J^{-1}$), then both M and N are **similar** and have the **same** eigenvalues. Our goal is to **diagonalize** M (i.e. make N to be **diagonal**), which will reduce equation (1) into **canonical form**.*

Proof. We have

$$U(Ax+By, Cx+Dy)=u(x,y),$$

and by the chain rule we have

$$u_x=AU_\xi+CU_\eta, \quad u_y=BU_\xi+DU_\eta, \quad \begin{pmatrix} u_x \\ u_y \end{pmatrix}=J^T\begin{pmatrix} U_\xi \\ U_\eta \end{pmatrix}, \tag{7}$$

which is equivalent to the operator identities:

$$\frac{\partial}{\partial x}=A\frac{\partial}{\partial \xi}+C\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial y}=B\frac{\partial}{\partial \xi}+D\frac{\partial}{\partial \eta},$$

which can be written as

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}=J\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}, \quad J=\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{8}$$

One step furthermore, we get

$$\begin{cases} \frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)=\left(A\frac{\partial}{\partial \xi}+C\frac{\partial}{\partial \eta}\right)(\cdots)=A^2\frac{\partial^2}{\partial \xi^2}+2AC\frac{\partial^2}{\partial \xi\partial\eta}+C^2\frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)=\left(A\frac{\partial}{\partial \xi}+C\frac{\partial}{\partial \eta}\right)(\cdots)=AB\frac{\partial^2}{\partial \xi^2}+(AD+BC)\frac{\partial^2}{\partial \xi\partial\eta}+CD\frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)=\left(B\frac{\partial}{\partial \xi}+D\frac{\partial}{\partial \eta}\right)(\cdots)=B^2\frac{\partial^2}{\partial \xi^2}+2BD\frac{\partial^2}{\partial \xi\partial\eta}+D^2\frac{\partial^2}{\partial \eta^2}, \end{cases}$$

i.e. we have

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} \\ \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix}=J^T\begin{pmatrix} \frac{\partial^2}{\partial \xi^2} & \frac{\partial^2}{\partial \xi\partial\eta} \\ \frac{\partial^2}{\partial \xi\partial\eta} & \frac{\partial^2}{\partial \eta^2} \end{pmatrix}J, \tag{9}$$

which means

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix}=J^T\begin{pmatrix} U_\xi \\ U_\eta \end{pmatrix}, \quad \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}=J^T\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}J\ . \tag{10}$$

Thus the equation

$$Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right]+\text{(lower order terms)}=f(x,y)$$

becomes (note that for any two matrices $A,\ B$, we have the identity $Trace\,(AB)=Trace\,(BA)$ in linear algebra)

$$\begin{aligned} &Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\cdot\underline{J^T\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}J}\right]+\text{(lower order terms)} \\ &=Trace\left[\underline{J\begin{pmatrix} a & b \\ b & c \end{pmatrix}J^T}\cdot\begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix}\right]+\text{(lower order terms)}=F(\xi,\eta) \end{aligned} \tag{11}$$

The proof is done. □

Definition 1.5 *Since the matrix*

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is ***symmetric****, it has two* ***real eigenvalues*** λ_1 *and* λ_2*. If both of them are positive (or both are negative), then we say the equation (1) is* ***elliptic*** *(this is equivalent to* $\det M = ac - b^2 > 0$*). If one eigenvalue is positive and the other is negative, we say the equation is* ***hyperbolic*** *(this is equivalent to* $\det M = ac - b^2 < 0$*). If one eigenvalue is zero and the other is nonzero, we say the equation is* ***parabolic*** *(this is equivalent to* $\det M = ac - b^2 = 0$*). Note that by*

$$\det \underbrace{\left(J \begin{pmatrix} a & b \\ b & c \end{pmatrix} J^T \right)} = (\det J)^2 \det \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{where } \det J \neq 0,$$

we see that ***the type of the linear equation (2) is invariant under a linear change of variables.***

Since the matrix M in (6) is **symmetric,** by linear algebra theory, we can find **orthonormal** basis $\{v_1, v_2\}$ (they are **eigenvectors** corresponding to $\lambda_1, \ \lambda_2$) such that

$$P^T M P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad P^T = P^{-1}, \tag{12}$$

where P is the **orthogonal matrix** $P = (v_1, v_2)$ (v_1, v_2 are column vectors of P). Assume that $v_1 = (\alpha, \beta)$ and $v_2 = (p, q)$ and let

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ p & q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ p & q \end{pmatrix} = P^T, \tag{13}$$

i.e. $J = P^T$. We have

$$\begin{aligned} &Trace \left[\underbrace{J \begin{pmatrix} a & b \\ b & c \end{pmatrix} J^T} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] = Trace \left[\underbrace{P^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} P} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] \\ &= Trace \left[\underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] = \underbrace{\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta}}, \end{aligned} \tag{14}$$

which will reduce the leading terms $au_{xx} + 2bu_{xy} + cu_{yy} + \cdots$ of the PDE (1) into **canonical form** !!

Remark 1.6 *(**Omit this in class.**) (**Important.**) In case the original equation (1) is already a Laplace equation* $u_{xx} + u_{yy} = 0$*, then the matrix* $M = I$ *is the identity matrix and* ***any orthogonal matrix*** P *will make*

$$P^T M P = P^T I P = P^T P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = (v_1, v_2).$$

With this, (5) becomes

$$\begin{aligned} &Trace \left[\underbrace{J \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} J^T} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right], \quad J = P^T, \quad JJ^T = P^T P = I \\ &= Trace \left[JJ^T \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] = U_{\xi\xi} + U_{\eta\eta} = 0, \end{aligned} \tag{15}$$

*which means that under **any orthogonal linear change of variables** $(x, y) \longleftrightarrow (\xi, \eta)$, the new equation for $U(\xi, \eta)$ still has the **same** form*

$$U_{\xi\xi} + U_{\eta\eta} = 0. \tag{16}$$

That means if $u(x, y)$ is a solution of the Laplace equation $u_{xx} + u_{yy} = 0$ and $(x, y) \longleftrightarrow (\xi, \eta)$ is an orthogonal linear change of variable, then the corresponding new function $U(\xi, \eta)$ will also satisfy the same equation $U_{\xi\xi} + U_{\eta\eta} = 0$.

Remark 1.7 *(**Omit this in class.**) (**Important.**) The space of all 2×2 **orthogonal matrices** forms a **group**, which we denote it as $O(2)$. It is known that the group $O(2)$ is the set of **all rotations about the origin** O in $\mathbb{R}^2$ (they have $\det = 1$) together with **all reflections with respect to straight lines through the origin** O in $\mathbb{R}^2$ (they have $\det = -1$). In terms of the polar coordinates (r, θ), the first case corresponds to $\theta \to \theta + \theta_0$ (θ_0 may be positive or negative) and the second case, if we do reflection with respect to the x-axis, corresponds to $\theta \to -\theta$. Note that **the Laplace equation under polar coordinates** (r, θ) has the form*

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) U(r, \theta) = 0$$

and one can see that if $U(r, \theta)$ is a solution, so is $U(r, \theta + \theta_0)$ (rotation about the origin); the same for $U(r, -\theta)$ (reflection with respect to the x-axis).

By (14), we can conclude the following **classification** result for equation (1):

Lemma 1.8 *(**Canonical form.**) If the linear equation (2) is **elliptic**, then one can find a suitable linear change of variables (**using eigenvalues and eigenvectors**)*

$$\xi = Ax + By, \quad \eta = Cx + Dy, \quad A,\ B,\ C,\ D \text{ are constants,}$$

so that the equation for $U(\xi, \eta)$ has the form

$$U_{\xi\xi} + U_{\eta\eta} + (\text{lower order terms}) = F(\xi, \eta). \tag{17}$$

*For **hyperbolic** case, the equation has the form*

$$U_{\xi\xi} - U_{\eta\eta} + (\text{lower order terms}) = F(\xi, \eta), \tag{18}$$

*and for **parabolic** case, the equation has the form*

$$U_{\xi\xi} + (\text{lower order terms}) = F(\xi, \eta). \tag{19}$$

Remark 1.9 *The forms in (17), (18) and (19) are said to be in **canonical forms**. Another canonical form of the **hyperbolic** case is*

$$U_{\xi\eta} + (\text{lower order terms}) = F(\xi, \eta). \tag{20}$$

*One can show that an equation of the form $u_{xx} - u_{yy} = 0$ can be converted into an equation of the form $4U_{\xi\eta} = 0$ (by the change of variables $\xi = x + y,\ \eta = x - y$). Therefore, canonical form (18) and (20) are **equivalent**.*

Proof. For the ***elliptic*** case, by multiplying the equation by a minus sign if necessary, we may assume $\lambda_1 > 0,\ \lambda_2 > 0$ (both are eigenvalues of the coefficient matrix). By the change of variables (13), we can convert in into the form

$$\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}) = F(\xi, \eta), \quad \lambda_1 > 0,\ \lambda_2 > 0.$$

If we let

$$\tilde{\xi} = \frac{\xi}{\sqrt{\lambda_1}}, \quad \tilde{\eta} = \frac{\eta}{\sqrt{\lambda_2}}, \quad \tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = U\left(\sqrt{\lambda_1}\tilde{\xi}, \sqrt{\lambda_2}\tilde{\eta}\right), \tag{21}$$

then we have

$$\begin{aligned} &\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}), \quad \text{where} \quad \lambda_1 > 0, \quad \lambda_2 < 0 \\ &= \tilde{U}_{\tilde{\xi}\tilde{\xi}}\left(\tilde{\xi}, \tilde{\eta}\right) + \tilde{U}_{\tilde{\eta}\tilde{\eta}}\left(\tilde{\xi}, \tilde{\eta}\right) + (\text{lower order terms}) = \tilde{F}\left(\tilde{\xi}, \tilde{\eta}\right). \end{aligned}$$

Thus we have arrived at the form (17). For the **hyperbolic** case, we have $\lambda_1 > 0,\ \lambda_2 < 0$. Then we replace (21) by

$$\tilde{\xi} = \frac{\xi}{\sqrt{\lambda_1}}, \quad \tilde{\eta} = \frac{\eta}{\sqrt{-\lambda_2}}, \quad \tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = U\left(\sqrt{\lambda_1}\tilde{\xi}, \sqrt{-\lambda_2}\tilde{\eta}\right) \tag{22}$$

and get

$$\begin{aligned} &\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}), \quad \text{where} \quad \lambda_1 > 0, \quad \lambda_2 < 0 \\ &= \tilde{U}_{\tilde{\xi}\tilde{\xi}}\left(\tilde{\xi}, \tilde{\eta}\right) - \tilde{U}_{\tilde{\eta}\tilde{\eta}}\left(\tilde{\xi}, \tilde{\eta}\right) + (\text{lower order terms}) = \tilde{F}\left(\tilde{\xi}, \tilde{\eta}\right). \end{aligned}$$

For the **parabolic** case, by multiplying the equation by a minus sign if necessary, we may assume $\lambda_1 > 0,\ \lambda_2 = 0$. Then we replace (21) by

$$\tilde{\xi} = \frac{\xi}{\sqrt{\lambda_1}}, \quad \tilde{\eta} = \eta, \quad \tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = U\left(\sqrt{\lambda_1}\tilde{\xi}, \tilde{\eta}\right) \tag{23}$$

and get

$$\begin{aligned} &\lambda_1 U_{\xi\xi} + \lambda_2 U_{\eta\eta} + (\text{lower order terms}), \quad \text{where} \quad \lambda_1 > 0, \quad \lambda_2 = 0 \\ &= \tilde{U}_{\tilde{\xi}\tilde{\xi}}\left(\tilde{\xi}, \tilde{\eta}\right) + (\text{lower order terms}) = \tilde{F}\left(\tilde{\xi}, \tilde{\eta}\right) \end{aligned}$$

The proof is done. □

Definition 1.10 *In case equation (1) is parabolic with canonical form*

$$U_{\xi\xi} + (\textit{lower order terms}) = F(\xi, \eta), \tag{24}$$

*and there is **no** U_η term in (lower order terms) of (24), we say the equation is **degenerate**. Otherwise, we say it is **nondegenerate.** A degenerate parabolic equation is just a second order **ODE** in ξ of the form (view η as a parameter):*

$$U_{\xi\xi} + aU_\xi + bU = F(\xi, \eta), \quad a,\ b \textit{ are constants.} \tag{25}$$

*We **will not** study a degenerate parabolic equation. From now on, if we study a parabolic equation, we always assume that it is **nondegenerate**.*

1.1 Refined canonical form; getting rid of the first derivative terms.

One can go further to **get rid of the first derivative terms** in (17), (18) and (19) of Lemma 1.8. For simplicity, we can just look at two examples.

Example 1.11 ***(For elliptic and hyperbolic equations.)*** *Assume we have an* ***elliptic equation*** *in canonical form:*

$$U_{\xi\xi} + U_{\eta\eta} + 3U_\xi + 4U_\eta + 5U = F(\xi, \eta), \tag{26}$$

where we can write is as

$$\left(\underbrace{U_{\xi\xi} + 3U_\xi}\right) + \left(\underbrace{U_{\eta\eta} + 4U_\eta}\right) + 5U = F(\xi, \eta)$$

We let $v(\xi, \eta)$ *be the new function given by*

$$v(\xi, \eta) = e^{a\xi + b\eta} U(\xi, \eta)$$

for some constants $a,\ b$ *and we choose* $a = 3/2,\ b = 4/2$ *to get*

$$v(\xi, \eta) = e^{\frac{3}{2}\xi + \frac{4}{2}\eta} U(\xi, \eta) \tag{27}$$

and compute

$$v_\xi = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\frac{3}{2}U + U_\xi\right), \quad v_\eta = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\frac{4}{2}U + U_\eta\right) \tag{28}$$

and

$$v_{\xi\xi} = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\frac{9}{4}U + \underbrace{3U_\xi + U_{\xi\xi}}\right), \quad v_{\eta\eta} = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(4U + \underbrace{4U_\eta + U_{\eta\eta}}\right). \tag{29}$$

Hence we obtain

$$v_{\xi\xi} + v_{\eta\eta} = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left[\left(\underbrace{U_{\xi\xi} + 3U_\xi}\right) + \left(\underbrace{U_{\eta\eta} + 4U_\eta}\right) + \left(\frac{9}{4} + 4\right)U\right]$$

and conclude

$$\begin{aligned} v_{\xi\xi} + v_{\eta\eta} &= e^{\frac{3}{2}\xi + \frac{4}{2}\eta}\left(\underbrace{U_{\xi\xi} + 3U_\xi + U_{\eta\eta} + 4U_\eta + 5U} + \overbrace{\left(\frac{9}{4} + 4\right)U - 5U}\right) \\ &= \frac{5}{4}v + e^{\frac{3}{2}\xi + \frac{4}{2}\eta}F(\xi, \eta), \end{aligned}$$

i.e.

$$v_{\xi\xi} + v_{\eta\eta} - \frac{5}{4}v = \phi(\xi, \eta), \quad \textit{where} \;\; \phi(\xi, \eta) = e^{\frac{3}{2}\xi + \frac{4}{2}\eta}F(\xi, \eta). \tag{30}$$

The new equation for v ***has no first derivatives terms.*** *Note that, in general, one* ***cannot*** *choose* ***two*** *constants* a *and* b *to get rid of the* ***three*** *terms* $3U_\xi + 4U_\eta + 5U$*. Therefore, the term* $-(5/4)\,v$ *in (30)* ***cannot*** *be removed in general. The same result holds for the* ***hyperbolic equation****.*

Example 1.12 ***(For nondegenerate parabolic equations.)*** *Assume we have the* ***nondegenerate parabolic equation*** *in canonical form:*

$$U_{\xi\xi} + 3U_\xi + 4U_\eta + 5U = F(\xi, \eta). \tag{31}$$

where we can write is as

$$\left(\underbrace{U_{\xi\xi} + 3U_\xi}\right) + \left(\underbrace{4U_\eta + 5U}\right) = F(\xi, \eta). \tag{32}$$

Now we let $v(\xi, \eta)$ *be the new function given by*

$$v(\xi, \eta) = e^{\frac{3}{2}\xi + \lambda\eta} U(\xi, \eta), \quad \lambda \textit{ is a constant to be determined} \tag{33}$$

and compute

$$\begin{cases} v_\xi = e^{\frac{3}{2}\xi+\lambda\eta} \cdot \left(\frac{3}{2}U + U_\xi\right), \quad v_\eta = e^{\frac{3}{2}\xi+\lambda\eta} \cdot \left(\underbrace{\lambda U + U_\eta}\right), \\ v_{\xi\xi} = e^{\frac{3}{2}\xi+\lambda\eta} \left(\frac{9}{4}U + \underbrace{3U_\xi + U_{\xi\xi}}\right), \end{cases} \tag{34}$$

where, since there is no $U_{\eta\eta}$ *term in the original equation, we* ***do not have to*** *compute* $v_{\eta\eta}$ *(otherwise, we will get* $U_{\eta\eta}$ *and this does not make sense). Now, unlike the elliptic case in which we can compute* $v_{\eta\eta}$ *to produce the term* U_η *(see (29)), here to produce the term* $4U_\eta$ *in (32),* ***the only method*** *is to look at* $4v_\eta$ *and get*

$$4v_\eta = e^{\frac{3}{2}\xi+\lambda\eta} \cdot \left(\underbrace{4\lambda U + 4U_\eta}\right). \tag{35}$$

Now we conclude

$$\begin{aligned} v_{\xi\xi} + 4v_\eta &= e^{\frac{3}{2}\xi+\lambda\eta} \left(\frac{9}{4}U + \underbrace{3U_\xi + U_{\xi\xi}} + \underbrace{4\lambda U + 4U_\eta}\right) \\ &= e^{\frac{3}{2}\xi+\lambda\eta} \left(U_{\xi\xi} + 3U_\xi + 4U_\eta + \underbrace{\left(\frac{9}{4} + 4\lambda\right)U}\right). \end{aligned} \tag{36}$$

*By (**??**), if we choose* $\lambda = 11/16$*, we will have* $(9/4 + 4\lambda)\,U = 5U$ *and (36) becomes*

$$v_{\xi\xi} + 4v_\eta = \phi(\xi,\eta), \quad \text{where } \phi(\xi,\eta) = e^{\frac{3}{2}\xi+\frac{11}{16}\eta} F(\xi,\eta). \tag{37}$$

As a comparison, we see that we have got rid of the terms $3U_\xi$ *and* $5U$ *in (37). From the above computation, we also see that it is* ***impossible*** *to reduce the* ***nondegenerate parabolic equation*** *(31) into the form*

$$v_{\xi\xi} + cv = 0 \tag{38}$$

for some constant c*.*

By the above two examples, we can improve Lemma 1.8 as:

Theorem 1.13 *(**Refined canonical form.**) If the linear equation (2) is* ***elliptic****, then one can find a suitable linear change of variables (**using eigenvalues, eigenvectors and scalings**) and* ***multiply the solution*** *by some suitable* ***exponential function*** *so that, eventually, the equation has the form*

$$v_{\xi\xi} + v_{\eta\eta} = cv + \phi(\xi,\eta), \quad v = v(\xi,\eta), \tag{39}$$

for some constant $c \in (-\infty,\infty)$ *and some function* $\phi(\xi,\eta)$*. If the equation (2) is* ***hyperbolic****, the equation has the form*

$$v_{\xi\xi} - v_{\eta\eta} = cv + \phi(\xi,\eta), \quad v = v(\xi,\eta), \tag{40}$$

for some constant $c \in (-\infty,\infty)$ *and some function* $\phi(\xi,\eta)$*. If the equation (2) is* ***parabolic*** *and* ***nondegenerate****, the equation has the form*

$$v_{\xi\xi} = cv_\eta + \phi(\xi,\eta), \quad v = v(\xi,\eta), \tag{41}$$

for some constant $c \in (-\infty,\infty)$*,* $c \neq 0$*, and some function* $\phi(\xi,\eta)$*.*

Proof. The proof is now obvious. We omit it. □

Remark 1.14 ***(Important.)*** *The constant c in the* ***elliptic case*** *can be $c>0$ or $c=0$ or $c<0$. For $c>0$, we can make it equal to 1 by doing the change of variables*

$$\tilde{\xi}=\sqrt{c}\xi,\quad \tilde{\eta}=\sqrt{c}\eta,\quad \tilde{v}\left(\tilde{\xi},\tilde{\eta}\right)=v\left(\frac{\tilde{\xi}}{\sqrt{c}},\frac{\tilde{\eta}}{\sqrt{c}}\right)$$

and for $c<0$, we can make it equal to -1 by doing the change of variables

$$\tilde{\xi}=\sqrt{-c}\xi,\quad \tilde{\eta}=\sqrt{-c}\eta,\quad \tilde{v}\left(\tilde{\xi},\tilde{\eta}\right)=v\left(\frac{\tilde{\xi}}{\sqrt{-c}},\frac{\tilde{\eta}}{\sqrt{-c}}\right).$$

Thus in the ***elliptic case****, we may simply assume $c=1$ or 0 or -1. The constant c in the* ***hyperbolic case*** *can be $c>0$ or $c=0$ or $c<0$. For $c<0$, by switching the role of ξ and η, we may assume $c>0$ or $c=0$. Hence for the* ***hyperbolic case,*** *eventually, we can simply assume $c=1$ or 0. Finally, for the* ***parabolic case****, the constant $c\neq 0$ can be $c>0$ or $c<0$. So eventually we can simply assume $c=1$ or -1. However, since most parabolic equations come from physical phenomenon involving the behavior of some quantity $v(\xi,\eta)$ depending on space and time. So ξ will represent* ***space variable*** *(we rewrite it as x) and η will represent* ***time variable*** *(we rewrite it as t). In that case a* ***nondegenerate parabolic equation*** *in its* ***refined canonical form*** *looks like (assume $\phi(\xi,\eta)=0$ for simplicity)*

$$(1).\ v_t=v_{xx}\quad or\quad (2).\ v_t=-v_{xx}, \tag{42}$$

where, physically, the quantity v_{xx} describes the process due to ***diffusion*** *(say, from high temperature to low temperature, or from high concentration to low concentration, ... etc). We call (1) the* ***"forward heat equation" (or just heat equation)*** *and (2) the* ***"backward heat equation".*** *Since in reality, time cannot go backwards, so in a parabolic equation, we always fucus on the behavior of a solution $v(x,t)$* ***as time goes forwards****, i.e., as t is* ***increasing****. One can use simple examples to see that,* ***as time goes forwards****, the heat equation (1) will make solution better, while the backward heat equation (2) will make solution worse (look at $e^{-t}\sin x$ and $e^{t}\sin x$ respectively). Thus, as time goes forwards, equation (1) is* ***well-posed****, while (2) is* ***ill-posed****. Hence, we will focus only on (1).*

Finally, by the above remark, we conclude the following **final canonical form**:

Theorem 1.15 ***(Final canonical form.)*** *If the linear equation (2) is* ***elliptic****, then one can find a suitable linear change of variables (****using eigenvalues, eigenvectors and scalings****) and multiply the solution by some suitable* ***exponential function*** *so that, eventually, the equation has the form*

$$v_{\xi\xi}+v_{\eta\eta}=\begin{cases} v+\phi(\xi,\eta), \\ \phi(\xi,\eta), \\ -v+\phi(\xi,\eta), \end{cases} \tag{43}$$

where $v=v(\xi,\eta)$. If the equation (2) is ***hyperbolic****, the equation has the form*

$$v_{\xi\xi}-v_{\eta\eta}=\begin{cases} v+\phi(\xi,\eta), \\ \phi(\xi,\eta), \end{cases} \tag{44}$$

where $v=v(\xi,\eta)$. If the equation (2) is ***parabolic, nondegenerate*** *and* ***forward****, the equation has the form*

$$v_{\xi\xi}=v_{\eta}+\phi(\xi,\eta),\quad v=v(\xi,\eta). \tag{45}$$

Proof. The proof is now obvious. We omit it. □

Corollary 1.16 *If the equation (1) has the form*

$$au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y = 0, \tag{46}$$

*where a, ..., e are all **constants** with $a^2 + b^2 + c^2 > 0$, then one can reduce it into one of the canonical forms*

$$v_{\xi\xi} + v_{\eta\eta} = 0, \quad v_{\xi\xi} - v_{\eta\eta} = 0, \quad v_{\xi\xi} - v_\eta = 0, \quad v = v(\xi, \eta). \tag{47}$$

Definition 1.17 *Let $v = v(\xi, \eta)$. The equations $v_{\xi\xi} + v_{\eta\eta} = 0$, $v_{\xi\xi} - v_{\eta\eta} = 0$ (view η as time), $v_\eta = v_{\xi\xi}$ (view η as time), are called **Laplace equation (elliptic equation), wave equation (hyperbolic equation), and heat equation (nondegenerate forward parabolic equation),** respectively.*

Remark 1.18 *In this elementary course we will focus only on **Laplace equation, wave equation** and **heat equation**, or focus only on equation (48) below.*

2 General solutions of hyperbolic equations without lower order terms.

In this section, we look at equations of the following form with **no** lower order terms and $f(x, y) \equiv 0$, i.e.

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0, \quad u = u(x, y), \quad a,\ b,\ c \text{ are const.}, \tag{48}$$

where $a,\ b,\ c$ are constants with $a^2 + b^2 + c^2 > 0$. (48) can be written as

$$Trace\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right] = 0, \quad \det\begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2.$$

We want to find general C^2 solution $u = u(x, y)$ of (48) defined on $\mathbb{R}^2$. The **canonical form** of (48) is given by

$$\begin{cases} U_{\xi\xi} + U_{\eta\eta} = 0 \quad (\det M > 0,\ \text{elliptic}) \\ U_{\xi\xi} - U_{\eta\eta} = 0 \quad \text{or} \quad U_{\xi\eta} = 0 \quad (\det M < 0,\ \text{hyperbolic}) \\ U_{\xi\xi} = 0 \quad (\det M = 0,\ \text{parabolic but } \textbf{degenerate}), \end{cases} \tag{49}$$

where M is the coefficient matrix of (48). The result is that, **for hyperbolic and parabolic cases** in (49), **we can solve them easily (but not for elliptic equation)**. The method is either by a **change of variables** (**diagonalization method**) or by a **factorization method**.

Lemma 2.1 *Let $A,\ B,\ C,\ D$ be constants with $AD - BC \neq 0$. Consider the first order equation*

$$Au_x + Bu_y = g(Dx - Cy), \quad u = u(x, y) \tag{50}$$

*where $g(\cdot)$ is a given continuous function defined on $\mathbb{R}$. Then the **general solution** of (50) is given by*

$$u(x, y) = F(Bx - Ay) + G(Dx - Cy), \tag{51}$$

where $F(\cdot)$ is an arbitrary C^1 function defined on $\mathbb{R}$ and the C^1 function $G(\theta)$ satisfies

$$G'(\theta) = \frac{g(\theta)}{AD - BC}, \quad \forall\, \theta \in (-\infty, \infty). \tag{52}$$

*In particular, if the function $g(\cdot)$ on $\mathbb{R}$ is **arbitrary**, then the function $G(\cdot)$ on $\mathbb{R}$ is also **arbitrary**.*

Remark 2.2 *The condition $AD - BC \neq 0$ is necessary. The case $AD - BC = 0$ will be discussed later on. See (81).*

Remark 2.3 *Also note that $AD - BC \neq 0$ implies that the two families of lines $Bx - Ay = \lambda$, $Dx - Cy = \eta$ are* ***not*** *parallel. As a consequence of this, the two terms $F(Bx - Ay)$, $G(Dx - Cy)$ in (58) are essentially different.*

Proof. We do the linear change of variables

$$w = Bx - Ay, \qquad z = Dx - Cy, \quad \text{Jacobian is} \begin{vmatrix} B & -A \\ D & -C \end{vmatrix} = AD - BC \neq 0,$$

which is a global linear change of variables from xy-space to wz-space. Now the function $u(x, y)$ becomes $U(w, z)$ and we have

$$Au_x + Bu_y = A[U_w B + U_z D] + B[U_w(-A) + U_z(-C)] = (AD - BC)U_z = g(z),$$

which gives

$$U_z = \frac{g(z)}{AD - BC}, \quad AD - BC \neq 0$$

and so

$$U(w, z) = F(w) + G(z) = F(Bx - Ay) + G(Dx - Cy),$$

where $G'(z) = \frac{g(z)}{AD - BC}$. The proof is done. □

2.1 Solving hyperbolic equations; factorization method.

Lemma 2.4 *Let a, b, c be three given constants with $ac < b^2$ (same as $b^2 - ac > 0$). Then one can find constants A, B, C, D satisfying*

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c. \tag{53}$$

In particular, we have

$$(AD - BC)^2 = 4\left(b^2 - ac\right) > 0, \quad AD - BC \neq 0. \tag{54}$$

This means that, if we have $b^2 - ac > 0$, we can factor the second order homogeneous polynomial $ax^2 + 2bxy + cy^2$ as

$$ax^2 + 2bxy + cy^2 = (Ax + By)(Cx + Dy), \quad AD - BC \neq 0, \tag{55}$$

where the two lines $Ax + By = 0$, $Cx + Dy = 0$ on $\mathbb{R}^2$ are not parallel (same as $Bx - Ay = 0$, $Dx - Cy = 0$ are ***not*** *parallel).*

Remark 2.5 *If $ac > b^2$, then (53) and (55) cannot be satisfied (check it yourself). Therefore, the method in this section cannot be used to elliptic equations.*

Remark 2.6 *If $a = 1$, we can choose $A = C = 1$, and then solve*

$$B + D = 2b, \qquad BD = c$$

to get

$$B = b \pm \sqrt{b^2 - c}, \quad D = \frac{c}{b \pm \sqrt{b^2 - c}}, \quad \textit{for} \quad c \neq 0$$

and

$$B = 2b, \quad D = 0, \quad \textit{for} \quad c = 0.$$

Remark 2.7 *If* $a > 0,\ b = 0,\ c < 0$, *we can choose* $A = C = \sqrt{a}$ *and* $B = \sqrt{-c},\ D = -\sqrt{-c}$.

Proof. (**Read it yourself**. **We omit it.**) If $a = 0$, then by $ac = 0 < b^2$, we must have $b \neq 0$. The numbers

$$A = 1, \quad B = \frac{c}{2b}, \quad C = 0, \quad D = 2b,$$

satisfy (53). If $c = 0$, then we still have $b \neq 0$. The numbers

$$A = \frac{a}{2b}, \quad B = 1, \quad C = 2b, \quad D = 0,$$

satisfy (53). If $ac \neq 0$, then $b + \sqrt{b^2 - ac} \neq 0$ and $b - \sqrt{b^2 - ac} \neq 0$. The numbers

$$A = 1, \quad B = \frac{b + \sqrt{b^2 - ac}}{a}, \quad C = a, \quad D = \frac{ac}{b + \sqrt{b^2 - ac}}$$

satisfy (53). Finally, in each case we can see that $AD - BC \neq 0$. □

We now focus on the **hyperbolic** ($ac - b^2 < 0$) case in (48). That is, the eigenvalues $\lambda_1,\ \lambda_2$ of the coefficient matrix have different sign and we may assume $\lambda_1 > 0,\ \lambda_2 < 0$.

Lemma 2.8 *(**Factorization method for hyperbolic equation**.) Assume (48) is hyperbolic, i.e., $ac < b^2$. Then one can decompose it as*

$$au_{xx} + 2bu_{xy} + cu_{yy} = \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u\right] = 0 \tag{56}$$

for some constants $A,\ B,\ C,\ D$ satisfying

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0. \tag{57}$$

*In particular, the **general solution** of (48) is given by*

$$u(x, y) = F(Bx - Ay) + G(Dx - Cy), \quad (x, y) \in \mathbb{R}^2 \tag{58}$$

*for arbitrary C^2 **functions** $F(\cdot),\ G(\cdot)$ defined on $\mathbb{R}$.*

Proof. For a C^2 function u, we have

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u\right]$$
$$= A(Cu_x + Du_y)_x + B(Cu_x + Du_y)_y = ACu_{xx} + (AD + BC)u_{xy} + BDu_{yy}.$$

Now by Lemma 2.4, there are numbers $A,\ B,\ C,\ D$ satisfying

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0. \tag{59a}$$

Hence, for a C^2 function u, it satisfies $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ if and only if it satisfies

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u\right] = 0, \tag{60}$$

where $A,\ B,\ C,\ D$ satisfy (59a). We can find solutions of (60) by solving two first order PDE. Let

$$v = \left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u.$$

It satisfies $\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)v=0$. Hence $v(x,y)=f(Bx-Ay)$ for **arbitrary** C^1 function f and the equation for u becomes

$$Cu_x+Du_y=f(Bx-Ay),\quad f\in C^1.$$

By Lemma 2.1, the general solution for $u(x,y)$ is

$$u(x,y)=G(Dx-Cy)+F(Bx-Ay),\quad (x,y)\in\mathbb{R}^2,$$

where $F,\ G$ are two arbitrary C^2 functions defined on $(-\infty,\infty)$ (since we want $u(x,y)$ to be a C^2 solution, we must require $F,\ G$ to be C^2 functions). The proof is done. □

Remark 2.9 *(**Important.**) Lemma 2.8 says that to solve the second order hyperbolic equation, it suffices to solve **two first order equations.***

Definition 2.10 *We call the **2-parameter family of lines***

$$Bx-Ay=\lambda,\quad Dx-Cy=\eta,\quad AD-BC\neq 0$$

*where $\lambda,\ \eta$ are arbitrary constants, the **characteristic lines** of the hyperbolic equation (56).*

Example 2.11 *Consider the second order linear equation in two variables:*

$$u_{xx}-4u_{xy}-2u_{yy}=0,\quad (x,y)\in\mathbb{R}^2,\quad u=u(x,y):\mathbb{R}^2\to\mathbb{R}.$$

*What is the type (elliptic, hyperbolic, or parabolic) of this equation ? Use **factorization method** to find the general solution of the equation.*

<u>**Solution**</u>:

The equation has the form $au_{xx}+2bu_{xy}+cu_{yy}=0$, where $a=1,\ b=-2,\ c=-2$ and $ac-b^2=-6<0$. Therefore the equation is **hyperbolic**. We know that one can decompose it into two **first order equations** of the form (by Remark 2.6, we can choose $A=C=1$)

$$u_{xx}-4u_{xy}-2u_{yy}=(\partial_x+B\partial_y)(\partial_x+D\partial_y)u=u_{xx}+(B+D)u_{xy}+BDu_{yy}.$$

Thus we solve $B,\ D$ to satisfy the equation (note that now we have $A=C=1$)

$$B+D=-4,\quad BD=-2.$$

We obtain $(B,D)=\left(-2+\sqrt{6},-2-\sqrt{6}\right)$ or $(B,D)=\left(-2-\sqrt{6},-2+\sqrt{6}\right)$. Thus we choose $(B,D)=\left(-2+\sqrt{6},-2-\sqrt{6}\right)$ and get

$$u_{xx}-4u_{xy}-2u_{yy}=\left[\partial_x+\left(-2+\sqrt{6}\right)\partial_y\right]\left[\partial_x+\left(-2-\sqrt{6}\right)\partial_y\right]u \tag{61}$$

and the general solution is

$$\begin{aligned}u(x,y)&=F(Bx-Ay)+G(Dx-Cy)=F\left(\left(-2+\sqrt{6}\right)x-y\right)+G\left(\left(-2-\sqrt{6}\right)x-y\right)\\&=F\left(\left(\sqrt{6}-2\right)x-y\right)+G\left(\left(\sqrt{6}+2\right)x+y\right),\end{aligned} \tag{62}$$

where $F(z):\mathbb{R}\to\mathbb{R}$ and $G(z):\mathbb{R}\to\mathbb{R}$ are two arbitrary C^2 functions. □

Remark 2.12 *In terms of polynomial, the decomposition (61) is the same as*

$$x^2-4xy-2y^2=\left(x+\left(-2+\sqrt{6}\right)y\right)\left(x+\left(-2-\sqrt{6}\right)y\right). \tag{63}$$

2.2 Solving hyperbolic equations; change of variables method.

Remark 2.13 *By (66) below, we see that this method is essentially the same as the factorization method. Computationally, you can just use the factorization method. Note that solving the equation* $U_{\xi\eta} = 0$ *is the same as solving* ***two first order equations****. See equation (60) also.*

We can also use a change of variables method to solve the equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0, \quad \text{where} \quad ac < b^2. \tag{64}$$

Introduce the change of variables

$$\xi = Bx - Ay, \quad \eta = Dx - Cy, \quad \text{Jacobian} = \begin{vmatrix} B & -A \\ D & -C \end{vmatrix} = AD - BC, \tag{65}$$

where $A,\ B,\ C,\ D$ are constants satisfying

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0, \tag{66}$$

where we note that the existence of $A,\ B,\ C,\ D$ satisfying (66) is guaranteed by Lemma 2.4. Since $AD - BC \neq 0$, this change of variables is good on all $\mathbb{R}^2$. Let $U(\xi, \eta)$ be the function $u(x, y)$ in (ξ, η) variables. We have:

Lemma 2.14 *(**Change of variables method for hyperbolic equation**.) Under the change of variables (65), (66), the equation for* $U(\xi, \eta)$ *is given by*

$$-(AD - BC)^2 U_{\xi\eta} = 0 \quad (\textit{same as } U_{\xi\eta} = 0 \textit{ since } AD - BC \neq 0), \tag{67}$$

which has general solution given by $U(\xi, \eta) = F(\xi) + G(\eta)$ *for arbitrary* C^2 *functions (we want* $U(\xi, \eta)$ *to be a* C^2 *function)* $F(\xi)$ *and* $G(\eta)$ *defined on* $\mathbb{R}$*. As a consequence, the general solution* $u(x, y)$ *of (64) is*

$$u(x, y) = F(Bx - Ay) + G(Dx - Cy), \quad (x, y) \in \mathbb{R}^2,$$

for arbitrary C^2 *functions* $F(\xi)$ *and* $G(\eta)$ *defined on* $\mathbb{R}$*.*

Proof. We have

$$\begin{cases} u_x = BU_\xi + DU_\eta, \qquad u_y = -AU_\xi - CU_\eta, \\ u_{xx} = B^2U_{\xi\xi} + 2BDU_{\xi\eta} + D^2U_{\eta\eta}, \\ u_{xy} = -ABU_{\xi\xi} - (BC + AD)\, U_{\xi\eta} - CDU_{\eta\eta}, \\ u_{yy} = A^2U_{\xi\xi} + 2ACU_{\xi\eta} + C^2U_{\eta\eta} \end{cases}$$

and so

$$\begin{aligned} & au_{xx} + 2bu_{xy} + cu_{yy} \\ &= \begin{cases} a\left[B^2U_{\xi\xi} + 2BDU_{\xi\eta} + D^2U_{\eta\eta}\right] \\ +2b\left[-ABU_{\xi\xi} - (BC + AD)\, U_{\xi\eta} - CDU_{\eta\eta}\right] + c\left[A^2U_{\xi\xi} + 2ACU_{\xi\eta} + C^2U_{\eta\eta}\right] \end{cases} \\ &= \begin{cases} \left(\underline{aB^2 - 2bAB + cA^2}\right) U_{\xi\xi} \\ + \left(\underbrace{a2BD - 2bBC - 2bAD + c2AC}\right) U_{\xi\eta} + \left(\overbrace{aD^2 - 2bCD + cC^2}\right) U_{\eta\eta}. \end{cases} \end{aligned}$$

By

$$AC = a, \qquad AD + BC = 2b, \qquad BD = c, \qquad AD - BC \neq 0,$$

we have

$$\begin{cases} \underline{aB^2 - 2bAB + cA^2} = ACB^2 - (AD+BC)\,AB + BDA^2 = 0 \\ \overbrace{aD^2 - 2bCD + cC^2} = ACD^2 - (AD+BC)\,CD + BDC^2 = 0 \end{cases}$$

and

$$\begin{aligned} &\underbrace{a2BD - 2bBC - 2bAD + c2AC} \\ &= 2ACBD - (AD+BC)\,BC - (AD+BC)\,AD + BD2AC \\ &= 2ABCD - B^2C^2 - A^2D^2 = -(AD-BC)^2 \neq 0. \end{aligned}$$

Hence the equation for $U(\xi,\eta)$ is

$$-(AD-BC)^2\,U_{\xi\eta} = 0 \quad (\text{same as } U_{\xi\eta} = 0).$$

Its general solution is $U(\xi,\eta) = F(\xi) + G(\eta)$ for arbitrary C^2 functions $F(\xi)$ and $G(\eta)$ defined on $\mathbb{R}$. □

Example 2.15 *Consider the second order linear equation in two variables:*

$$u_{xx} - 4u_{xy} - 2u_{yy} = 0, \quad (x,y) \in \mathbb{R}^2, \quad u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}.$$

Use ***change of variables method*** *to reduce it to canonical form (*$U_{\xi\xi} - U_{\eta\eta} = 0$ *or* $U_{\xi\eta} = 0$*) and then solve it.*

<u>**Solution**</u>:

Recall that the numbers $A,\ B,\ C,\ D$ satisfying

$$AC = a = 1, \qquad AD + BC = 2b = -4, \qquad BD = c = -2, \tag{68}$$

are given by $A = C = 1,\ B = -2+\sqrt{6},\ D = -2-\sqrt{6}$. By Lemma 2.14, if we do the change of variables

$$\begin{cases} \xi = Bx - Ay = \left(-2+\sqrt{6}\right)x - y \\ \eta = Dx - Cy = \left(-2-\sqrt{6}\right)x - y, \end{cases} \tag{69}$$

the new for $U(\xi,\eta)$ is given by $U_{\xi\eta}(\xi,\eta) = 0$. The general solution for $U(\xi,\eta)$ is $U(\xi,\eta) = F(\xi) + G(\eta)$. Hence the general solution for $u(x,y)$ is

$$\begin{aligned} u(x,y) &= F(Bx - Ay) + G(Dx - Cy) = F\left(\left(-2+\sqrt{6}\right)x - y\right) + G\left(\left(-2-\sqrt{6}\right)x - y\right) \\ &= F\left(\left(\sqrt{6}-2\right)x - y\right) + G\left(\left(\sqrt{6}+2\right)x + y\right), \end{aligned} \tag{70}$$

where $F(z) : \mathbb{R} \to \mathbb{R}$ and $G(z) : \mathbb{R} \to \mathbb{R}$ are two arbitrary C^2 functions. □

2.3 Solving hyperbolic equations; diagonalization method (eigenvalue-eigenvector method).

Remark 2.16 *Interesting question:* ***Can you find the relation between this method and the factorization method ?***

Example 2.17 *Consider the second order linear equation in two variables:*

$$u_{xx} - 4u_{xy} - 2u_{yy} = 0, \quad (x,y) \in \mathbb{R}^2, \quad u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}.$$

Use ***diagonalization method*** *to reduce it to canonical form (*$U_{\xi\xi} - U_{\eta\eta} = 0$*) and then solve it.*

Solution:

One can write the equation as

$$Trace\left[\begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right] = 0.$$

The eigenvalues of the coefficient matrix are $\lambda_1 = 2,\ \lambda_2 = -3$ with corresponding **orthonormal** eigenvectors

$$v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{71}$$

By (13) and (14), we introduce the change of variables

$$\xi = -\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y, \quad \eta = \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y.$$

Then, in terms of the variables (ξ, η), we have

$$2U_{\xi\xi} - 3U_{\eta\eta} = 0.$$

Finally, we let $\tilde{\xi} = \frac{1}{\sqrt{2}}\xi,\ \tilde{\eta} = \frac{1}{\sqrt{3}}\eta$, we have the final **canonical form:**

$$\tilde{U}_{\tilde{\xi}\tilde{\xi}} - \tilde{U}_{\tilde{\eta}\tilde{\eta}} = 0 \text{ (which can be decomposed as } \left(\frac{\partial}{\partial\tilde{\xi}} + \frac{\partial}{\partial\tilde{\eta}}\right)\left[\left(\frac{\partial}{\partial\tilde{\xi}} - \frac{\partial}{\partial\tilde{\eta}}\right)\tilde{U}\right] = 0 \text{)}$$

and its general solution is

$$\tilde{U}\left(\tilde{\xi}, \tilde{\eta}\right) = F\left(\tilde{\xi} - \tilde{\eta}\right) + G\left(-\tilde{\xi} - \tilde{\eta}\right) \text{ (same as } G\left(\tilde{\xi} + \tilde{\eta}\right) \text{)},$$

where $F(z)$ and $G(z)$ are two arbitrary C^2 functions. Hence the general solution for $u(x, y)$ is

$$u(x,y) = \begin{cases} F\left(\frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)\right) \\ +G\left(\frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)\right) \end{cases} \tag{72}$$

$$= \begin{cases} \underbrace{F\left(\left(-\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{15}}\right)y\right)} \\ +\underline{G\left(\left(-\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{15}}\right)y\right)}. \end{cases}$$

By the identity

$$\frac{-\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{15}}}{\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{15}}} = \frac{2\sqrt{15} + \sqrt{10}}{-\sqrt{15} + 2\sqrt{10}} = \sqrt{6} + 2$$

we can write $\underbrace{F\left(\left(-\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{15}}\right)y\right)}$ as $\underbrace{F\left(\left(\sqrt{6}+2\right)x + y\right)}$ and by the identity

$$\frac{-\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{15}}}{\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{15}}} = \frac{-2\sqrt{15} + \sqrt{10}}{\sqrt{15} + 2\sqrt{10}} = -\sqrt{6} + 2$$

we can write $\underline{G\left(\left(-\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{15}}\right)x + \left(\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{15}}\right)y\right)}$ as $\underbrace{G\left(\left(-\sqrt{6}+2\right)x + y\right)}$. Thus the general solution can also be expressed as

$$u(x,y) = F\left(\left(\sqrt{6}+2\right)x + y\right) + G\left(\left(-\sqrt{6}+2\right)x + y\right), \tag{73}$$

which is the same as (62) and (70). □

Remark 2.18 ***(Omit this in class.)*** ***(Interesting.)*** *One can use eigenvalues and eigenvectors to express the general solution (72) as*

$$u(x,y)=\begin{cases} F\left(\frac{1}{\sqrt{\lambda_1}}\left\langle\begin{pmatrix}x\\y\end{pmatrix},\ v_1\right\rangle-\frac{1}{\sqrt{-\lambda_2}}\left\langle\begin{pmatrix}x\\y\end{pmatrix},\ v_2\right\rangle\right)\\ +G\left(\frac{1}{\sqrt{\lambda_1}}\left\langle\begin{pmatrix}x\\y\end{pmatrix},\ v_1\right\rangle+\frac{1}{\sqrt{-\lambda_2}}\left\langle\begin{pmatrix}x\\y\end{pmatrix},\ v_2\right\rangle\right),\end{cases}\tag{74}$$

where $\lambda_1=2,\ \lambda_2=-3$ *and* $v_1,\ v_2$ *are corresponding* ***orthonormal eigenvectors*** *given by (71).*

3 General solutions of parabolic equations without lower order terms.

3.1 Solving parabolic equations; factorization method.

We now come to the **parabolic** case for the equation

$$au_{xx}+2bu_{xy}+cu_{yy}=0,\quad \text{where}\quad ac=b^2.\tag{75}$$

Here we may assume $a,\ b,\ c$ are **all nonzero** (otherwise we will get into a trivial case). By the identity $ac=b^2>0$, we know that $a,\ c$ have the same sign. By multiplying the equation by a minus sign if necessary, we may assume that $a>0,\ c>0$. However, b can be either $b>0$ or $b<0$.

As there is no lower order terms in (75), the parabolic equation is **degenerate**. Hence it is essentially an ODE. We have the following:

Lemma 3.1 ***(Factorization method for parabolic equation.)*** *Assume the equation (75) is parabolic, i.e.* $ac=b^2$*, with* $a>0,\ b\neq 0,\ c>0$*. Then one can decompose it as*

$$au_{xx}+2bu_{xy}+cu_{yy}=\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)\left[\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)u\right]=0\tag{76}$$

for some constants $A>0,\ B\neq 0$*. More precisely, we have*

$$\begin{cases} A=\sqrt{a},\quad B=\sqrt{c},\quad \text{if}\quad b>0\\ A=\sqrt{a},\quad B=-\sqrt{c},\quad \text{if}\quad b<0.\end{cases}\tag{77}$$

The ***general solution*** *of* ***(75)*** *can be expressed as either one of the following two forms*

$$(1).\ \ u(x,y)=F(Bx-Ay)+\frac{x}{A}G(Bx-Ay)\tag{78}$$

or

$$(2).\ \ u(x,y)=F(Bx-Ay)+\frac{y}{B}G(Bx-Ay)\tag{79}$$

for arbitrary C^2 *functions* $F(z),\ G(z)$ *defined on* $\mathbb{R}$*.*

Remark 3.2 *If* $a=0$ *(then* $b=0,\ c\neq 0$*) or* $c=0$ *(then* $b=0,\ a\neq 0$*), then we are in a trivial case. The equation now has the form* $cu_{yy}=0$ *or* $au_{xx}=0$*. We have not much to discuss.*

Remark 3.3 *The two forms in (78) and (79) are the same due to the identity*

$$\begin{aligned}&\frac{x}{A}G(Bx-Ay)\\ &=\frac{(Bx-Ay)+Ay}{AB}G(Bx-Ay)\\ &=\frac{(Bx-Ay)}{AB}G(Bx-Ay)\ \ \text{(absorb this term into } F(Bx-Ay)\text{)}+\frac{y}{B}G(Bx-Ay).\end{aligned}\tag{80}$$

Proof. For a C^2 function u, (76) is the same as

$$A\left(Au_x+Bu_y\right)_x+B\left(Au_x+Bu_y\right)_y=A^2u_{xx}+2ABu_{xy}+B^2u_{yy}=0.$$

Hence we need to solve

$$A^2=a,\quad AB=b,\quad B^2=c,$$

which is **solvable** due to $ac-b^2=0$. If $a>0,\ b>0,\ c>0$, then we can choose $A=\sqrt{a},\ B=\sqrt{c}$. If $a>0,\ b<0,\ c>0$, then we can choose $A=\sqrt{a},\ B=-\sqrt{c}$.

Let $w=\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)u$. Then by $\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)w=0$, we see that $w=G\left(Bx-Ay\right)$ for some arbitrary C^1 function $G\left(z\right)$ defined on $\mathbb{R}$. Next we solve

$$\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)u\left(x,y\right)=G\left(Bx-Ay\right). \tag{81}$$

Since $A>0$ and $B\neq 0$, we can do the change of variables $w=Bx-Ay,\ z=x$, to get

$$Au_x+Bu_y=A\left[U_wB+U_z\right]+BU_w\left(-A\right)=\underbrace{AU_z=G\left(w\right)}$$

and obtain the general solution for $U\left(w,z\right)$:

$$U\left(w,z\right)=F\left(w\right)+\frac{z}{A}G\left(w\right),$$

which gives

$$u\left(x,y\right)=F\left(Bx-Ay\right)+\frac{x}{A}G\left(Bx-Ay\right),\quad \text{if}\ \ A\neq 0,$$

where now $F\left(z\right),\ G\left(z\right)$ are two arbitrary C^2 functions defined on $\mathbb{R}$. This is the form (78).

Similarly, we can also do the change of variables $w=Bx-Ay,\ z=y$, to get

$$Au_x+Bu_y=A\left(U_wB\right)+B\left[U_w\left(-A\right)+U_z\right]=\underbrace{BU_z=G\left(w\right)}$$

which gives the general solution of the form (79). □

Remark 3.4 *Note that, in solving (81), Lemma 2.1 is* ***not*** *applicable here.*

Remark 3.5 *We can also do the change of variables $w=Bx-Ay,\ z=Ax+By$ in (81) and get*

$$Au_x+Bu_y=A\left[U_wB+U_zA\right]+B\left[U_w\left(-A\right)+U_zB\right]=\underbrace{\left(A^2+B^2\right)U_z=G\left(w\right)}.$$

We now have

$$U\left(w,z\right)=F\left(w\right)+\frac{z}{A^2+B^2}G\left(w\right), \tag{82}$$

which gives the ***symmetric*** *form*

$$u\left(x,y\right)=F\left(Bx-Ay\right)+\frac{Ax+By}{A^2+B^2}G\left(Bx-Ay\right). \tag{83}$$

The ***three forms*** *(78), (79) and (83) are all* ***equivalent*** *due to the following identities:*

$$\frac{Ax+By}{A^2+B^2}=\frac{1}{A^2+B^2}\left[-\frac{B\left(Bx-Ay\right)}{A}+\frac{\left(A^2+B^2\right)x}{A}\right] \tag{84}$$

and

$$\frac{Ax+By}{A^2+B^2}=\frac{1}{A^2+B^2}\left[\frac{A\left(Bx-Ay\right)}{B}+\frac{\left(A^2+B^2\right)y}{B}\right]. \tag{85}$$

Definition 3.6 *Unlike the hyperbolic case, we have only* ***1-parameter family of characteristic lines***

$$Bx-Ay=\lambda \tag{86}$$

for the parabolic equation (75), where λ is an arbitrary constant.

3.2 Solving parabolic equations; diagonalization method.

What happens if we use **diagonalization method** to solve a parabolic equation ($ac = b^2$) ? One can check that the coefficient matrix of equation (75) has two eigenvalues $\lambda_1 = a + c,\ \lambda_2 = 0$. The corresponding orthonormal eigenvectors are

$$v_1 = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a \\ b \end{pmatrix}, \qquad v_2 = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} b \\ -a \end{pmatrix}.$$

By (14), under the change of variables

$$\xi = \frac{1}{\sqrt{a^2+b^2}} (ax+by), \qquad \eta = \frac{1}{\sqrt{a^2+b^2}} (bx-ay), \tag{87}$$

equation (75) can be reduced to

$$Trace \left[\begin{pmatrix} a+c & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\xi\eta} & U_{\eta\eta} \end{pmatrix} \right] = 0.$$

Since $a + c \neq 0$, the above is same as

$$U_{\xi\xi} = 0, \tag{88}$$

which implies

$$U(\xi, \eta) = \xi g(\eta) + f(\eta) \tag{89}$$

for arbitrary C^2 functions $f(\eta)$ and $g(\eta)$. The corresponding $u(x, y)$ is given by

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{a^2+b^2}} (ax+by)\, g\left(\frac{1}{\sqrt{a^2+b^2}} (bx-ay) \right) + f\left(\frac{1}{\sqrt{a^2+b^2}} (bx-ay) \right) \\ &= H(bx-ay) + \frac{ax+by}{\sqrt{a^2+b^2}} K(bx-ay) \end{aligned} \tag{90}$$

for arbitrary C^2 functions $H(\eta)$ and $K(\eta)$. We can compare (90) with (83). Assume we are in the case $a > 0,\ b > 0,\ c > 0$, then we have $A = \sqrt{a},\ B = \sqrt{c}$ and so $b = \sqrt{ac} = AB$. Hence

$$\begin{aligned} & H(bx-ay) + \frac{ax+by}{\sqrt{a^2+b^2}} K(bx-ay) \\ &= H\left(ABx - A^2 y\right) + \frac{A^2 x + ABy}{\sqrt{A^4 + A^2B^2}} K\left(ABx - A^2y\right) = F(Bx - Ay) + \frac{Ax+By}{A^2+B^2} G(Bx - Ay) \end{aligned} \tag{91}$$

for another two arbitrary C^2 functions $F(\eta)$ and $G(\eta)$. **Therefore, (90) is the same as (83).** The check for the case $a > 0,\ b < 0,\ c > 0$ is similar. Therefore, both methods are actually equivalent.

4 Hyperbolic and parabolic equations of the form $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ with initial conditions.

If we write the equation $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ as $au_{xx} + 2bu_{xt} + cu_{tt} = 0$ and view x as space variable, t as time variable, then a pair of initial conditions (at time $t = 0$) of the form

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad x \in (-\infty, \infty) \tag{92}$$

can determine the solution **uniquely**, i.e. the solution exists and is unique. Here $\phi(x)$ and $\psi(x)$ are two given functions defined on $(-\infty, \infty)$. We will demonstrate this by direct computations.

Remark 4.1 *(**Useful motivation.**) Roughly speaking, a hyperbolic equation (imagine it is a wave equation) comes from **Newtonian mechanics**, hence as long as the **initial position** and **initial velocity** are known, the whole process of motion is **uniquely** determined.*

Lemma 4.2 *(**Hyperbolic equation with initial conditions.**) Let $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$ be two given functions. Assume $ac < b^2$ in equation $au_{xx} + 2bu_{xt} + cu_{tt} = 0$. Consider the **hyperbolic equation with initial conditions:***

$$\begin{cases} \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial t}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}\right)u\right] = 0, & u = u(x,t) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in (-\infty,\infty), \end{cases} \tag{93}$$

*where A, B, C, D are constants satisfying $AD - BC \neq 0$ and **here we also assume that** $B \neq 0$ **and** $D \neq 0$ (they are the coefficients of the operator $\frac{\partial}{\partial t}$). Then the initial value problem (93) has a unique solution $u(x,t) \in C^2(\mathbb{R}^2)$. Moreover, the solution is given by*

$$u(x,t) = \frac{AD}{AD-BC}\phi\left(x - \frac{C}{D}t\right) - \frac{BC}{AD-BC}\phi\left(x - \frac{A}{B}t\right) + \frac{BD}{AD-BC}\int_{x-\frac{A}{B}t}^{x-\frac{C}{D}t}\psi(s)\,ds, \tag{94}$$

*where $(x,t) \in \mathbb{R}^2$. Note: since we assume $B \neq 0$ and $D \neq 0$, the line $t = 0$ is **not a characteristic line**, which is good.*

Remark 4.3 *(**Important.**) (**2-parameter families of characteristic lines for hyperbolic equations.**) The assumption $B \neq 0$ and $D \neq 0$ is **essential** in the formula (94). There are **2-parameter families of characteristic lines** for the hyperbolic equation*

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial t}\right)\left[\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}\right)u\right] = 0, \tag{95}$$

namely the lines

$$Bx - At = const., \quad \text{and} \quad Dx - Ct = const.. \tag{96}$$

If we have $B = 0$, then by $AD - BC \neq 0$, we must have $A \neq 0$ and $D \neq 0$. Hence the line $t = const.$ is a characteristic line. Similarly, if we have $D = 0$, then $B \neq 0$ and $C \neq 0$, and again the line $t = const.$ is a characteristic line. By this, the initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty,\infty)$$

*happen to occur on the **characteristic line** $t = const.$ (i.e. $t = 0$). In general, for $B = 0$, we have either **no** solution to the initial value problem or **infinitely many solutions** to the initial value problem. The same conclusion holds for the case $D = 0$. **We will leave this as a homework problem for you to verify**. Also see Remark 4.9.*

Proof. The general solution of the equation is

$$u(x,t) = F(Bx - At) + G(Dx - Ct) \tag{97}$$

for arbitrary C^2 functions $F(\cdot)$, $G(\cdot)$ defined on $\mathbb{R}$ and we need to solve

$$\begin{cases} F(Bx) + G(Dx) = \phi(x) \\ -AF'(Bx) - CG'(Dx) = \psi(x). \end{cases} \tag{98}$$

Differentiate the first equation with respect to x to get the system of equations:

$$\begin{cases} BF'(Bx) + DG'(Dx) = \phi'(x) \\ -AF'(Bx) - CG'(Dx) = \psi(x), \end{cases}$$

i.e.

$$\begin{pmatrix} B & D \\ -A & -C \end{pmatrix}\begin{pmatrix} F'(Bx) \\ G'(Dx) \end{pmatrix} = \begin{pmatrix} \phi'(x) \\ \psi(x) \end{pmatrix}.$$

Hence we get

$$\begin{pmatrix} F'(Bx) \\ G'(Dx) \end{pmatrix} = \frac{1}{AD-BC}\begin{pmatrix} -C & -D \\ A & B \end{pmatrix}\begin{pmatrix} \phi'(x) \\ \psi(x) \end{pmatrix}$$

and so

$$F'(Bx) = \frac{-1}{AD-BC}\left(C\phi'(x) + D\psi(x)\right), \qquad G'(Dx) = \frac{1}{AD-BC}\left(A\phi'(x) + B\psi(x)\right),$$

which are the same as (note that we assume $B \neq 0$ and $D \neq 0$)

$$\begin{cases} \frac{d}{dx}F(Bx) = BF'(Bx) = \frac{-B}{AD-BC}\left(C\phi'(x) + D\psi(x)\right), \\ \frac{d}{dx}G(Dx) = DG'(Dx) = \frac{D}{AD-BC}\left(A\phi'(x) + B\psi(x)\right). \end{cases} \tag{99}$$

If we let $\tilde{\psi}(x)$ be an **antiderivative** of $\psi(x)$ (it is not unique), we get

$$\begin{cases} F(Bx) = \frac{-B}{AD-BC}\left(C\phi(x) + D\tilde{\psi}(x)\right) + C_1 \\ G(Dx) = \frac{D}{AD-BC}\left(A\phi(x) + B\tilde{\psi}(x)\right) + C_2 \end{cases} \tag{100}$$

for some integration constants $C_1,\ C_2$. Now by the first equation of (98), we must have $C_1 + C_2 = 0$. Therefore, we conclude (note that we assume $B \neq 0$ and $D \neq 0$)

$$\begin{aligned} &u(x,t) \\ &= F(Bx - At) + G(Dx - Ct) = F\left(B\left(x - \frac{A}{B}t\right)\right) + G\left(D\left(x - \frac{C}{D}t\right)\right) \\ &= \begin{cases} \frac{-B}{AD-BC}\left[C\phi\left(x - \frac{A}{B}t\right) + D\tilde{\psi}\left(x - \frac{A}{B}t\right)\right] \\ +\frac{D}{AD-BC}\left[A\phi\left(x - \frac{C}{D}t\right) + B\tilde{\psi}\left(x - \frac{C}{D}t\right)\right] \end{cases} \\ &= \frac{AD}{AD-BC}\phi\left(x - \frac{C}{D}t\right) - \frac{BC}{AD-BC}\phi\left(x - \frac{A}{B}t\right) + \frac{BD}{AD-BC}\left[\underbrace{\tilde{\psi}\left(x - \frac{C}{D}t\right) - \tilde{\psi}\left(x - \frac{A}{B}t\right)}\right]. \end{aligned}$$

Note that the antiderivative $\tilde{\psi}(x)$ is **not unique**. However, the quantity

$$\tilde{\psi}\left(x - \frac{C}{D}t\right) - \tilde{\psi}\left(x - \frac{A}{B}t\right) \tag{101}$$

is **unique** and is **independent of** the choice of the antiderivative $\tilde{\psi}(x)$. For convenience, we can choose $\tilde{\psi}(x) = \int_0^x \psi(s)\,ds$ and obtain

$$\tilde{\psi}\left(x - \frac{C}{D}t\right) - \tilde{\psi}\left(x - \frac{A}{B}t\right) = \int_{x-\frac{A}{B}t}^{x-\frac{C}{D}t} \psi(s)\,ds. \tag{102}$$

Thus the solution formula for $u(x,t)$ is given by

$$u(x,t) = \frac{AD}{AD-BC}\phi\left(x - \frac{C}{D}t\right) - \frac{BC}{AD-BC}\phi\left(x - \frac{A}{B}t\right) + \frac{BD}{AD-BC}\int_{x-\frac{A}{B}t}^{x-\frac{C}{D}t} \psi(s)\,ds. \tag{103}$$

The solution is defined on all $(x,t)\in\mathbb{R}^2$. Since $\phi\in C^2(\mathbb{R})$ and $\psi\in C^1(\mathbb{R})$, we have $u(x,t)\in C^2(\mathbb{R}\times\mathbb{R})$. The solution (103) satisfies the initial value problem (93).

To see uniqueness of the solution (103), we may look at the case $\phi(x)=\psi(x)\equiv 0$ for all $x\in(-\infty,\infty)$ and from the above derivation, we must have $u(x,t)\equiv 0$ for all $(x,t)\in\mathbb{R}^2$. Hence the uniqueness follows. □

Remark 4.4 *By direct check, one can see that (103) is indeed a solution of (93). First, the solution $u(x,t)$ has the form $F(Bx-At)+G(Dx-Ct)$. Hence it must be a solution. Second, for any $x\in\mathbb{R}$, we have*

$$u(x,0)=\frac{AD}{AD-BC}\phi(x)-\frac{BC}{AD-BC}\phi(x)+\frac{BD}{AD-BC}\int_x^x\psi(s)\,ds=\phi(x)$$

and

$$\begin{aligned} u_t(x,0)&=\begin{cases}\frac{AD}{AD-BC}\left(-\frac{C}{D}\right)\phi'(x)-\frac{BC}{AD-BC}\left(-\frac{A}{B}\right)\phi'(x)\\ +\frac{BD}{AD-BC}\left(-\frac{C}{D}\right)\psi(x)-\frac{BD}{AD-BC}\left(-\frac{A}{B}\right)\psi(x)\end{cases}\\ &=\frac{BD}{AD-BC}\left(-\frac{C}{D}\right)\psi(x)-\frac{BD}{AD-BC}\left(-\frac{A}{B}\right)\psi(x)=\psi(x).\end{aligned}$$

Thus $u(x,t)$ given by (103) is indeed a solution of the equation satisfying the initial conditions.

Remark 4.5 ***(Domain of dependence for hyperbolic equations.)*** *By (103), the* ***domain of dependence interval*** *of the point (x_0,t_0) is the interval $\left[x_0-\frac{A}{B}t_0,x_0-\frac{C}{D}t_0\right]$ (or the interval $\left[x_0-\frac{C}{D}t_0,x_0-\frac{A}{B}t_0\right]$ if $x_0-\frac{C}{D}t_0$ is smaller) lying on the x-axis. Only the values of $\phi(x)$ and $\psi(x)$ on the interval will determine the value of u at the point (x_0,t_0).*

In case equation (93) is parabolic (i.e. $A=C,\ B=D$) and has the same initial conditions, we can still solve it (note that now the equation is **degenerate parabolic**).

Lemma 4.6 ***(Parabolic equation with initial conditions.)*** *Let $\phi\in C^3(\mathbb{R})$ and $\psi\in C^2(\mathbb{R})$ be two given functions. Assume $ac=b^2$ in equation (48) (we view y as time and here we denote it as t). Consider the* ***parabolic equation with initial conditions (at time*** $t=0$***):***

$$\begin{cases}\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial t}\right)\left[\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial t}\right)u\right]=0, & u=u(x,t)\\ u(x,0)=\phi(x), \quad u_t(x,0)=\psi(x), & x\in(-\infty,\infty),\end{cases}\tag{104}$$

where $A,\ B$ are constants with $B\neq 0$ (this is same as in Lemma 4.2; B is the coefficient of the operator $\frac{\partial}{\partial t}$). Then the initial value problem (104) has a ***unique*** *solution $u(x,t)\in C^2(\mathbb{R}^2)$. Moreover, the solution is given by*

$$u(x,t)=\phi\left(x-\frac{A}{B}t\right)+\frac{t}{B}\left[A\phi'\left(x-\frac{A}{B}t\right)+B\psi\left(x-\frac{A}{B}t\right)\right],\tag{105}$$

where $(x,t)\in\mathbb{R}^2$. Note: there is only ***1-parameter family of characteristic lines*** *$Bx-At=$ const. for the parabolic equation in (104). Since we assume $B\neq 0$, the line $t=0$ is* ***not a characteristic line****, which is good.*

Proof. Since $B\neq 0$, we choose the general solution to have the form (see (79))

$$u(x,t)=F(Bx-At)+\frac{t}{B}G(Bx-At),\quad B\neq 0,\tag{106}$$

then we have

$$\begin{cases} F(Bx) = \phi(x) \\ -AF'(Bx) + \frac{1}{B}G(Bx) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

which gives

$$\begin{cases} BF'(Bx) = \phi'(x) \\ -ABF'(Bx) + G(Bx) = B\psi(x), \quad x \in \mathbb{R}, \end{cases}$$

and we conclude

$$\begin{cases} F(Bx) = \phi(x) \\ G(Bx) = B\psi(x) + A\phi'(x). \end{cases} \tag{107}$$

By the above we will get

$$\begin{aligned} u(x,t) &= F(Bx - At) + \frac{t}{B}G(Bx - At) = F\left(B\left(x - \frac{A}{B}t\right)\right) + \frac{t}{B}G\left(B\left(x - \frac{A}{B}t\right)\right) \\ &= \phi\left(x - \frac{A}{B}t\right) + \frac{t}{B}\left[B\psi\left(x - \frac{A}{B}t\right) + A\phi'\left(x - \frac{A}{B}t\right)\right], \end{aligned} \tag{108}$$

Since we assume $\phi \in C^3(\mathbb{R})$ and $\psi \in C^2(\mathbb{R})$, the solution $u(x,t)$ given by (108) satisfies $u(x,t) \in C^2(\mathbb{R} \times \mathbb{R})$. □

Remark 4.7 *By direct check, one can see that (116) is indeed a solution of (104).*

Remark 4.8 ***(Important.)*** *Note that the value of the solution u at (x_0, t_0) depends only on the initial data ϕ and ψ at the point $x_0 - \frac{A}{B}t_0$.*

Remark 4.9 ***(1-parameter family of characteristic lines for parabolic equations.) (This is a continuation of Remark 4.3.)*** *In case $B = 0$ (of course, we must have $A \neq 0$), equation becomes*

$$\begin{cases} u_{xx}(x,t) = 0, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty, \infty), \end{cases} \tag{109}$$

and the general solution for $u(x,t)$ is $u(x,t) = h(t)x + g(t)$ for arbitrary functions of $h(t)$ and $g(t)$, $t \in (-\infty, \infty)$. Now we need to require

$$\begin{cases} u(x,0) = h(0)x + g(0) = \phi(x), \quad x \in (-\infty, \infty), \\ u_t(x,0) = h'(0)x + g'(0) = \psi(x), \quad x \in (-\infty, \infty). \end{cases} \tag{110}$$

Clearly, in general, (110) has ***no solution****. But if $\phi(x)$ and $\psi(x)$ are of the form $ax+b$, $cx+d$, then (110) has* ***infinitely many solutions****. Again, the reason for this is that* ***the data is prescribed on the line $t = 0$, which is a characteristic line.***

Remark 4.10 ***(Omit this in class.)*** *Here is another proof of Lemma 4.6 using formula (78). We divide the discussion into two cases. In case $A = 0$, the equation becomes $u_{tt} = 0$ with general solution given by $u(x,t) = g(x) + th(x)$ for arbitrary C^2 functions $h(x)$ and $g(x)$. To satisfy the initial conditions, the unique solution is given by $u(x,t) = \phi(x) + t\psi(x)$, $x \in (-\infty, \infty)$. In case $A \neq 0$, we choose the general solution to have the form*

$$u(x,t) = F(Bx - At) + \frac{x}{A}G(Bx - At), \quad A \neq 0, \tag{111}$$

then by the initial conditions, we need to solve

$$\begin{cases} \underbrace{F(Bx) + \frac{x}{A} G(Bx) = \phi(x)} \\ -AF'(Bx) - xG'(Bx) = \psi(x), \quad x \in \mathbb{R}, \end{cases} \tag{112}$$

which gives

$$\begin{cases} BF'(Bx) + \frac{1}{A} G(Bx) + \frac{x}{A} BG'(Bx) = \phi'(x) \\ -AF'(Bx) - xG'(Bx) = \psi(x), \quad x \in \mathbb{R}, \end{cases} \tag{113}$$

and therefore

$$\begin{cases} ABF'(Bx) + G(Bx) + xBG'(Bx) = A\phi'(x) \\ -ABF'(Bx) - xBG'(Bx) = B\psi(x), \quad x \in \mathbb{R}. \end{cases} \tag{114}$$

From the above, we first obtain

$$\underbrace{G(Bx) \ = A\phi'(x) + B\psi(x)}$$

and substitute it into the first identity of (112), we can get

$$F(Bx) = \phi(x) - \frac{x}{A}\left[A\phi'(x) + B\psi(x)\right], \qquad G(Bx) \ = A\phi'(x) + B\psi(x), \tag{115}$$

which gives

$$\begin{cases} F(Bx - At) = F\left(B\frac{Bx-At}{B}\right) = \phi\left(\frac{Bx-At}{B}\right) - \frac{1}{A}\left(\frac{Bx-At}{B}\right)\left[A\phi'\left(\frac{Bx-At}{B}\right) + B\psi\left(\frac{Bx-At}{B}\right)\right] \\ G(Bx - At) = G\left(B\frac{Bx-At}{B}\right) = A\phi'\left(\frac{Bx-At}{B}\right) + B\psi\left(\frac{Bx-At}{B}\right) \end{cases}$$

and so

$$\begin{aligned} u(x,t) &= F(Bx - At) + \frac{x}{A} G(Bx - At) \\ &= \begin{cases} \phi\left(\frac{Bx-At}{B}\right) - \frac{1}{A}\left(\frac{Bx-At}{B}\right)\left[A\phi'\left(\frac{Bx-At}{B}\right) + B\psi\left(\frac{Bx-At}{B}\right)\right] \\ +\frac{x}{A}\left[A\phi'\left(\frac{Bx-At}{B}\right) + B\psi\left(\frac{Bx-At}{B}\right)\right] \end{cases} \\ &= \phi\left(x - \frac{A}{B}t\right) + \frac{t}{B}\left[A\phi'\left(x - \frac{A}{B}t\right) + B\psi\left(x - \frac{A}{B}t\right)\right], \quad (x,t) \in \mathbb{R}^2, \end{aligned} \tag{116}$$

which is the same as (108). From this proof we see that, whether $A = 0$ or not is not important. In either case we can get unique solution. The key point is that we must require $B \neq 0$.

Example 4.11 *Find the solution $u(x,t)$ satisfying:*

$$u_{xx} + 2u_{xt} + u_{tt} = 0, \quad u(x,0) = x^2, \quad u_t(x,0) = e^x, \quad x \in \mathbb{R}.$$

Solution:

By (78) in Lemma 3.1 and the identity

$$u_{xx} + 2u_{xt} + u_{tt} = (\partial_x + \partial_t)\left[(\partial_x + \partial_t) u\right] = 0, \quad A = B = 1,$$

we see that the general solution of this degenerate parabolic equation has the form

$$u(x,t) = F(x-t) + xG(x-t)$$

for arbitrary C^2 functions $F(\cdot)$, $G(\cdot)$ defined on $\mathbb{R}$. Hence we need to solve

$$\begin{cases} F(x) + xG(x) = x^2 \\ -F'(x) - xG'(x) = e^x \end{cases}$$

and obtain

$$\begin{cases} F'(x) + G(x) + xG'(x) = 2x \\ -F'(x) - xG'(x) = e^x \end{cases}$$

and then

$$G(x) = 2x + e^x.$$

Next, by the identity $F(x) + xG(x) = x^2$ we have

$$F(x) = x^2 - xG(x) = x^2 - x(2x + e^x) = -x^2 - xe^x.$$

The answer for $u(x,t)$ is

$$\begin{aligned} u(x,t) &= F(x-t) + xG(x-t) \\ &= -(x-t)^2 - (x-t)e^{x-t} + x\left[2(x-t) + e^{x-t}\right] = (x-t)(x+t) + te^{x-t} \end{aligned}$$

□

5 The wave equation with initial conditions.

Remark 5.1 *One can see W. A. Strauss PDE book (second edition), p. 11-13, for a brief explanation of how to derive wave equation from a flexible, elastic homogeneous string which undergoes relatively small transverse vibrations. However, it is difficult to understand his explanation.*

In this section, we look at **one-dimensional** wave equation for the function $u(x,t)$, given by

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{117}$$

Under some assumptions (small amplitude, etc ...), the equation describes **the motion of a vibrating string**, where $u(x,t)$ represents the position of the string (I will not derive this in class). Here $c > 0$ is a constant given by

$$c = \sqrt{\frac{T}{\rho}}, \tag{118}$$

where T is the **tension** of the string and ρ is the **density** of the string (both are assumed to be constants, not very realistic at all).

Remark 5.2 *By a change of variable in time:*

$$x = x, \quad \tilde{t} = ct,$$

the function $v\left(x,\tilde{t}\right) = u(x,t) = u\left(x, \frac{\tilde{t}}{c}\right)$ *will satisfy (117) with* $c = 1$. *Hence, the two equations* $u_{tt} = c^2 u_{xx}$ *and* $u_{tt} = u_{xx}$ *are equivalent. Some books discuss wave equation in the form* $u_{tt} = u_{xx}$ *only.*

As (117) is a physical equation, it has initial conditions. They are the **initial position** and **initial velocity** of the string, given by

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R}. \tag{119}$$

Remark 5.3 ***(Useful motivation.)*** *Roughly speaking, the wave equation comes from* ***Newtonian mechanics****, hence as long as the initial position and initial velocity are known, the whole process of motion is uniquely determined.*

With the above conditions, the solution $u(x,t)$ satisfying (117) and (119) **exists** and is **unique** (which can be seen from (103)). Moreover, if we change $\phi(x)$ and $\psi(x)$ a little bit, then the corresponding solution will also change a little bit (we shall see this soon). In this sense, we say that the problem (117) and (119) is **well-posed**.

Since one can factorize the equation (117) as

$$\left(c\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\left[\left(c\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)u\right]=0,\quad u=u(x,t), \tag{120}$$

by previous discussion we know the general solution of (117), defined on the whole space $\mathbb{R}^2$, is given by

$$u(x,t)=F(x+ct)+G(x-ct),\quad (x,t)\in(-\infty,\infty)\times(-\infty,\infty) \tag{121}$$

for arbitrary C^2 functions $F,\ G$ defined on $\mathbb{R}$.

Definition 5.4 *Any line of the form $x+ct=const.$ or $x-ct=const.$ is called a* ***characteristic line*** *of the wave equation (117). A wave equation has* ***2-parameter family*** *of characteristic lines.*

Remark 5.5 ***(The geometric meaning of the wave equation.)*** *The solution $u(x,t)$ given by (121) consists of two* ***traveling waves*** *moving in opposite directions (positive and negative x-direction) with the same speed c (the graph of $F(x+ct)$ moves to the left and the graph of $G(x-ct)$ moves to the right; draw a picture on blackboard). Moreover, since $c=\sqrt{T/\rho}$, if the tension T is large and the density ρ is small, then the traveling wave speed is large. This matches with physical observation.*

Without remembering the formula in Lemma 4.2, one can easily derive the solution formula satisfying the conditions (119). We need to require

$$\begin{cases} u(x,0)=F(x)+G(x)=\phi(x) \\ u_t(x,0)=cF'(x)-cG'(x)=\psi(x). \end{cases} \tag{122}$$

By this we obtain

$$F'(x)=\frac{c\phi'(x)+\psi(x)}{2c},\qquad G'(x)=\frac{c\phi'(x)-\psi(x)}{2c}, \tag{123}$$

and so

$$F(x)=\frac{\phi(x)}{2}+\frac{1}{2c}\int_0^x\psi(s)\,ds+\delta,\qquad G(x)=\frac{\phi(x)}{2}-\frac{1}{2c}\int_0^x\psi(s)\,ds+\varepsilon \tag{124}$$

with suitable constants $\delta,\ \varepsilon$. Here $\delta+\varepsilon=0$ by (122). Hence we get the **unique solution** given by

$$u(x,t)=F(x+ct)+G(x-ct)=\underbrace{\frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds}. \tag{125}$$

We can conclude the following:

Lemma 5.6 *Assume ϕ and ψ in the initial conditions (119) satisfy $\phi\in C^2(\mathbb{R})$ and $\psi\in C^1(\mathbb{R})$. Then the function $u(x,t)$ given by (125) is the* ***unique*** *C^2 solution of the initial value problem*

$$\begin{cases} u_{tt}(x,t)=c^2u_{xx}(x,t) \\ u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x),\quad x\in(-\infty,\infty) \end{cases} \tag{126}$$

defined on the domain $(x,t)\in\mathbb{R}^2$.

Proof. (**Due to Lemma 4.2, we can omit the proof of this lemma.**) Given $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$, we define $u(x,t)$ as in (125) for $(x,t) \in \mathbb{R}^2$. It clearly satisfies $u(x,0) = \phi(x)$ for all $x \in (-\infty, \infty)$. We also have

$$u_t(x,t) = \frac{c}{2}\left[\phi'(x+ct) - \phi'(x-ct)\right] + \frac{1}{2}\left[\psi(x+ct) + \psi(x-ct)\right]$$

for all $(x,t) \in \mathbb{R}^2$, and so $u(x,t)$ satisfies $u_t(x,0) = \psi(x)$ for all $x \in (-\infty,\infty)$. Finally, note that

$$u_{tt}(x,t) = \frac{c^2}{2}\left[\phi''(x+ct) + \phi''(x-ct)\right] + \frac{c}{2}\left[\psi'(x+ct) - \psi'(x-ct)\right]$$

and

$$u_{xx}(x,t) = \frac{1}{2}\left[\phi''(x+ct) + \phi''(x-ct)\right] + \frac{1}{2c}\left[\psi'(x+ct) - \psi'(x-ct)\right],$$

which implies $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ for all $(x,t) \in \mathbb{R}^2$. Therefore, $u(x,t)$ given by (125) is indeed a C^2 solution of (126) on the domain $(x,t) \in \mathbb{R}^2$.

To check **uniqueness**, assume we have two C^2 solutions $u_1(x,t)$ and $u_2(x,t)$ of (126) on $\mathbb{R}^2$. Then the function $u(x,t) = u_1(x,t) - u_2(x,t)$ is a C^2 solution of the problem

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) \\ u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in (-\infty,\infty) \end{cases}$$

on $\mathbb{R}^2$. Since we know the general solution of $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ on $\mathbb{R}^2$ has the form $u(x,t) = F(x+ct) + G(x-ct)$ for some C^2 functions $F(\cdot)$, $G(\cdot)$ defined on $\mathbb{R}$, we have

$$\begin{cases} u(x,0) = F(x) + G(x) = 0 \\ u_t(x,0) = cF'(x) - cG'(x) = 0, \quad x \in (-\infty,\infty) \end{cases} \tag{127}$$

which implies $F'(x) = G'(x) = 0$ on $\mathbb{R}$. Therefore, both $F(\cdot)$, $G(\cdot)$ are **constant** functions, and we must have $u(x,t) \equiv 0$ on $\mathbb{R}^2$. The uniqueness property is verified. □

Remark 5.7 *(**Important.**) We call (125)* ***d'Alembert solution****. It was due to him in 1746. Note that the function*

$$\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] \tag{128}$$

is an ***even function*** *in t and the function*

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \tag{129}$$

is an ***odd function*** *in t.*

Remark 5.8 *(**Omit this in class**) If $\phi \in C^k(\mathbb{R})$ and $\psi \in C^{k-1}(\mathbb{R})$, then u given by (125) represents a classical solution $u \in C^k(\mathbb{R} \times [0,\infty))$ of the initial value problem, but* ***its regularity is not smoother (or worse) in general. Thus the wave equation does not produce instantaneous smoothing of the initial data as the heat equation does.***

5.1 Domain of dependence and influence of the initial conditions for wave equation.

For convenience of discussion, we confine to **nonnegative** time $t \geq 0$ (this is not really essential). Recall that the solution of the initial value problem (126) is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, \quad (x,t) \in (-\infty,\infty) \times [0,\infty). \tag{130}$$

For each fixed time $t = t_0 \in [0, \infty)$ and fixed $x_0 \in (-\infty, \infty)$, we get an interval of the form $[x_0 - ct_0, x_0 + ct_0]$. The value of $u(x_0, t_0)$ depends only on ϕ at $x_0 - ct_0$, $x_0 + ct_0$, and ψ on the interval $[x_0 - ct_0, x_0 + ct_0]$. The values of ϕ and ψ **outside** the interval $[x_0 - ct_0, x_0 + ct_0]$ **will not** affect the value of $u(x_0, t_0)$.

Definition 5.9 *The interval $[x_0 - ct_0, x_0 + ct_0]$ lying on the x-axis is called the* ***domain of dependence interval*** *of the point (x_0, t_0).*

Remark 5.10 ***The above also says that for our wave equation "disturbances" or "signals" only travel with speed*** *c. To understand this more clearly, see the definition of* ***"domain of influence".***

In view of this, for each fixed $x_0 \in (-\infty, \infty)$, there is a region $R \subset xt$-plane (**unbounded closed set**, lying on the upper half xt-plane) so that the values of u on this region can be **affected** by the values of $\phi(x_0)$ and $\psi(x_0)$. This region $R \subset xt$-plane is called the **domain of influence** of the point $(x_0, 0)$ (equivalently, a point $p \in R$ if and only if its domain of dependence interval **contains** the point x_0). The value of u at any point (x, t) lying **out of** of the region R is **not** affected by the values of $\phi(x_0)$ and $\psi(x_0)$. The **domain of influence** region R can be described as

$$R = \{(x, t) \in \mathbb{R} \times [0, \infty) : x - ct \leq x_0 \text{ and } x + ct \geq x_0\}, \tag{131}$$

where the two half-lines $x - ct = x_0$, $t \geq 0$, and $x + ct = x_0$, $t \geq 0$, intersect at the point $(x_0, 0)$.

Remark 5.11 *Draw a picture for the region R (or see Figure 1 in p. 39 of Strauss's undergraduate PDE book).*

Outside the **domain of influence** of the point $(x_0, 0)$, the value of $u(x, t)$ is not affected by the values of $\phi(x_0)$ and $\psi(x_0)$. In view of this, we have the following obvious fact:

Lemma 5.12 *For any given $\phi(x)$ and $\psi(x)$, the* ***domain of influence*** *of the interval (lying on the x-axis) $|x| \leq \sigma$ is the region (lying on $\mathbb{R} \times [0, \infty)$) $|x| \leq \sigma + ct$. In particular, if $\phi(x) \equiv \psi(x) \equiv 0$ for $|x| > \sigma$ (i.e. both $\phi(x)$ and $\psi(x)$ have* ***compact support*** *), then $u(x, t) \equiv 0$ on the region $|x| \geq \sigma + ct$.*

Remark 5.13 *Draw a picture for the above lemma.*

Corollary 5.14 *Assume both $\phi(x)$ and $\psi(x)$ have* ***compact support****. Then $u(x, t)$ given by (125) has compact support in x for each fixed t (however, the support of $u(x, t)$ will become larger if t gets larger).*

Remark 5.15 *(**Important.**) If we write the solution as (see (125))*

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = F(x + ct) + G(x - ct)$$

for some suitable function $F(\cdot)$ and $G(\cdot)$, then, in general, $F(\cdot)$ and $G(\cdot)$ ***do not*** *have compact support. For example, take $\phi(x) \equiv 0$ and*

$$\psi(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \leq x \leq \pi \\ 0, & x > 0, \end{cases}$$

then we have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\left(\tilde{\psi}(x+ct) - \tilde{\psi}(x-ct)\right),$$

where $\tilde{\psi}(x)$ *is an antiderivative of* $\psi(x)$, *given by*

$$\tilde{\psi}(x) = \begin{cases} -1, & x < 0 \\ -\cos x, & 0 \le x \le \pi \\ 1, & x > 0. \end{cases}$$

In such a case, the functions $F(\cdot)$ *and* $G(\cdot)$ ***do not*** *have compact support.*

Example 5.16 *Consider the wave equation with initial conditions:*

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & c > 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in (-\infty,\infty). \end{cases}$$

Assume that $\phi = \psi \equiv 0$ *on the interval* $[-1-c, 1+c]$. *Determine the region* $R \subset \mathbb{R} \times [0,\infty)$ *such that* $u(x,t) \equiv 0$ *on* R.

Solution:

The region R is the trianglular-shaped region given by

$$R = \{(x,t) : x - ct \ge -1 - c\} \bigcap \{(x,t) : x + ct \le 1 + c\} \bigcap \{(x,t) : t \ge 0\}.$$

□

Remark 5.17 *Draw a picture for the above answer.*

5.2 Space-time separable solutions of the wave equation.

We look for certain special solutions of the wave equation $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ with the **separable form:**

$$u(x,t) = f(x)\,g(t),$$

where $f(x)$ and $g(t)$ are C^2 functions defined on $(-\infty,\infty)$. Plug it into the equation to get

$$f(x)\,g''(t) = c^2 f''(x)\,g(t)$$

and then we get the identity (assume the denominators are nonzero)

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)}.$$

The above identity cannot hold unless there is some constant K such that

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = K, \quad \forall\, x \in \text{domain of } f, \quad \forall\, t \in \text{domain of } g.$$

There are three possibilities for the constant K : $K = \lambda^2 > 0,\ K = 0,\ K = -\lambda^2 < 0$. For the first case, we have

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda^2, \quad \lambda > 0,$$

which gives

$$g(t) = c_1 e^{\lambda ct} + c_2 e^{-\lambda ct}, \quad f(x) = d_1 e^{\lambda x} + d_2 e^{-\lambda x}, \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty), \tag{132}$$

where $c_1, \ c_2, \ d_1, \ d_2$ are integration constants. For the second case, we have

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = 0,$$

which gives

$$g(t) = c_1 t + c_2, \quad f(x) = d_1 x + d_2, \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty). \tag{133}$$

For the third case, we have

$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = -\lambda^2, \quad \lambda > 0,$$

which gives

$$g(t) = c_1 \sin(\lambda ct) + c_2 \cos(\lambda ct), \quad f(x) = d_1 \sin(\lambda x) + d_2 \cos(\lambda x), \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty). \tag{134}$$

Thus we have:

Lemma 5.18 *(**Classification of space-time separable solutions of the wave equation.**) The following are the **space-time separable** solutions of the wave equation*

$$u(x,t) = \begin{cases} \left(c_1 e^{\lambda ct} + c_2 e^{-\lambda ct}\right)\left(d_1 e^{\lambda x} + d_2 e^{-\lambda x}\right) \\ (c_1 t + c_2)(d_1 x + d_2) \\ [c_1 \sin(\lambda ct) + c_2 \cos(\lambda ct)]\,[d_1 \sin(\lambda x) + d_2 \cos(\lambda x)], \quad x \in (-\infty, \infty), \quad t \in (-\infty, \infty) \end{cases}$$

and there are no others. Here λ $(\lambda > 0)$, $c_1, \ c_2, \ d_1, \ d_2$ are all arbitrary constants.

Remark 5.19 *(**Important.**) The space-time separable solutions are important if we want to use **Fourier series** to express solutions of the wave equation.*

Example 5.20 *For each $n \in \mathbb{N}$ and $L > 0$, there is a solution, defined on $\mathbb{R}^2$, of the wave equation of the form (choose $\lambda = n\pi/L$ in the above lemma)*

$$u_n(x,t) = \left[A_n \sin\left(\frac{n\pi c}{L} t\right) + B_n \cos\left(\frac{n\pi c}{L} t\right)\right] \sin\left(\frac{n\pi x}{L}\right), \ A_n, \ B_n \text{ are const.,} \tag{135}$$

*which satisfies the **initial-boundary conditions (fixed-end condition)***

$$\begin{cases} u_n(x,0) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad (u_n)_t(x,0) = A_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right), \quad \forall\, x \in [0,L] \quad \textit{(initial cond.)} \\ u_n(0,t) = u_n(L,t) = 0, \quad \forall\, t \in [0,\infty) \quad \textit{(boundary cond.).} \end{cases} \tag{136}$$

*We call it a **harmonic** of the wave equation $u_{tt} = c^2 u_{xx}$ with **fixed ends** at $x = 0$ and $x = L$ (by Fourier series expansion applying to solutions of the wave equation with fixed ends, any such solution can be expressed as an **infinite sum** of **different harmonics**). For each $n \in \mathbb{N}$, the initial data $u_n(x,0) = B_n \sin\left(\frac{n\pi x}{L}\right)$ in (135) has n zeros on the interval $[0,L]$. Note that a wave equation with **fixed-end** boundary condition is **not enough to ensure a unique solution**. We still need the initial physical conditions $u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), \ x \in [0,L]$, to guarantee a* *unique solution (see F. John, PDE, 4th edition, p. 42-44 for a geometrical explanation). One can rewrite* $u_n(x,t)$ *as*

$$u_n(x,t) = R_n\left[\cos\theta_n \sin\left(\frac{n\pi c}{L}t\right) + \sin\theta_n \cos\left(\frac{n\pi c}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$
$$= R_n \sin\left(\frac{n\pi c}{L}t + \theta_n\right)\sin\left(\frac{n\pi x}{L}\right), \tag{137}$$

where $R_n = \sqrt{A_n^2 + B_n^2}$ *and the angle* θ_n *satisfies* $\cos\theta_n = \frac{A_n}{R_n}$, $\sin\theta_n = \frac{B_n}{R_n}$. *Note that the value of* $u_n(x,t)$ *lies between* $-R_n$ *and* R_n. *We call* R_n *the* ***amplitude*** *of* $u_n(x,t)$ *and call* θ_n *the* ***phase*** *of* $u_n(x,t)$. *The solution (137), defined on* $[0,L]$, *is* ***periodic*** *in time with*

$$u_n\left(x, t + \frac{2L}{nc}\right) = u_n(x,t), \quad \forall\, x \in [0,L], \quad t \in \mathbb{R}.$$

The time $\frac{2L}{nc}$ *is called the* ***period*** *of the solution.*

5.3 Conservation of the total energy for wave equation.

Lemma 5.21 *(**Equipartition of energy for wave equations with compact support initial data.**) Assume both* $\phi(x)$ *and* $\psi(x)$ *have* ***compact support*** *(this assumption is essential) in (126) and* $u \in C^2(\mathbb{R}\times(-\infty,\infty))$ *solves the initial value problem (126). Define the* ***kinetic energy*** *and the* ***potential energy*** *for the solution* $u(x,t)$ *as*

$$k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx \quad \text{and} \quad p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx, \quad t \in (-\infty,\infty). \tag{138}$$

Then we have (1). $k(t) + c^2 p(t)$ *is a* ***constant*** *for all time* $t \in (-\infty,\infty)$. (2). *Moreover, for* $t \in (-\infty,\infty)$ *such that* $|t|$ *is* ***large enough***, *we have* $k(t) = c^2 p(t)$. *In particular, we have*

$$k(t) = c^2 p(t) = const. \tag{139}$$

when $|t|$ *is large enough.*

Remark 5.22 *By the conclusion* (1), *we see that the energy is equal to*

$$k(t) + c^2 p(t) = k(0) + c^2 p(0)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,0)\,dx + \frac{c^2}{2}\int_{-\infty}^{\infty} u_x^2(x,0)\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\psi^2(x)\,dx + \frac{c^2}{2}\int_{-\infty}^{\infty}(\varphi'(x))^2\,dx,$$

where we see that the improper integrals are actually proper integrals since $\phi(x)$ *and* $\psi(x)$ *have* ***compact support*** *...*

Proof. (1). By Corollary 5.14, we know that $u(x,t)$ given by (125) have compact support in x **for each fixed** $t \in (-\infty,\infty)$. Computing

$$\begin{cases} u_t(x,t) = \frac{c}{2}\left[\phi'(x+ct) - \phi'(x-ct)\right] + \frac{1}{2}\left[\psi(x+ct) + \psi(x-ct)\right] \\ u_x(x,t) = \frac{1}{2}\left[\phi'(x+ct) + \phi'(x-ct)\right] + \frac{1}{2c}\left[\psi(x+ct) - \psi(x-ct)\right], \end{cases} \tag{140}$$

we can see that both $u_t(x,t)$ and $u_x(x,t)$ also have **compact support** in x for each fixed $t \in (-\infty,\infty)$. Hence the two **improper** integrals in (138) both **converge** (both are actually ***proper*** integrals). **Hence, the differentiation with respect to time can commute with the integral.** That is, we have

$$\frac{d}{dt}\left[k(t)+c^2p(t)\right]=\frac{d}{dt}\left[\frac{1}{2}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx+\frac{c^2}{2}\int_{-\infty}^{\infty}u_x^2(x,t)\,dx\right]$$
$$=\int_{-\infty}^{\infty}u_t(x,t)\,u_{tt}(x,t)\,dx+c^2\int_{-\infty}^{\infty}u_x(x,t)\,u_{xt}(x,t)\,dx$$
$$=c^2\int_{-\infty}^{\infty}\left[u_t(x,t)\,u_{xx}(x,t)+u_x(x,t)\,u_{xt}(x,t)\right]dx$$
$$=c^2\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[u_t(x,t)\,u_x(x,t)\right]dx=c^2\left[u_x(x,t)\,u_t(x,t)\right]\Big|_{x=-\infty}^{x=\infty}=0,\quad\forall\, t\in(-\infty,\infty),$$

where we have used the fact that both $u_t(x,t)$ and $u_x(x,t)$ also have compact support in x for each fixed $t\in(-\infty,\infty)$. The proof of (1) is done.

Remark 5.23 *(**Omit this in class.**) If you have difficulty assuring the identity*

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx=2\int_{-\infty}^{\infty}u_t(x,t)\,u_{tt}(x,t)\,dx,$$

you can use definition and mean value theorem to see that

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx$$
$$=\lim_{h\to0}\int_{-\infty}^{\infty}\frac{u_t^2(x,t+h)-u_t^2(x,t)}{h}dx=\lim_{h\to0}\int_{-\infty}^{\infty}2u_t(x,t+\theta(h))\,u_{tt}(x,t+\theta(h))\,dx,$$

where $t+\theta(h)$ lies between t and $t+h$ and in the above two limits the time t is fixed. As we will let $h\to0$, we may assume that $h\in[-1,1]$ and so $t+\theta(h)\in[t-1,t+1]$ for all $h\in[-1,1]$. One can find a large number $M>0$ so that

$$2u_t(x,t+\theta(h))\,u_{tt}(x,t+\theta(h))\equiv0\ \text{for all}\ |x|\geq M\ \text{and all}\ h\in[-1,1],$$

which gives

$$\int_{-\infty}^{\infty}2u_t(x,t+\theta(h))\,u_{tt}(x,t+\theta(h))\,dx=\int_{-M}^{M}2u_t(x,t+\theta(h))\,u_{tt}(x,t+\theta(h))\,dx,\ \forall\ h\in[-1,1].$$

*Since the integrand $2u_t(x,s)\,u_{tt}(x,s)$ is a **continuous function** in $(x,s)\in[-M,M]\times[t-1,t+1]$, we have*

$$\lim_{s\to t}\int_{-M}^{M}2u_t(x,s)\,u_{tt}(x,s)\,dx=\int_{-M}^{M}2u_t(x,t)\,u_{tt}(x,t)\,dx,$$

which is an elementary fact in Advanced Calculus. The above implies

$$\lim_{h\to0}\int_{-\infty}^{\infty}2u_t(x,t+\theta(h))\,u_{tt}(x,t+\theta(h))\,dx$$
$$=\lim_{s\to t}\int_{-M}^{M}2u_t(x,s)\,u_{tt}(x,s)\,dx=\int_{-M}^{M}2u_t(x,t)\,u_{tt}(x,t)\,dx=\int_{-\infty}^{\infty}2u_t(x,t)\,u_{tt}(x,t)\,dx$$

and we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_t^2(x,t)\,dx$$
$$=\lim_{h\to0}\int_{-\infty}^{\infty}2u_t(x,t+\theta(h))\,u_{tt}(x,t+\theta(h))\,dx=\int_{-\infty}^{\infty}2u_t(x,t)\,u_{tt}(x,t)\,dx.$$

The proof is done.

For (2), by (140), we have

$$u_t^2(x,t) = \begin{cases} \frac{c^2}{4}\left[\underbrace{(\phi'(x+ct))^2 + (\phi'(x-ct))^2} - 2\phi'(x+ct)\phi'(x-ct)\right] \\ +\frac{1}{4}\left[\underbrace{\psi^2(x+ct) + \psi^2(x-ct)} + 2\psi(x+ct)\psi(x-ct)\right] \\ +\frac{c}{2}\left(\begin{array}{l} \underbrace{\phi'(x+ct)\psi(x+ct)} + \phi'(x+ct)\psi(x-ct) \\ -\phi'(x-ct)\psi(x+ct) - \underbrace{\phi'(x-ct)\psi(x-ct)} \end{array}\right) \end{cases}$$

and

$$c^2u_x^2(x,t) = \begin{cases} \frac{c^2}{4}\left[\underbrace{(\phi'(x+ct))^2 + (\phi'(x-ct))^2} + 2\phi'(x+ct)\phi'(x-ct)\right] \\ +\frac{1}{4}\left[\underbrace{\psi^2(x+ct) + \psi^2(x-ct)} - 2\psi(x+ct)\psi(x-ct)\right] \\ +\frac{c}{2}\left(\begin{array}{l} \underbrace{\phi'(x+ct)\psi(x+ct)} - \phi'(x+ct)\psi(x-ct) \\ +\phi'(x-ct)\psi(x+ct) - \underbrace{\phi'(x-ct)\psi(x-ct)} \end{array}\right). \end{cases}$$

By above, it suffices to show that

$$\begin{aligned} &\int_{-\infty}^{\infty}\left(\begin{array}{l} \frac{c^2}{4}\left[-2\phi'(x+ct)\phi'(x-ct)\right] + \frac{1}{4}\left[2\psi(x+ct)\psi(x-ct)\right] \\ +\frac{c}{2}\left[\phi'(x+ct)\psi(x-ct) - \phi'(x-ct)\psi(x+ct)\right] \end{array}\right)dx \\ = &\int_{-\infty}^{\infty}\left(\begin{array}{l} \frac{c^2}{4}\left[2\phi'(x+ct)\phi'(x-ct)\right] + \frac{1}{4}\left[-2\psi(x+ct)\psi(x-ct)\right] \\ +\frac{c}{2}\left[-\phi'(x+ct)\psi(x-ct) + \phi'(x-ct)\psi(x+ct)\right] \end{array}\right)dx. \end{aligned}$$

which is equivalent to showing that

$$\int_{-\infty}^{\infty}\left(\begin{array}{l} -c^2\phi'(x+ct)\phi'(x-ct) + \psi(x+ct)\psi(x-ct) \\ +c\phi'(x+ct)\psi(x-ct) - c\phi'(x-ct)\psi(x+ct) \end{array}\right)dx = 0. \tag{141}$$

Since, for any $x \in \mathbb{R}$, the two points $p = x+ct,\ q = x-ct$ have distance $2c|t|$. They **both** can not stay in the support of ϕ and ψ for large $|t|$ for **any** $x \in \mathbb{R}$. Hence if $|t| \geq 0$ is large enough, we must have

$$\begin{aligned} &\phi'(x+ct)\phi'(x-ct) = \psi(x+ct)\psi(x-ct) \\ &= \phi'(x+ct)\psi(x-ct) = \phi'(x-ct)\psi(x+ct) = 0 \quad \textbf{for all} \quad x \in (-\infty,\infty). \end{aligned}$$

The proof is done. □

Another energy property for the wave equation is the following:

Lemma 5.24 *(**Conservation of energy for wave equations with fixed-end condition.**) Assume $u \in C^2(\mathbb{R}^2)$ is a solution of the wave equation $u_{tt}(x,t) = c^2u_{xx}(x,t)$ and there is some $L > 0$ such that*

$$u(0,t) = u(L,t) = 0, \quad \forall\ t \in (-\infty,\infty), \tag{142}$$

*i.e. $u(x,t)$ satisfies "**fixed-end**" condition, then the **total energy over the interval** $[0,L]$:*

$$\frac{1}{2}\left(\int_0^L u_t^2(x,t)\,dx + c^2\int_0^L u_x^2(x,t)\,dx\right), \quad t \in (-\infty,\infty) \tag{143}$$

*is **independent** of time.*

Proof. We first note that by (142) we have $u_t(0,t) = u_t(L,t) = 0$ for all $t \in (-\infty, \infty)$. Compute

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\left(\int_0^L u_t^2(x,t)\,dx + c^2\int_0^L u_x^2(x,t)\,dx\right)\right] \\
&= \int_0^L u_t u_{tt} dx + c^2 \int_0^L u_x u_{xt} dx = c^2 \int_0^L (u_t u_{xx} + u_x u_{xt})\,dx \\
&= c^2 \int_0^L \frac{\partial}{\partial x}(u_t u_x)\,dx = c^2 \underbrace{u_t(L,t)} u_x(L,t) - c^2 \underbrace{u_t(0,t)} u_x(0,t) = 0, \quad \forall t \in (-\infty, \infty).
\end{aligned}$$

Hence the total energy over the interval $[0, L]$ is independent of time. □

6 Nonhomogeneous wave equation.

We now consider the nonhomogeneous wave equation

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), \quad (x,t) \in \mathbb{R}^2 \tag{144}$$

with initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty, \infty). \tag{145}$$

Here the functions $f(x,t) \in C^1(\mathbb{R}^2)$, $\phi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$ are all given.

Remark 6.1 ***(Important.)*** *Note that here we assume $f(x,t) \in C^1(\mathbb{R}^2)$ instead of $f(x,t) \in C^0(\mathbb{R}^2)$. We will explain the reason later on.*

By linearity, it suffices to look at the case when $\phi(x) = \psi(x) = 0$ due to the following observation:

Lemma 6.2 *If $u(x,t) \in C^2(\mathbb{R}^2)$ solves the problem*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R} \end{cases} \tag{146}$$

and $v(x,t) \in C^2(\mathbb{R}^2)$ solves the problem

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = 0, & (x,t) \in \mathbb{R}^2 \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{147}$$

then $w(x,t) = u(x,t) + v(x,t) \in C^2(\mathbb{R}^2)$ solves the problem

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{148}$$

Since we already know how to solve (147), it suffices to focus on (146). We want to derive a **solution formula** for $u(x,t) \in C^2(\mathbb{R}^2)$ satisfying (146).

We shall use the **change of variables method (characteristic coordinates method)** to solve (146). This method is quite **straightforward** and **natural**. We first note the following simple fact:

Lemma 6.3 *Let $p(x,y)$ be a C^1 function defined on $\mathbb{R}^2$ satisfying*

$$p(\lambda,\lambda) = 0 \quad \text{and} \quad p_x(\lambda,\lambda) = p_y(\lambda,\lambda), \quad \forall\, \lambda \in \mathbb{R}. \tag{149}$$

Then we must have

$$p_x(\lambda,\lambda) = p_y(\lambda,\lambda) = 0, \quad \forall\, \lambda \in \mathbb{R}. \tag{150}$$

Conversely, if it satisfies

$$p(\lambda,\lambda) = 0 \quad \text{and} \quad p_x(\lambda,\lambda) = 0, \quad \forall\, \lambda \in \mathbb{R}, \tag{151}$$

then we must also have $p_y(\lambda,\lambda) = 0$ also for all $\lambda \in \mathbb{R}$. ***Hence condition (149) is equivalent to condition (151).***

Proof. Assume $p(\lambda,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Then by the chain rule

$$0 = \frac{d}{d\lambda} p(\lambda,\lambda) = \frac{\partial p}{\partial x}(\lambda,\lambda)\frac{d\lambda}{d\lambda} + \frac{\partial p}{\partial y}(\lambda,\lambda)\frac{d\lambda}{d\lambda} = p_x(\lambda,\lambda) + p_y(\lambda,\lambda).$$

The rest is clear. □

Recall that (see Section 2.2) when we solve the homogeneous wave equation $u_{tt} - c^2 u_{xx} = 0$, we can use the change of variables (by use of the two characteristic lines)

$$\xi = x + ct, \qquad \eta = x - ct \tag{152}$$

to reduce the equation into the simple form (let $\tilde{u}(\xi,\eta) = u(x,t)$). By

$$\begin{cases} u_t = c\tilde{u}_\xi - c\tilde{u}_\eta, & u_x = \tilde{u}_\xi + \tilde{u}_\eta \\ u_{tt} = c^2\tilde{u}_{\xi\xi} - 2c^2\tilde{u}_{\xi\eta} + c^2\tilde{u}_{\eta\eta}, & u_{xx} = \tilde{u}_{\xi\xi} + 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}, \end{cases} \tag{153}$$

the new equation for $\tilde{u}(\xi,\eta)$ is

$$-4c^2\tilde{u}_{\xi\eta}(\xi,\eta) = 0, \tag{154}$$

where, in (154), we have used the identity $\tilde{u}_{\xi\eta} = \tilde{u}_{\eta\xi}$ for a C^2 solution. Hence we obtain the general solution for $\tilde{u}(\xi,\eta)$:

$$\tilde{u}(\xi,\eta) = F(\xi) + G(\eta) \tag{155}$$

and then the general C^2 solution of $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x,t) = F(x+ct) + G(x-ct) \tag{156}$$

for arbitrary C^2 functions $F(z)$ and $G(z)$ defined on $(-\infty,\infty)$.

For the nonhomogeneous equation $u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t)$ with initial conditions in (146), we can do the same change of variables and the equation for $\tilde{u}(\xi,\eta)$ becomes

$$-4c^2\tilde{u}_{\xi\eta}(\xi,\eta) = f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right). \tag{157}$$

To solve (157), we also need to convert the initial conditions for $u(x,t)$ into the initial conditions for $\tilde{u}(\xi,\eta)$. Since we have $u(x,t) = \tilde{u}(x+ct, x-ct)$, the condition $u(x,0) = 0$ for all $x \in (-\infty,\infty)$ becomes

$$\tilde{u}(\lambda,\lambda) = 0, \quad \forall\, \lambda \in (-\infty,\infty). \tag{158}$$

Next, by

$$u_t(x,t) = c\tilde{u}_\xi(x+ct, x-ct) - c\tilde{u}_\eta(x+ct, x-ct), \tag{159}$$

the condition $u_t(x,0)=0$ for all $x\in(-\infty,\infty)$ becomes

$$\tilde{u}_\xi(\lambda,\lambda)=\tilde{u}_\eta(\lambda,\lambda),\quad \forall\,\lambda\in(-\infty,\infty). \tag{160}$$

Now by Lemma 6.3, the initial conditions in (158) and (160) are equivalent to:

$$(1).\ \tilde{u}(\lambda,\lambda)=0,\qquad (2).\ \tilde{u}_\xi(\lambda,\lambda)=0,\qquad \forall\,\lambda\in(-\infty,\infty). \tag{161}$$

Thus the initial value problem (146) for the new function $\tilde{u}(\xi,\eta)$ becomes the following:

$$\begin{cases} \tilde{u}_{\xi\eta}(\xi,\eta)=\dfrac{\partial}{\partial\eta}\left(\tilde{u}_\xi(\xi,\eta)\right)=-\dfrac{1}{4c^2}f\left(\dfrac{\xi+\eta}{2},\ \dfrac{\xi-\eta}{2c}\right), & \forall\ (\xi,\eta)\in\mathbb{R}^2 \\ \tilde{u}(\xi,\xi)=0,\quad \tilde{u}_\xi(\xi,\xi)=0,\quad \forall\,\xi\in\mathbb{R}. \end{cases} \tag{162}$$

We first integrate the equation with respect to η to get $\tilde{u}_\xi(\xi,\eta)$. We need to take the condition $\tilde{u}_\xi(\xi,\xi)=0$ into consideration. In calculus, if we want to find a one-variable function $H(\eta)$ satisfying $H'(\eta)=g(\eta),\ H(\xi)=0$, then the answer is **unique** and is given by

$$H(\eta)=\int_\xi^\eta g(\sigma)\,d\sigma. \tag{163}$$

Motivated by the above, we write the equation in (162) as

$$\frac{\partial}{\partial\sigma}\left(\tilde{u}_\xi(\xi,\sigma)\right)=-\frac{1}{4c^2}f\left(\frac{\xi+\sigma}{2},\ \frac{\xi-\sigma}{2c}\right)$$

and apply the integral $\int_\xi^\eta d\sigma$ onto it to get

$$\tilde{u}_\xi(\xi,\eta)=\frac{\partial}{\partial\xi}\tilde{u}(\xi,\eta)=-\frac{1}{4c^2}\int_\xi^\eta f\left(\frac{\xi+\sigma}{2},\ \frac{\xi-\sigma}{2c}\right)d\sigma,\quad \tilde{u}_\xi(\xi,\xi)=0. \tag{164}$$

Next, we integrate the above with respect to ξ to get $\tilde{u}(\xi,\eta)$ and we need to take the condition $\tilde{u}(\xi,\xi)=0$ into consideration. By the same trick as in (163), if we want to find $G(\xi)$ satisfying $G'(\xi)=p(\xi),\ G(\eta)=0$, then the answer is **unique** and is given by

$$G(\xi)=\int_\eta^\xi p(r)\,dr. \tag{165}$$

Therefore, by similar trick, we write the equation in (164) as

$$\frac{\partial}{\partial r}\tilde{u}(r,\eta)=-\frac{1}{4c^2}\int_r^\eta f\left(\frac{r+\sigma}{2},\ \frac{r-\sigma}{2c}\right)d\sigma$$

and apply the integral $\int_\eta^\xi dr$ onto it to get

$$\tilde{u}(\xi,\eta)=\int_\eta^\xi\left[-\frac{1}{4c^2}\int_r^\eta f\left(\frac{r+\sigma}{2},\frac{r-\sigma}{2c}\right)d\sigma\right]dr, \tag{166}$$

which is the same as

$$\tilde{u}(\xi,\eta)=\frac{1}{4c^2}\int_\eta^\xi\left[\int_\eta^r f\left(\frac{r+\sigma}{2},\frac{r-\sigma}{2c}\right)d\sigma\right]dr,\quad \forall\ (\xi,\eta)\in\mathbb{R}^2. \tag{167}$$

It satisfies

$$\tilde{u}(\xi,\xi)=0,\quad \tilde{u}_\xi(\xi,\xi)=0,\quad \forall\,\xi\in\mathbb{R}. \tag{168}$$

Therefore, we conclude that the unique solution of problem (162) is given by the formula (167). To get a better picture for the domain of integration in (ξ, η)-plane, we write (167) as

$$\tilde{u}(\xi_0, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \left[\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c} \right) d\eta \right] d\xi, \quad \forall \ (\xi_0, \eta_0) \in \mathbb{R}^2 \tag{169}$$

or as the double integral

$$\tilde{u}(\xi_0, \eta_0) = \frac{1}{4c^2} \iint_{\tilde{\triangle}} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c} \right) d\xi d\eta. \tag{170}$$

The domain of integration $\tilde{\triangle}$ in the (ξ, η)-plane for the above double integral is given by (for convenience, in the picture below we assume $\eta_0 < \xi_0$)

$\tilde{\triangle}$: DRAW A PICTURE HERE IN (ξ, η)-PLANE !!!.

We note that $\tilde{\triangle}$ is the region inside a **right triangle** bounded by the three lines $L_1 : \xi = \xi_0$, $L_2 : \eta = \eta_0$, and $L_3 : \xi = \eta$ (the point (ξ_0, η_0) is on $L_1 \bigcap L_2$). Going back to (x,t)-plane, by the relation

$$\xi = x + ct, \quad \eta = x - ct \quad \Longleftrightarrow \quad x = \frac{\xi+\eta}{2}, \quad t = \frac{\xi-\eta}{2c},$$

we have the following correspondence (denote the point corresponding to (ξ_0, η_0) as (x_0, t_0)):

$$\begin{cases} (1).\ (\xi_0, \eta_0) = (x_0 + ct_0, x_0 - ct_0) \Longleftrightarrow (x_0, t_0) = \left(\frac{\xi_0+\eta_0}{2}, \frac{\xi_0-\eta_0}{2c} \right) \text{ (top point of the triangle)}, \\ (2).\ L_1 : \xi = \xi_0 \Longleftrightarrow \text{ line } x + ct = x_0 + ct_0 \text{ (characteristic line with negative slope)}, \\ (3).\ L_2 : \eta = \eta_0 \Longleftrightarrow \text{ line } x - ct = x_0 - ct_0 \text{ (characteristic line with positive slope)}, \\ (4).\ L_3 : \xi = \eta \Longleftrightarrow \text{ line } t = 0 \text{ (the } x\text{-axis)}. \end{cases} \tag{171}$$

Denote the region of (171) as $\triangle \subset (x,t)$ plane, which is a triangle with top point at (x_0, t_0). Its picture is given by

$\triangle$: DRAW A PICTURE HERE IN (x, t)-PLANE !!!.

As a consequence of the above, from Advanced Calculus, we have the change of variables identity for **double integrals** (the **absolute value** of the Jacobian $J = \partial(\xi, \eta) / \partial(x, t)$ is $2c$, $c > 0$) :

$$\tilde{u}(\xi_0, \eta_0) = \frac{1}{4c^2} \iint_{\tilde{\triangle}} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c} \right) d\xi d\eta = \frac{1}{2c} \iint_{\triangle} f(x,t)\, dxdt = u(x_0, t_0). \tag{172}$$

The last step is to express the double integral $\iint_{\triangle} f(x,t)\, dxdt$ as an **iterated integral**. Based on the shape of $\triangle$, it is easier to integrate with respect to x first and then with respect to t. We have

$$u(x_0, t_0) = \frac{1}{2c} \iint_{\triangle} f(x,t)\, dxdt = \frac{1}{2c} \int_0^{t_0} \left(\int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(x,t)\, dx \right) dt, \quad (x_0, t_0) \in \mathbb{R}^2. \tag{173}$$

Back to general $u(x,t)$, we can express the solution as

$$u(x,t) = \frac{1}{2c} \int_0^{t} \left(\int_{x - c(t-s)}^{x + c(t-s)} f(\theta, s)\, d\theta \right) ds, \quad (x,t) \in \mathbb{R}^2. \tag{174}$$

At this moment, we are ready to state **the main theorem in this section**, which is:

Theorem 6.4 *(**Solution for nonhomogeneous wave equation with zero initial data.**) Assume $f \in C^1(\mathbb{R}^2)$. Then the function $u(x,t)$ given by*

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta, s)\, d\theta \right) ds, \quad (x,t) \in \mathbb{R}^2 \tag{175}$$

*lies in the space $C^2(\mathbb{R}^2)$ and is a C^2 **solution** of the initial value problem*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R} \end{cases} \tag{176}$$

on the domain $(x,t) \in \mathbb{R}^2$. Moreover, the $C^2(\mathbb{R}^2)$ solution of the problem (176) is uniquely given by (175).

Proof. One first give a direct proof that, for $f \in C^1$, the function $u(x,t)$ given by (175) is indeed a C^2 **solution** of problem (176) defined on $\mathbb{R}^2$. For this purpose, we recall the following **derivative formula** from Calculus: Assume $\alpha(x),\ \beta(x) : [a,b] \to [c,d]$ are differentiable with respect to $x \in [a,b]$ and $h(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ is a C^1 function. Then we have the identity (also see Remark 6.5 below)

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} h(x,y)\, dy = h(x,\beta(x))\,\beta'(x) - h(x,\alpha(x))\,\alpha'(x) + \int_{\alpha(x)}^{\beta(x)} h_x(x,y)\, dy. \tag{177}$$

Now let $u(x,t)$ be the function given by (175). We clearly have

$$u(x,0) = \frac{1}{2c}\int_0^0 (\cdots)\, ds = 0, \quad \forall\, x \in \mathbb{R} \tag{178}$$

and

$$\begin{aligned} u_x(x,t) &= \frac{1}{2c}\frac{d}{dx}\left[\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\, d\theta\right) ds\right] = \frac{1}{2c}\int_0^t \frac{\partial}{\partial x}\left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\, d\theta\right) ds \\ &= \frac{1}{2c}\int_0^t \left[f(x+c(t-s),s) - f(x-c(t-s),s)\right] ds \end{aligned} \tag{179}$$

and

$$u_{xx}(x,t) = \frac{1}{2c}\int_0^t \underbrace{\frac{\partial}{\partial x}\left[f(x+c(t-s),s) - f(x-c(t-s),s)\right]}\, ds. \tag{180}$$

On the other hand, we have (let $h(x,t,s) = \int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\, d\theta$ and apply the identity (177))

$$\begin{aligned} u_t(x,t) &= \frac{1}{2c}\frac{d}{dt}\int_0^t (h(x,t,s))\, ds \\ &= h(x,t,t) \ \text{(this is zero)} + \frac{1}{2c}\int_0^t \frac{\partial}{\partial t}(h(x,t,s))\, ds \\ &= \frac{1}{2c}\int_0^t \frac{\partial}{\partial t}\left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\, d\theta\right) ds \\ &= \frac{1}{2c}\int_0^t \left[f(x+c(t-s),s)\,c - f(x-c(t-s),s)(-c)\right] ds \\ &= \frac{1}{2}\int_0^t \left[f(x+c(t-s),s) + f(x-c(t-s),s)\right] ds, \end{aligned} \tag{181}$$

which gives

$$u_t(x,0) = \frac{1}{2c}\int_0^0 (\cdots)\,ds = 0, \quad \forall\, x \in \mathbb{R} \tag{182}$$

and also

$$u_{tt}(x,t) = f(x,t) + \frac{1}{2}\int_0^t \underbrace{\frac{\partial}{\partial t}\left[f(x+c(t-s),s) + f(x-c(t-s),s)\right]}\,ds. \tag{183}$$

By the identity

$$\underbrace{\frac{\partial}{\partial t}\left[f(x+c(t-s),s) + f(x-c(t-s),s)\right]} = \underbrace{c\frac{\partial}{\partial x}\left[f(x+c(t-s),s) - f(x-c(t-s),s)\right]},$$

we conclude

$$u_{tt}(x,t) = f(x,t) + c^2 u_{xx}(x,t), \quad (x,t) \in \mathbb{R}^2. \tag{184}$$

The check is done.

Now we check uniqueness. If there is another C^2 solution $w(x,t)$ of the problem (176) on $\mathbb{R}^2$, we can look at the difference

$$v(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta\right) ds - w(x,t), \quad (x,t) \in \mathbb{R}^2.$$

Then $v \in C^2(\mathbb{R}^2)$ and it satisfies

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = 0, & (x,t) \in \mathbb{R}^2 \\ v(x,0) = 0, \quad v_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

By Lemma 5.6, we must have $v(x,t) \equiv 0$ on $\mathbb{R}^2$. The proof is done. □

Remark 6.5 *(**Change the order of differentiation and integration**.) If $h(x,y)$ and $\frac{\partial h}{\partial x}(x,y)$ are both in $C^0([a,b]\times[c,d])$, then the function*

$$H(x) := \int_c^d h(x,y)\,dy, \quad x \in [a,b]$$

is differentiable with respect to $x \in [a,b]$ and satisfies

$$H'(x) = \frac{d}{dx}\left(\int_c^d h(x,y)\,dy\right) = \int_c^d \frac{\partial h}{\partial x}(x,y)\,dy, \quad \forall\, x \in [a,b]. \tag{185}$$

In particular, we note that $H'(x)$ is continuous on $[a,b]$ due to $\frac{\partial h}{\partial x} \in C^0([a,b]\times[c,d])$.

Remark 6.6 *(**Important observation.**) In calculus, if $f(s) \in C^0(\mathbb{R})$ is a **continuous** function on $\mathbb{R}$, and we **integrate it twice over** $\mathbb{R}$, then the result will be a C^2 function on $\mathbb{R}$, i.e.*

$$u(t) = \int_0^t \left(\int_0^\rho f(s)\,ds\right) d\rho \in C^2(\mathbb{R}). \tag{186}$$

Note that here $\dim \mathbb{R} = 1$ and the times of integration over the domain space $\mathbb{R}$ is 2. However, if $f \in C^0(\mathbb{R}^2)$ and we look at

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta,s)\,d\theta\right) ds = \frac{1}{2c}\iint_\triangle f(\theta,s)\,d\theta ds \ \textit{(top point of } \triangle \textit{ is } (x,t)\textit{)}, \tag{187}$$

*then we have $\dim \mathbb{R}^2 = 2$ and the times of integration over the domain space $\mathbb{R}^2$ is 1 (2 iterated integrals over $\mathbb{R}$ is equal to 1 **double integral** over the domain space $\mathbb{R}^2$). **Therefore, we expect** $u(x,t)$ **given by (187) to lie in the space** $C^1(\mathbb{R}^2)$ **only (and this is so in general !!!).** Finally, if we assume $f \in C^1(\mathbb{R}^2)$, then $u(x,t)$ given by (187) will lie in the space $C^2(\mathbb{R}^2)$. In Evans PDE book, p. 81, Theorem 4, the author also assumes that $f \in C^1(\mathbb{R}^2)$ in order for $u(x,t)$ to be a C^2 solution of the problem (176) on $\mathbb{R}^2$.*

Remark 6.7 *(Omit this in class.)* ***(Important.)*** *By (181), it is easy to give an example of* $f \in C^0(\mathbb{R}^2)$ *but with* $u \notin C^2(\mathbb{R}^2)$. *We can take* $c = 1$ *and*

$$f(p,q) = |p + q - 1|, \quad (p,q) \in \mathbb{R}^2 \tag{188}$$

and get

$$\begin{aligned} 2u_t(x,t) &= \int_0^t [f(x + (t-s), s) + f(x - (t-s), s)]\, ds \\ &= \int_0^t |x + t - 1|\, ds + \int_0^t |x - t + 2s - 1|\, ds \\ &= \underbrace{|x + t - 1| \cdot t}+ \int_0^t |x - t + 2s - 1|\, ds, \quad (x,t) \in \mathbb{R}^2. \end{aligned} \tag{189}$$

We look at the behavior of $u(x,t)$ *near the point* $(x,t) = (-4,5)$. *At* $t = 5$, *we have*

$$|x + t - 1| \cdot t = 5|x + 4|, \tag{190}$$

which is ***not differentiable*** *at* $x = -4$. *For the second term in (189), at* $t = 5$, *we get*

$$\int_0^t |x - t + 2s - 1|\, ds = \int_0^5 |x + 2s - 6|\, ds = \frac{1}{2}\int_{x-6}^{x+4} |z|\, dz, \tag{191}$$

which is a ***differentiable function*** *of* $x \in (-\infty, \infty)$ *(since the integrand is a continuous function on* $\mathbb{R}$*). Therefore, we conclude that the function*

$$2u_t(x,5) = 5|x + 4| + \frac{1}{2}\int_{x-6}^{x+4} |z|\, dz \tag{192}$$

is ***not differentiable*** *at* $x = -4$.

By Theorem 6.4, we can state the following:

Theorem 6.8 *Consider the following initial value problem for nonhomogeneous wave equation:*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & (x,t) \in \mathbb{R}^2 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{193}$$

where $f \in C^1(\mathbb{R}^2)$, $\phi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$, *are given functions. Then the solution* $u(x,t) \in C^2(\mathbb{R}^2)$ *of (193) is* ***unique****, defined on* $\mathbb{R}^2$, *and is given by the formula*

$$u(x,t) = \begin{cases} \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds \\ +\frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta, s)\, d\theta\right) ds, \quad (x,t) \in \mathbb{R}^2. \end{cases} \tag{194}$$

Remark 6.9 ***(Important.)*** *By the representation formula (194), we can conclude the following: The* ***domain of dependence*** *of the solution* u *at the point* (x_0, t_0), $t_0 > 0$, *is given by a* ***compact triangular set*** $\triangle$ *in the* (x,t) *plane, enclosed by the three lines:*

$$\text{line } t = 0 \text{ (}x\text{-axis)}; \quad \text{line } x + ct = x_0 + ct_0; \quad \text{line } x - ct = x_0 - ct_0. \tag{195}$$

In the above lemma we assume $f \in C^1(\mathbb{R}^2)$ so that $u \in C^2(\mathbb{R}^2)$. **However, if** $f(x,t)$ **depends only on** x **(or only on** t**), then it suffices to assume that** $f \in C^0(\mathbb{R})$. **We have:**

Corollary 6.10 *Assume $f(x,t)$ in (176) is a* **continuous** *function depending only on x, i.e., $f(x,t) = f(x) \in C^0(\mathbb{R})$. Let $G(x)$ be a function satisfying $G''(x) = f(x)$ for all $x \in \mathbb{R}$, then the function $u(x,t)$ in (175) has the form*

$$u(x,t) = \frac{1}{2c^2}\left[G(x+ct) + G(x-ct) - 2G(x)\right], \qquad (x,t) \in \mathbb{R}^2, \tag{196}$$

where $u \in C^2(\mathbb{R}^2)$. Note that the solution $u(x,t)$ depends on both space and time.

Remark 6.11 *One can check that the solution (196) does not depend on the function $G(x)$ as long as it satisfies $G''(x) = f(x)$ for all $x \in \mathbb{R}$. That is, if we replace $G(x)$ by $\tilde{G}(x) = G(x) + ax + b$ for some constants a, b, then we still have $\tilde{G}''(x) = f(x)$. Using this $\tilde{G}(x)$ in (196) will give the same answer $u(x,t)$.*

Proof. The proof is straightforward. Let $F(x)$ be such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$ and $G(x)$ be such that $G'(x) = F(x)$ for all $x \in \mathbb{R}$ (and so $G''(x) = f(x)$). We have

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta)\, d\theta\right) ds = \frac{1}{2c}\int_0^t \left[F(x + c(t-s)) - F(x - c(t-s))\right] ds.$$

By

$$\frac{d}{ds}\left(\frac{-1}{c}G(x + c(t-s))\right) = F(x + c(t-s)), \quad \frac{d}{ds}\left(\frac{1}{c}G(x - c(t-s))\right) = F(x - c(t-s)),$$

we get

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left(\frac{-1}{c}G(x + c(t-s))\right)\Bigg|_{s=0}^{s=t} - \frac{1}{2c}\left(\frac{1}{c}G(x - c(t-s))\right)\Bigg|_{s=0}^{s=t} \\ &= -\frac{1}{2c^2}G(x) + \frac{1}{2c^2}G(x+ct) - \frac{1}{2c^2}G(x) + \frac{1}{2c^2}G(x-ct) \\ &= \frac{1}{2c^2}\left[G(x+ct) + G(x-ct) - 2G(x)\right], \qquad (x,t) \in \mathbb{R}^2. \end{aligned}$$

The proof is done. □

Corollary 6.12 *Assume $f(x,t)$ in (176) is a* **continuous** *function depending only on t, i.e., $f(x,t) = f(t) \in C^0(\mathbb{R})$. Then the function $u(x,t)$ in (175) has the form*

$$u(x,t) = \int_0^t f(s)(t-s)\, ds = \int_0^t \left(\int_0^\rho f(s)\, ds\right) d\rho \in C^2(\mathbb{R}^2), \qquad \forall\ (x,t) \in \mathbb{R}^2. \tag{197}$$

Note that $u(x,t)$ depends only on time t and is a C^2 function of $t \in \mathbb{R}$.

Remark 6.13 ***(Interesting observation !!)*** *A remarkable thing is that for a* **continuous** *function $f(t) \in C^0(\mathbb{R})$, the function $u(t) := \int_0^t f(s)(t-s)\, ds$ is a C^2 function of $t \in (-\infty, \infty)$ (not just a C^1 function). Note that $\int_0^t f(s)(t-s)\, ds$ is a* **convolution-type integral** *and we have*

$$\frac{d}{dt}\int_0^t f(s)(t-s)\, ds = \int_0^t f(s)\, ds \quad \text{and} \quad \frac{d^2}{dt^2}\int_0^t f(s)(t-s)\, ds = f(t), \quad \forall\ t \in \mathbb{R}. \tag{198}$$

On the other hand, since $u(t)$ satisfies $u''(t) = f(t)$, $u(0) = 0$, $u'(0) = 0$, we can also express $u(t)$ as the **double integral**

$$u(t) = \int_0^t \left(\int_0^\rho f(s)\, ds\right) d\rho, \quad \forall\ t \in \mathbb{R}. \tag{199}$$

Therefore, we have

$$\int_0^t f(s)(t-s)\,ds = \int_0^t \left(\int_0^\rho f(s)\,ds \right) d\rho, \tag{200}$$

which can be verified using the ***change of order of integration in the*** ρs***-plane****, i.e.*

$$\int_0^t \left(\int_0^\rho f(s)\,ds \right) d\rho = \int_0^t \left(\int_s^t f(s)\,d\rho \right) ds = \int_0^t f(s)(t-s)\,ds. \tag{201}$$

From (200), we see that $\int_0^t f(s)(t-s)\,ds$ *is a* C^2 *function of* $t \in (-\infty, \infty)$.

Proof. By (175), we have

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\theta, s)\,d\theta \right) ds = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(s)\,d\theta \right) ds = \underbrace{\int_0^t f(s)(t-s)\,ds}.$$

We have

$$\begin{cases} u_t(x,t) = f(t)(t-t) + \int_0^t \frac{\partial}{\partial t}(f(s)(t-s))\,ds = \int_0^t f(s)\,ds, \quad u_{tt}(x,t) = f(t), \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u_x(x,t) = u_{xx}(x,t) = 0, \quad u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(t). \end{cases}$$

□

<u>The above will be the coverage of the midterm exam on 2022-4-18.</u>