Revised on 2022-6-3

Remark 0.1 *This part consists of parabolic equations only.*

Remark 0.2 *This note is based on "Lecture-notes-on-PDE-2019-third-part.tex".*

1 The heat equation.

Recall that if we study the second order equation

$$au_{xx}+2bu_{xy}+cu_{yy}+2du_x+2eu_y+ku=f\left(x,y\right),\quad u=u\left(x,y\right) \tag{1}$$

where a, ..., k are all **constants** and $f\left(x,y\right)$ is a given function defined on some open set $\Omega\subseteq\mathbb{R}^2$, then we have the following classification result:

Theorem 1.1 *(**Refined canonical form.**) If the linear equation (1) is **elliptic**, then one can find a suitable linear change of variables (**using eigenvalues, eigenvectors and scalings**) and multiply the solution by some **exponential function** so that, eventually, the equation has the form*

$$v_{\xi\xi}+v_{\eta\eta}+cv=\phi\left(\xi,\eta\right),\quad v=v\left(\xi,\eta\right), \tag{2}$$

*for some constant $c\in\left(-\infty,\infty\right)$ and some function $\phi\left(\xi,\eta\right)$. If the equation (1) is **hyperbolic**, the equation has the form*

$$v_{\xi\xi}-v_{\eta\eta}+cv=\phi\left(\xi,\eta\right),\quad v=v\left(\xi,\eta\right), \tag{3}$$

*for some constant $c\in\left(-\infty,\infty\right)$ and some function $\phi\left(\xi,\eta\right)$. If the equation (1) is **parabolic** and **nondegenerate**, the equation has the form*

$$v_{\xi\xi}+cv_{\eta}=\phi\left(\xi,\eta\right),\quad v=v\left(\xi,\eta\right), \tag{4}$$

for some constant $c\neq 0$ and some function $\phi\left(\xi,\eta\right)$.

Remark 1.2 *(**Important.**) The constant c in the **elliptic case** can be $c>0$, or $c=0$, or $c<0$. For $c>0$, we can make it equal to 1 by doing the change of variables*

$$\tilde{\xi}=\sqrt{c}\xi,\quad \tilde{\eta}=\sqrt{c}\eta,\quad \tilde{v}\left(\tilde{\xi},\tilde{\eta}\right)=v\left(\frac{\xi}{\sqrt{c}},\frac{\eta}{\sqrt{c}}\right),$$

and for $c<0$, we can make it equal to -1 by doing the change of variables

$$\tilde{\xi}=\sqrt{-c}\xi,\quad \tilde{\eta}=\sqrt{-c}\eta,\quad \tilde{v}\left(\tilde{\xi},\tilde{\eta}\right)=v\left(\frac{\xi}{\sqrt{-c}},\frac{\eta}{\sqrt{-c}}\right).$$

*Thus in the **elliptic case**, we may simply assume $c=1$, or 0, or -1. The constant c in the **hyperbolic case** can be $c>0$, or $c=0$, or $c<0$. For $c<0$, by switching the role of ξ and η, we may assume $c>0$, or $c=0$. Hence for the **hyperbolic case,** eventually, we can simply assume $c=1$, or 0. Finally, for the **parabolic case**, the constant $c\neq 0$ can be $c>0$, or $c<0$. So eventually we can simply assume $c=1$, or -1. However, since most parabolic equations come from physical phenomenon involving the behavior of some quantity $v\left(\xi,\eta\right)$ depending on space and time. So ξ will represent **space variable** and η will represent **time variable**. In that case a **model parabolic equation** looks like (assume $\phi\left(\xi,\eta\right)=0$ for simplicity)*

$$\left(1\right).\ v_t=v_{xx}\quad \textit{or}\quad \left(2\right).\ v_t=-v_{xx}. \tag{5}$$

We call (1) *the* ***"forward heat equation" (or just heat equation)*** *and* (2) *the* ***"backward heat equation".*** *Since in reality, time cannot go backwards, so in a parabolic equation, we always fucus on the behavior of a solution* $v(x,t)$ ***as time goes forwards****, i.e., as* t *is* ***increasing****. One can use simple examples to see that,* ***as time goes forwards****, the heat equation* (1) *will make solution better, while the backward heat equation* (2) *will make solution worse (look at* $e^{-t}\sin x$ *and* $e^{t}\sin x$ *respectively). Thus, as time goes forwards, equation* (1) *is well-posed, while* (2) *is ill-posed. In this course, we will focus only on* (1) *(on the other hand,* ***as time goes backwards****,* (1) *will make solution worse and* (2) *will make solution better...).*

Definition 1.3 *Let* $v = v(\xi,\eta)$. *The equations* $v_{\xi\xi} + v_{\eta\eta} = 0$, $v_{\xi\xi} - v_{\eta\eta} = 0$, $v_{\xi\xi} - v_{\eta} = 0$, *are called* ***Laplace equation (elliptic equation), wave equation (hyperbolic equation), and heat equation (nondegenerate parabolic equation),*** *respectively.*

By Theorem 1.1 and Remark 1.2, we study the following **one-dimensional heat equation** (**nondegenerate parabolic equation**) (we focus on equation (1) in (5)):

$$u_t = u_{xx}, \quad u = u(x,t), \quad x \in \mathbb{R}. \tag{6}$$

For higher dimensional heat equation, it has the form

$$u_t = \triangle u, \quad u = u(x_1, \dots, x_n, t), \quad (x_1, \dots, x_n) \in \mathbb{R}^n,$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \text{ is the Laplace operator in } \mathbb{R}^n.$$

We will focus only on the one-dimensional case, i.e. $n = 1$, in this course.

2 Physical motivation for the heat equation.

We will give a brief explanation why the equation $u_t = \triangle u$ is called the **heat equation**. The reason is that it describes the behavior of the **temperature function** in the **heat flow phenomenon**. We look at the case $n = 3$ and let $\Omega \subset \mathbb{R}^3$ be a bounded "**heated domain**". At any time $t \in (0,\infty)$, let $u(x,y,z,t)$ be the temperature at the point $\mathbf{x} = (x,y,z) \in \Omega$. The **total heat** inside the domain Ω at time $t \in (0,\infty)$ is given by

$$H(t) = \int_{\Omega} u(\mathbf{x},t)\, d\mathbf{x}, \quad t \in (0,\infty). \tag{7}$$

The change of total heat inside Ω is given by (here we assume we can differentiate under the integral sign, which is actually so in most situations)

$$\frac{dH}{dt}(t) = \int_{\Omega} \frac{\partial u}{\partial t}(\mathbf{x},t)\, d\mathbf{x}, \quad t \in (0,\infty). \tag{8}$$

On the other hand, by physical experiment, the French mathematician J. Fourier discovered that the heat will flow from hot to cold regions in a way that is **proportional to the gradient of the temperature everywhere**, i.e., proportional to the quantity

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right), \tag{9}$$

with certain proportion constant κ (heat conductivity).

Moreover, due to the **conservation law of energy**, if there is a change in the total heat inside Ω, it must be due to the heat flowing out or flowing into Ω through **the boundary** $\partial\Omega$ (which is a surface in $\mathbb{R}^3$). By **conservation law** and **Fourier's law**, we also have the identity

$$\frac{dH}{dt}(t) = \int_{\partial\Omega} \kappa\left(\nabla u(\mathbf{x},t)\cdot \mathbf{N}(\mathbf{x})\right) dS \text{ (surface integral in } \mathbb{R}^3), \quad t\in(0,\infty), \tag{10}$$

where $\mathbf{N}(\mathbf{x})$ is the **unit outward normal** of $\partial\Omega$ at $\mathbf{x}\in\partial\Omega$ and $\nabla u(\mathbf{x},t)\cdot\mathbf{N}(\mathbf{x})$ is the inner product in $\mathbb{R}^3$ between the two vectors $\nabla u(\mathbf{x},t)$ and $\mathbf{N}(\mathbf{x})$. By (8) and (10) and the classical **divergence theorem**, we have the identity

$$\int_\Omega \frac{\partial u}{\partial t}(\mathbf{x},t)\,d\mathbf{x} = \int_{\partial\Omega} \kappa\left(\nabla u(\mathbf{x},t)\cdot\mathbf{N}(\mathbf{x})\right) dS = \int_\Omega \kappa\,\mathrm{div}\left(\nabla u(\mathbf{x},t)\right) d\mathbf{x} = \int_\Omega \kappa \triangle u(\mathbf{x},t)\,d\mathbf{x}$$

and so we conclude the **integral identity** on Ω :

$$\int_\Omega \left[\frac{\partial u}{\partial t}(\mathbf{x},t) - \kappa\triangle u(\mathbf{x},t)\right] d\mathbf{x} = 0, \quad \forall\, t\in(0,\infty). \tag{11}$$

Finally, we note that the analysis leading to the identity (11) is independent of the domain Ω, i.e. on any subdomain $\tilde{\Omega}\subset\Omega\subset\mathbb{R}^3$, as long as the heat flow phenomenon obeys the **conservation law** and **Fourier's law** in $\tilde{\Omega}$, we always have the identity

$$\int_{\tilde{\Omega}} \left[\frac{\partial u}{\partial t}(\mathbf{x},t) - \kappa\triangle u(\mathbf{x},t)\right] d\mathbf{x} = 0, \quad \forall\, t\in(0,\infty). \tag{12}$$

Since the domain $\tilde{\Omega}\subset\Omega\subset\mathbb{R}^3$ in (12) is arbitrary, we must have

$$\frac{\partial u}{\partial t}(\mathbf{x},t) = \kappa\triangle u(\mathbf{x},t), \quad \forall\ (\mathbf{x},t)\in\Omega\times(0,\infty), \tag{13}$$

which is the heat equation if we do suitable scaling to make $\kappa = 1$.

3 The 1-dimensional heat equation.

Most of the time (but not always) we will focus only on the 1-dimensional heat equation in this course ("1-dimensional" means space dimension n is 1). Unlike the 1-dimensional wave equation $u_{tt} = u_{xx}$, the heat equation $u_t = u_{xx}$ is much more difficult to solve. It is not difficult to guess some special solutions of $u_t = u_{xx}$, like

$$u(x,t) = x,\quad 2t+x^2,\quad x^3+6xt,\quad e^{t+x},\quad e^{t-x},\quad e^{-t}\cos x,\quad e^{-t}\sin x,\quad e^t\cosh x,\quad e^t\sinh x, \tag{14}$$

, etc. (note that $e^t\cosh x$ and $e^t\sinh x$ are linear combinations of e^{t+x} and e^{t-x}). All of the above solutions are defined on $\mathbb{R}\times\mathbb{R}$. One can check that the only **space-time separable solutions** of the heat equation are "**essentially**" of the form

$$u(x,t) = 1,\quad x,\quad e^{t+x},\quad e^{t-x},\quad e^{-t}\cos x,\quad e^{-t}\sin x \tag{15}$$

and no others. Also note that the solutions $u(x,t) = x,\ 2t+x^2,\ x^3+6xt$ are **polynomial solutions** with $u(x,0) = x,\ x^2,\ x^3$. There is a formula for a polynomial solution with $u(x,0) = x^n$ for any $n\in\mathbb{N}$. We will discuss this later on.

There are several major differences between the wave equation and the heat equation:

1. There is smoothing effect for heat equation, but not so in wave equation. We will discuss this later on.

2. For wave equation, if $u(x,t)$ is a solution, so is the function $u(x,-t)$, but for the heat equation, if $u(x,t)$ is a solution, the function $u(x,-t)$ is, in general, no longer a solution. Thus for the heat equation $u_t = u_{xx}$, one cannot reverse the direction of time.

3. (**Scaling property.**) If $u(x,t)$ is a solution of the heat equation, so is the function $\tilde{u}(x,t) = u(\lambda x, \lambda^2 t)$ for any constant $\lambda \neq 0$ (for the wave equation, if $u(x,t)$ is a solution, so is the function $\tilde{u}(x,t) = u(\lambda x, \lambda t)$ for any constant $\lambda \neq 0$).

Example 3.1 *(**Interesting solutions.**) We have the following interesting solutions of $u_t = u_{xx}$. They are all defined on $\mathbb{R} \times (-\infty, \infty)$.*

$$u(x,t) = e^{-t}\cos x, \quad e^{-t}\sin x \quad (\textit{\textbf{space-periodic} solutions, } u(x+2\pi,t) = u(x,t))$$

and

$$u(x,t) = e^{\pm\frac{x}{\sqrt{2}}}\cos\left(t \pm \frac{x}{\sqrt{2}}\right), \quad e^{\pm\frac{x}{\sqrt{2}}}\sin\left(t \pm \frac{x}{\sqrt{2}}\right) \quad (\textit{\textbf{time-periodic} solutions, } u(x,t+2\pi) = u(x,t))$$

and

$$u(x,t) = \begin{cases} x^2 + 2t, \quad x^3 + 6xt, \\ x^4 + 12x^2 t + 12t^2, \quad x^5 + 20x^3 t + 60xt^2, \ \dots. \end{cases} \quad (\textit{\textbf{polynomial} solutions}),$$

*where **we note the important property that** t **is like** x^2 (see Remark 3.7 below), and*

$$u(x,t) = e^{t+x}, \quad e^{t-x} \quad (\textit{\textbf{traveling wave} solutions}).$$

*Note that a function $u(x,t)$ of the form $u(x,t) = h(x - \lambda t)$ for some constant $\lambda \in \mathbb{R}$ is usually called a **traveling wave** function. To understand this terminology, you can plot the graphs of $u(x,0)$, $u(x,1)$, $u(x,2)$, $u(x,3)$, ..., and see that the graph of $u(x,0)$ is moving along the x-direction as time goes on.*

3.1 Polynomial solutions of the 1-dimensional heat equation.

If we do not impose any "**side condition**" on the heat equation $u_t = u_{xx}$, then on $\mathbb{R}^2$ it has infinitely many solutions. Recall that for the Laplace equation on $\mathbb{R}^2$, we have a family of polynomial solutions known as "**harmonic polynomials**". They are $1, x, y, xy, x^2 - y^2,$, etc., and they are all defined on $\mathbb{R}^2$. In terms of polar coordinates (r,θ) in the plane they have the forms $r^n \cos n\theta$, $r^n \sin n\theta$ for $n \in \mathbb{N} \bigcup \{0\}$. These solutions are important because we can use them to construct the **Poisson Integral Formula** on the disc.

For the heat equation $u_t(x,t) = u_{xx}(x,t)$ on $(x,t) \in \mathbb{R}^2$, there are also "**heat polynomials**" defined on the whole space $(x,t) \in \mathbb{R}^2$. In below, we show you how to derive them.

Consider the 1-dimensional heat equation $u_t(x,t) - u_{xx}(x,t) = 0$ with **initial data** (data at $t = 0$)

$$u(x,0) = p_0(x), \quad x \in (-\infty, \infty), \tag{16}$$

where $p_0(x)$ is a **polynomial** defined on $x \in (-\infty, \infty)$ with degree $n \in \mathbb{N} \bigcup \{0\}$. We try to look for a **space-time polynomial solution** $u(x,t)$ of the heat equation of the form

$$u(x,t) = p_0(x) + p_1(x)t + p_2(x)t^2 + p_3(x)t^3 + \cdots,$$

where each $p_i(x)$ is also a polynomial in $x \in (-\infty, \infty)$.

We compute

$$u_t(x,t) = p_1(x) + 2p_2(x)t + 3p_3(x)t^2 + \cdots$$

and

$$u_{xx}(x,t) = p_0''(x) + p_1''(x)t + p_2''(x)t^2 + p_3''(x)t^3 + \cdots,$$

and by comparing the coefficient functions (because we want $u_t(x,t) = u_{xx}(x,t)$), we require

$$\begin{cases} p_1(x) = p_0''(x), \\ p_2(x) = \frac{1}{2}p_1''(x) = \frac{1}{2}p_0''''(x), \\ p_3(x) = \frac{1}{3}p_2''(x) = \frac{1}{3!}p_0^{(6)}(x), \\ \cdots \\ p_k(x) = \frac{1}{3}p_2''(x) = \frac{1}{k!}p_0^{(2k)}(x), \\ \cdots. \end{cases} \tag{17}$$

Since $p_0(x)$ is a polynomial with finite degree $n \in \mathbb{N}$, the above process will stop at some k (i.e. $p_0^{(2k)}(x)$ will become 0 for some $k \in \mathbb{N}$). Moreover, we see that all of the other polynomials $p_1(x),\ p_2(x),\ p_3(x),\ ....$, can be **uniquely determined** by $p_0(x)$, which is the **initial condition** of the heat equation. Therefore, if the polynomial $p_0(x)$ is given in advance, we can find a **unique polynomial solution** of the heat equation $u_t = u_{xx}$ defined on $(x,t) \in \mathbb{R}^2$ satisfying (16).

We look at some simple examples.

Example 3.2 *Take $p_0(x) = x$. Then $p_1(x) = p_0''(x) = 0$ and so on. The function $u(x,t) = x$ is a polynomial solution of the heat equation.*

Example 3.3 *Take $p_0(x) = x^2$. Then $p_1(x) = p_0''(x) = 2$ and $p_2(x) = 0$ and so on. The function*

$$u(x,t) = p_0(x) + p_1(x)t = x^2 + 2t \tag{18}$$

is a polynomial solution of the heat equation.

Example 3.4 *Take $p_0(x) = x^3$. Then $p_1(x) = p_0''(x) = 6x$ and $p_2(x) = 0$ and so on. The function*

$$u(x,t) = p_0(x) + p_1(x)t = x^3 + 6xt \tag{19}$$

is a polynomial solution of the heat equation.

Example 3.5 *Take $p_0(x) = x^4$. Then $p_1(x) = p_0''(x) = 12x^2$ and $p_2(x) = 12$ and $p_3(x) = 0$ and so on. The function*

$$u(x,t) = p_0(x) + p_1(x)t + p_2(x)t^2 = x^4 + 12x^2t + 12t^2 \tag{20}$$

is a polynomial solution of the heat equation.

Example 3.6 *Take $p_0(x) = x^5$. Then $p_1(x) = p_0''(x) = 20x^3$ and $p_2(x) = 60x$ and $p_3(x) = 0$ and so on. The function*

$$u(x,t) = p_0(x) + p_1(x)t + p_2(x)t^2 = x^5 + 20x^3t + 60xt^2 \tag{21}$$

is a polynomial solution of the heat equation.

Remark 3.7 *In all of the above examples, note that* t *is like* x^2 *(so that each term has the same degree !!). Therefore, in the solution*

$$u(x,t) = x^5 + 20x^3 t + 60xt^2,$$

we see that each term has "degree 5".

Remark 3.8 *(**You will understand this remark later on.**) If we use the **representation formula** (you will see it later on)*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} p_0(y)\,dy, \quad t > 0, \tag{22}$$

*we will get the **same** answer on the domain* $(x,t) \in (-\infty,\infty) \times (0,\infty)$. *Note that the integral (22) converges for any polynomial* $p_0(y)$. *Moreover, differentiation can move into the integral sign.*

3.2 Finding the fundamental solution of the heat equation with the help of polynomial solutions.

Until now, we have found lots of **polynomial solutions** of the heat equation $u_t = u_{xx}$ on $(x,t) \in \mathbb{R}^2$, namely

$$x^2 + 2t, \quad x^3 + 6xt, \quad x^4 + 12x^2 t + 12t^2, \quad x^5 + 20x^3 t + 60xt^2, \quad \cdots, \text{ etc.} \tag{23}$$

Restricted onto the domain $\mathbb{R} \times (0,\infty)$, each of the polynomial solution can be expressed as the form

$$u(x,t) = g(t)\, h\left(\frac{x^2}{t}\right), \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

for some functions $g(t)$, $h(\theta)$ defined on $t \in (0,\infty)$, $\theta \in [0,\infty)$. For example, we can express

$$x^2 + 2t = g(t)\, h\left(\frac{x^2}{t}\right), \quad \text{where} \quad g(t) = t, \quad h(\theta) = \theta + 2$$

and

$$\begin{aligned} x^3 + 6xt &= t^{3/2}\left(\left(\frac{x}{\sqrt{t}}\right)^3 + 6\left(\frac{x}{\sqrt{t}}\right)\right) \\ &= g(t)\, h\left(\frac{x^2}{t}\right), \quad \text{where} \quad g(t) = t^{3/2}, \quad h(\theta) = \left(\sqrt{\theta}\right)^3 + 6\sqrt{\theta} \end{aligned}$$

and

$$\begin{aligned} x^4 + 12x^2 t + 12t^2 &= t^2\left(\left(\frac{x^2}{t}\right)^2 + 12\left(\frac{x^2}{t}\right) + 12\right) \\ &= g(t)\, h\left(\frac{x^2}{t}\right), \quad \text{where} \quad g(t) = t^2, \quad h(\theta) = \theta^2 + 12\theta + 12. \end{aligned}$$

Therefore, we can plug the general form $u(x,t) = g(t)\, h\left(\frac{x^2}{t}\right)$ into the heat equation $u_t = u_{xx}$ and see if we can find new interesting solutions. Compute

$$u_t(x,t) = g'(t)\, h\left(\frac{x^2}{t}\right) - g(t)\, h'\left(\frac{x^2}{t}\right)\frac{x^2}{t^2}, \qquad u_x(x,t) = g(t)\, h'\left(\frac{x^2}{t}\right)\frac{2x}{t}$$

and

$$u_{xx}(x,t) = g(t)\, h''\left(\frac{x^2}{t}\right)\frac{4x^2}{t^2} + g(t)\, h'\left(\frac{x^2}{t}\right)\frac{2}{t}.$$

We hope to have the identity

$$\underline{g'(t)\, h\left(\frac{x^2}{t}\right)} \underbrace{-g(t)\, h'\left(\frac{x^2}{t}\right)\frac{x^2}{t^2}} = \underbrace{g(t)\, h''\left(\frac{x^2}{t}\right)\frac{4x^2}{t^2}} \underline{+g(t)\, h'\left(\frac{x^2}{t}\right)\frac{2}{t}}, \quad \forall\ (x,t) \in \mathbb{R}\times(0,\infty), \tag{24}$$

which is **possible** if we require

$$\begin{cases} -h'(\theta) = 4h''(\theta), & \theta \in [0,\infty) \\ g'(t)\, h(\theta) = \left(g(t)\,\frac{2}{t}\right) h'(\theta), & \theta \in [0,\infty), \quad t \in (0,\infty). \end{cases} \tag{25}$$

Solving the first equation, we get the general solution $h(\theta) = A + Be^{-\frac{\theta}{4}}$ for arbitrary constants $A,\ B$ and we choose $A = 0,\ B = 1$ and plug $h(\theta) = e^{-\frac{\theta}{4}}$ into the second equation to get the equation for g :

$$g'(t) = -\frac{1}{2t} g(t), \tag{26}$$

which gives the general solution $g(t) = \frac{C}{\sqrt{t}}$ for arbitrary constant C. Therefore, we see that

$$u(x,t) = g(t)\, h\left(\frac{x^2}{t}\right) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}, \quad (x,t) \in \mathbb{R}\times(0,\infty) \tag{27}$$

is a **new solution** of the heat equation on $\mathbb{R}\times(0,\infty)$. Note that this solution is different from any solution you encountered before. □

Remark 3.9 *If $g(t)$ and $h(\theta)$ are from a polynomial solution $u(x,t)$, then they will satisfy (24) too.*

Remark 3.10 *(**Important.**) If we use the fact: if $u_j(x,t)$ is a solution for the one-dimensional heat equation $u_t = u_{xx}$ on $\mathbb{R}\times(0,\infty)$, then the function*

$$u(\mathbf{x},t) = u_1(x_1,t)\ \cdots\ u_n(x_n,t), \quad \mathbf{x} = (x_1,\ \ldots,x_n) \in \mathbb{R}^n \tag{28}$$

is a solution of the heat equation $u_t = \triangle u$ on $\mathbb{R}^n\times(0,\infty)$. With this, by (27), we will obtain the solution

$$u(\mathbf{x},t) = \frac{1}{\sqrt{t}} e^{-\frac{x_1^2}{4t}}\ \cdots\ \frac{1}{\sqrt{t}} e^{-\frac{x_n^2}{4t}} = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0 \tag{29}$$

of the heat equation $u_t = \triangle u$ on $\mathbb{R}^n\times(0,\infty)$.

By (29), we now define the following (for **normalization** purpose, we divide the solution in (29) by the constant $(4\pi)^{n/2}$; see Lemma 3.17):

Definition 3.11 *The function*

$$\Phi(\mathbf{x},t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}, & \mathbf{x} \in \mathbb{R}^n, \quad t > 0 \\ 0, \quad \mathbf{x} \in \mathbb{R}^n, & t \le 0. \end{cases} \tag{30}$$

*is called the **fundamental solution** of the heat equation. For each fixed time t, it is **radial** in $\mathbf{x} \in \mathbb{R}^n$. Moreover it satisfies the heat equation $\partial_t u = \triangle u$ in $\mathbb{R}^{n+1}\setminus\{(0,0)\}$ and is **invariant** under the **space-time scaling** $\Phi(\mathbf{x},t) \to \lambda^n \Phi(\lambda\mathbf{x},\lambda^2 t)$, i.e. we have*

$$\lambda^n \Phi\left(\lambda\mathbf{x},\lambda^2 t\right) = \Phi(\mathbf{x},t), \quad \forall\ \lambda > 0, \quad \forall\ (\mathbf{x},t) \in \mathbb{R}^{n+1}. \tag{31}$$

Remark 3.12 *(1). The* ***only singularity*** *of* Φ *is at the point* $(0,0)$, *i.e.* $\Phi(\mathbf{x},t)\in C^\infty(\mathbb{R}^{n+1}\backslash\{(0,0)\})$ *and it is* ***not continuous*** *at* $(0,0)$. *To understand the property* $\Phi(\mathbf{x},t)\in C^\infty(\mathbb{R}^{n+1}\backslash\{(0,0)\})$ *you need to know the fact that for each fixed* $\mathbf{x}_0\neq 0\in\mathbb{R}^n$, *the function*

$$\psi(t)=\begin{cases}\dfrac{1}{(4\pi t)^{n/2}}e^{-\frac{|\mathbf{x}_0|^2}{4t}}, & t>0\\[2mm] 0, \quad t\leq 0\end{cases} \tag{32}$$

is a C^∞ *function of* $t\in(-\infty,\infty)$. *On the other hand, for* $\mathbf{x}_0=0$, $\psi(t)$ *becomes*

$$\psi(t)=\begin{cases}\dfrac{1}{(4\pi t)^{n/2}}, & t>0\\[2mm] 0, \quad t\leq 0,\end{cases}$$

with $\lim_{t\to 0^+}\psi(t)=+\infty$ *and so it is not continuous at* $t=0$. *(2).* $\Phi(\mathbf{x},t)$ *satisfies the heat equation* $\partial_t\Phi=\triangle\Phi$ *in* $\mathbb{R}^{n+1}\backslash\{(0,0)\}$. *There is an easy way to check this on* $\mathbb{R}^n\times(0,\infty)$. *Let* $v=\ln\Phi$. *Then* Φ *satisfies the heat equation* $\partial_t\Phi=\triangle\Phi$ *if and only if* $v=\ln\Phi$ *satisfies the equation* $\partial_t v=\triangle v+|\nabla v|^2$ *(this is an exercise for you to check). Therefore we check the later equation. We have*

$$v=\ln\Phi=-\frac{n}{2}\ln(4\pi t)-\frac{|\mathbf{x}|^2}{4t},\quad t>0.$$

and then

$$\frac{\partial v}{\partial t}=-\frac{n}{2t}+\frac{|\mathbf{x}|^2}{4t^2}.$$

Also

$$\triangle v=-\frac{n}{2t},\quad |\nabla v|^2=\frac{|\mathbf{x}|^2}{4t^2}.$$

Hence we have $\partial_t v=\triangle v+|\nabla v|^2$. *(3). Exercise: check that we have* $\partial_t\Phi(\mathbf{x},0)=\triangle\Phi(\mathbf{x},0)$ *for all* $\mathbf{x}\in\mathbb{R}^n\backslash\{0\}$, $\mathbf{x}\neq 0$.

3.3 Basic properties of the fundamental solution.

In order to study the **initial value problem** for the heat equation (see (59) below) and to derive its solution formula, we need to discuss several important properties for the fundamental solution $\Phi(\mathbf{x},t)$ given in (30). One can use this fundamental solution to give a **representation formula** (**solution formula**) for the solution of (59) (this is similar to the **Poisson Integral Formula** for Laplace equation on the disc).

As a comparison, recall that for the Laplace equation $\triangle u(\mathbf{x})=0$ in $\mathbb{R}^n$ there is a **radial solution** (with a **singularity** at the origin of $\mathbb{R}^n$, i.e. $\mathbf{x}=0$) of the form

$$u(\mathbf{x})=\begin{cases}A|\mathbf{x}|^{2-n}+B, & n>2,\ \text{where}\quad \mathbf{x}\in\mathbb{R}^n\backslash\{0\}\\ A\log|\mathbf{x}|+B, & n=2,\quad \text{where}\quad \mathbf{x}\in\mathbb{R}^2\backslash\{0\},\end{cases}$$

where A, B are arbitrary constants. It plays an important role in the theory of Laplace equation.

For the heat equation $u_t=\triangle u$, the fundamental solution $\Phi(\mathbf{x},t)$ given in (30) is also a **radial solution** (**radial in space** $\mathbb{R}^n$**, not in space-time** $\mathbb{R}^{n+1}$), which, similar to the elliptic case, has a **singularity at the origin** of $\mathbb{R}^{n+1}$, i.e. at $(x,t)=(0,0)$.

In the following, we will discuss several properties of the fundamental solution $\Phi(\mathbf{x},t)$ for the case $n=1$. These properties are all valid for general $n>1$, but for simplicity of proof, here we focus only on the case $n=1$.

Lemma 3.13 *Let*

$$\Phi(x,t)=\begin{cases}\dfrac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, & x\in\mathbb{R},\quad t>0\\ 0, \quad x\in\mathbb{R}, & t\le 0.\end{cases} \tag{33}$$

Then $\Phi(x,t)\in C^\infty(\mathbb{R}^2\backslash\{(0,0)\})$ *and it satisfies the heat equation* $\partial_t u=\triangle u$ *in* $\mathbb{R}^2\backslash\{(0,0)\}$.

Proof. Since

$$\Phi(0,t)=\begin{cases}\dfrac{1}{\sqrt{4\pi t}}, & t>0\\ 0, & t\le 0,\end{cases}$$

we see that $\Phi(x,t)$ is *not* continuous at $(0,0)$. Moreover, we have

$$\lim_{t\to 0^+}\Phi(0,t)=\lim_{t\to 0^+}\frac{1}{\sqrt{4\pi t}}=\infty.$$

To check that $\Phi(x,t)\in C^\infty(\mathbb{R}^2\backslash\{(0,0)\})$, it suffices to look at the behavior of $\Phi(x,t)$ on the set $S=\{(x,0)\in\mathbb{R}^2: x\neq 0\}$. By the limit

$$\lim_{t\to 0^+}\left(\frac{1}{t^\alpha}e^{-\frac{\beta}{t}}\right)=0,\quad \forall \text{ const. } \alpha,\ \beta>0,$$

one can check that $\Phi(x,t)$ is C^∞ at any point of S. Computing

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right)&=\frac{1}{\sqrt{4\pi t}}\left(-\frac{x}{2t}\right)e^{-\frac{x^2}{4t}},\quad t>0\\ \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right)&=\frac{1}{\sqrt{4\pi t}}\left(-\frac{1}{2t}+\frac{x^2}{4t^2}\right)e^{-\frac{x^2}{4t}},\quad t>0\\ \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right)&=\frac{1}{\sqrt{4\pi}}\left(-\frac{1}{2}t^{-3/2}\right)e^{-\frac{x^2}{4t}}+\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\frac{x^2}{4t^2},\quad t>0\end{aligned} \tag{34}$$

we see that $\Phi(x,t)$ satisfies the heat equation on $\mathbb{R}\times(0,\infty)$. Clearly it also satisfies the heat equation on $\mathbb{R}\times(-\infty,0)$. At any point $(x_0,0)\in S,\ x_0\neq 0$, we have $\Phi_{xx}(x_0,0)=0$. Also note that

$$\lim_{t\to 0^-}\frac{\Phi(x_0,t)-\Phi(x_0,0)}{t}=0\quad (x_0\neq 0)$$

and

$$\lim_{t\to 0^+}\frac{\Phi(x_0,t)-\Phi(x_0,0)}{t}=\lim_{t\to 0^+}\left(\frac{1}{t}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x_0^2}{4t}}\right)=0\quad (x_0\neq 0,\ x_0^2>0),$$

and so we have $\Phi_t(x_0,0)=0$. The proof is done. □

Lemma 3.14 *For any fixed* $\varepsilon>0$, *we have*

$$\lim_{t\to 0^+}\Phi(x,t)=0\quad \textbf{\textit{uniformly}} \textit{ in the region } \{x\in\mathbb{R}:|x|\ge\varepsilon\}. \tag{35}$$

Also

$$\lim_{|x|\to\infty}\Phi(x,t)=0\quad \textbf{\textit{uniformly}} \textit{ in the region } t\in(-\infty,\infty). \tag{36}$$

Remark 3.15 *We also have*

$$\lim_{t\to\infty}\Phi(x,t)=0\quad \textbf{\textit{uniformly}} \textit{ in } x\in(-\infty,\infty). \tag{37}$$

This is easy due to

$$|\Phi(x,t)|=\left|\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right|\le\frac{1}{\sqrt{4\pi t}}\quad \textit{for all}\quad x\in(-\infty,\infty),\quad t>0. \tag{38}$$

Remark 3.16 *Draw a picture for* $\Phi(x,t)$ *with* $t \to 0^+$.

Proof. For (35), we have for $t > 0$ the inequality

$$0 < \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{\varepsilon^2}{4t}}, \quad \forall\ |x| \geq \varepsilon$$

and the conclusion follows. For (36), it suffices to focus on $t \in (0,\infty)$ since $\Phi(x,t) \equiv 0$ for all $x \in \mathbb{R},\ t \leq 0$. For fixed $x \in \mathbb{R},\ x \neq 0$, the maximum value of the positive function

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad t \in (0,\infty)$$

over $t \in (0,\infty)$, is attained at the point $t = x^2/2$ with maximum value equal to

$$\frac{1}{|x|\sqrt{2\pi}} e^{-\frac{1}{2}}. \tag{39}$$

This is due to the identity

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right) = \frac{1}{\sqrt{4\pi t}} \frac{1}{2t}\left(\frac{x^2}{2t} - 1\right) e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R}, \quad t \in (0,\infty).$$

The result follows. □

Lemma 3.17 *For each fixed* $t > 0$, *we have*

$$\int_{-\infty}^{\infty} \Phi(x,t)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\,dx = 1, \quad t > 0. \tag{40}$$

Moreover, the convergence of the integral is ***uniform*** *with respect to* $t \in (0,T)$ *for any fixed* $T > 0$ *(but not uniform with respect to* $t \in (0,\infty)$*).*

Remark 3.18 *Draw a picture for* $\Phi(x,t)$ *(for small* $t > 0$ *and for large* $t > 0$*) and show the property* $\int_{-\infty}^{\infty} \Phi(x,t)\,dx = 1$ *for all* $t > 0$.

Remark 3.19 *(**Helpful interpretation ...**) For fixed* $t > 0$, *if we let*

$$F_N(t) = \int_{-N}^{N} \Phi(x,t)\,dx, \quad t \in (0,T), \quad N \in \mathbb{N},$$

then the convergence of the integral is ***uniform*** *with respect to* $t \in (0,T)$ *can be interpreted as*

$$\lim_{N\to\infty} F_N(t) = 1 \ \ \text{uniformly in} \ \ t \in (0,T).$$

Remark 3.20 *Similarly, we have*

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}\,dx = 1, \quad \forall\ y \in \mathbb{R}, \quad t > 0 \tag{41}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}\,dy = 1, \quad \forall\ x \in \mathbb{R}, \quad t > 0. \tag{42}$$

Proof. We first recall the following improper integral identity from calculus:

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}. \tag{43}$$

By a change of variables (let $s = \alpha y + \beta$), we have

$$\int_{-\infty}^{\infty} e^{-(\alpha y+\beta)^2} dy = \frac{\sqrt{\pi}}{\alpha}, \quad \forall \ \beta \in \mathbb{R}, \quad \alpha > 0. \tag{44}$$

Letting $x = \sqrt{4t}s$, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-s^2} \sqrt{4t} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1.$$

Next, let $T > 0$ be a fixed time. For any $\varepsilon > 0$, then there exists a large $M > 0$ (M depends only on ε and T) such that **for all** $t \in (0, T)$ we have the estimate (again, let $x = \sqrt{4t}s$)

$$0 < \int_{M}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} dx = \frac{1}{\sqrt{\pi}} \int_{\frac{M}{\sqrt{4t}}}^{\infty} e^{-s^2} ds < \frac{1}{\sqrt{\pi}} \int_{\frac{M}{\sqrt{4T}}}^{\infty} e^{-s^2} ds < \varepsilon, \quad \forall \, t \in (0, T).$$

The same result holds for the integral $\int_{-\infty}^{-M} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} dx$. Therefore, the convergence of the integral is uniform with respect to $t \in (0, T)$ for any fixed $T > 0$. □

Remark 3.21 *By the integral identity*

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} = \left(\sqrt{\pi}\right)^n,$$

one can also obtain the identity

$$\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} d\mathbf{x} = 1 \tag{45}$$

for each $t > 0$.

Lemma 3.22 *For **fixed** $\delta > 0$, we have*

$$\lim_{t \to 0^+} \int_{|y-x|>\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy = 0 \quad \textbf{\textit{uniformly}} \textit{ in } \ x \in \mathbb{R}, \tag{46}$$

which means that the values of the fundamental solution $\Phi(x-y,t)$ (view it as a function of y with parameter x) concentrate around x as $t \to 0^+$.

Remark 3.23 *For fixed $\delta > 0$, the quantity*

$$\int_{|y-x|>\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy$$

*is a function of $(x,t) \in \mathbb{R} \times (0,\infty)$ (denote it as $F(x,t)$). The above lemma says that $\lim_{t\to 0^+} F(x,t) = 0$ **uniformly** in $x \in \mathbb{R}$.*

Proof. Let $y = x + \sqrt{4t}s$. Then

$$\lim_{t \to 0^+} \int_{|y-x|>\delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy = \lim_{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_{|s|>\delta/\sqrt{4t}} e^{-s^2} ds = 0. \tag{47}$$

Note that the right hand side of (47) does not depend on $x \in \mathbb{R}$. Hence we have convergence to zero uniformly in $x \in \mathbb{R}$. The proof is done. □

The following lemma is crucial in solving the initial value problem (59) below.

Lemma 3.24 *Let $\phi(x)$ be a* ***bounded*** *function defined on $(-\infty,\infty)$ and is* ***continuous*** *at $x = x_0$. Then we have*

$$\lim_{(x,t)\to(x_0,0^+)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy=\phi(x_0). \tag{48}$$

In particular, we also have

$$\lim_{t\to 0^+}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_0-y)^2}{4t}}\phi(y)\,dy=\phi(x_0). \tag{49}$$

Remark 3.25 *The above two limits have different meaning. In the first limit, $(x,t)\to(x_0,0^+)$ means that $(x,t)\in\mathbb{R}\times(0,\infty)$ approaches the point $(x_0,0)\in\mathbb{R}\times\{0\}$ in the plane $\mathbb{R}^2$, while maintaining $t>0$. In the second limit, we take $x=x_0$ in the integrand and look at the limit $t\to 0$, still maintaining $t>0$. Note that (48) is a 2-dimensional limit, but (49) is just a 1-dimensional limit.*

Proof. Let

$$u(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy,\quad (x,t)\in\mathbb{R}\times(0,\infty). \tag{50}$$

For any $\varepsilon>0$, we choose $\delta>0$ such that $|\phi(y)-\phi(x_0)|<\varepsilon$ if $|y-x_0|<2\delta$. Let $M=\sup_{\mathbb{R}}|\phi|$. If $|x-x_0|<\delta$, then

$$\begin{aligned}
|u(x,t)-\phi(x_0)| &= \left|\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}}(\phi(y)-\phi(x_0))\,dy\right| \\
&\le \frac{1}{\sqrt{4\pi t}}\left(\int_{|y-x|<\delta}e^{-\frac{(x-y)^2}{4t}}|\phi(y)-\phi(x_0)|\,dy+\int_{|y-x|\ge\delta}e^{-\frac{(x-y)^2}{4t}}|\phi(y)-\phi(x_0)|\,dy\right) \\
&\le \frac{1}{\sqrt{4\pi t}}\left(\int_{|y-x_0|<2\delta}e^{-\frac{(x-y)^2}{4t}}|\phi(y)-\phi(x_0)|\,dy+2M\int_{|y-x|\ge\delta}e^{-\frac{|x-y|^2}{4t}}dy\right) \\
&\le \varepsilon\left(\int_{|y-x_0|<2\delta}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy\right)+2M\left(\int_{|y-x|\ge\delta}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy\right)
\end{aligned} \tag{51}$$

Therefore, by (42) and (46), if $t>0$ is small enough and $|x-x_0|<\delta$, (51) will imply

$$|u(x,t)-\phi(x_0)|\le\varepsilon+2M\varepsilon,$$

Hence we have

$$\lim_{(x,t)\to(x_0,0^+)}u(x,t)=\phi(x_0)$$

and (48) is proved. (49) is a consequence of (48). □

Lemma 3.26 *Let $\phi(y)$ be a* ***continuous bounded*** *function defined on $(-\infty,\infty)$. Then we have*

$$\begin{aligned}
&\left(\frac{\partial^{m+n}}{\partial t^m\partial x^n}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy\right)(x_0,t_0) \\
&=\int_{-\infty}^{\infty}\left[\left(\frac{\partial^{m+n}}{\partial t^m\partial x^n}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\right)(x_0,t_0,y)\right]dy
\end{aligned} \tag{52}$$

for all $(x_0,t_0)\in(-\infty,\infty)\times(0,\infty)$ and all $m,\ n\in\mathbb{N}\bigcup\{0\}$. In particular, the function

$$u(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy,\quad (x,t)\in(-\infty,\infty)\times(0,\infty), \tag{53}$$

satisfies

$$\begin{cases}(1).\ u(x,t)\in C^\infty((-\infty,\infty)\times(0,\infty)) \\ (2).\ u_t(x,t)=u_{xx}(x,t),\quad \forall\ (x,t)\in(-\infty,\infty)\times(0,\infty).\end{cases} \tag{54}$$

Remark 3.27 ***(Important.)*** *To understand the proof of Lemma 3.26, you need to know when a differentiation (say $\frac{\partial}{\partial x}$) and an improper integral (say of the form $\int_{-\infty}^{\infty} g(x,y)\,dy$ or $\int_0^{\infty} g(x,y)\,dy$ for some differentiable function $g(x,y)$) can* ***commute****. For your convenience, here I provide two results in the following:*

1. Let $f(x,y) \in C^0(I \times [0,\infty))$, where $I \subseteq \mathbb{R}$ is an **arbitrary connected interval** and assume that the improper integral $\int_0^{\infty} f(x,y)\,dy$ **converges uniformly to a function** $F(x)$ **on** I. Then $F(x)$ is continuous on I. This means that we have the identity

$$\lim_{x\to x_0}\int_0^{\infty} f(x,y)\,dy = \int_0^{\infty} f(x_0,y)\,dy, \quad \forall\ x_0 \in I. \tag{55}$$

   The same conclusion holds if we replace $\int_0^{\infty} f(x,y)\,dy$ by $\int_{-\infty}^{\infty} f(x,y)\,dy$.

2. Let $f(x,y) \in C^0(I \times [0,\infty))$, where $I \subseteq \mathbb{R}$ is an **arbitrary connected interval** and assume that the improper integral $\int_0^{\infty} f(x,y)\,dy$ converges to a function $F(x)$ on I (**does not have to be uniform**) and $\frac{\partial f}{\partial x} \in C^0(I \times [0,\infty))$ and $\int_0^{\infty} \frac{\partial f}{\partial x}(x,y)\,dy$ **converges uniformly** on I,Then $F(x)$ is differentiable with respect to $x \in I$ and

$$F'(x) = \int_0^{\infty} \frac{\partial f}{\partial x}(x,y)\,dy, \quad \forall\ x \in I. \tag{56}$$

   In particular, $F(x)$ is also continuous on I. Moreover, if I is a **finite interval**, then $\int_0^{\infty} f(x,y)\,dy$ also **converges uniformly** on I. The same conclusion holds if we replace $\int_0^{\infty} f(x,y)\,dy$ by $\int_{-\infty}^{\infty} f(x,y)\,dy$ and $\int_0^{\infty} \frac{\partial f}{\partial x}(x,y)\,dy$ by $\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x,y)\,dy$.

   Note: Compare with Rudin's Advanced Calculus book (Principle of Mathematical Analysis, 3rd edition) Theorem 7.17 in p. 152. In terms of series of functions, Rudin's Theorem 7.17 can be stated as: Let $\{f_n\}$ be a sequence of differentiable functions on $[a,b]$ such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for some $x_0 \in [a,b]$ and assume that the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a,b]$ to a function $h(x)$, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$ to a function $f(x)$, which is differentiable, and we have

$$f'(x) = h(x), \quad \forall\ x \in [a,b].$$

Proof. For any fixed $m,\ n \in \mathbb{N}\bigcup\{0\}$ and fixed $(x_0,t_0) \in (-\infty,\infty)\times(0,\infty)$ the function

$$\left(\frac{\partial^{m+n}}{\partial t^m \partial x^n}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\right)(x_0,t_0,y), \quad y \in (-\infty,\infty)$$

decays exponentially in the variable y as $|y| \to \infty$. In fact, it also decays exponentially in the variable y as $|y| \to \infty$ for all (x,t) in some neighborhood R of (x_0,t_0). For example, one can take R as

$$R = \left\{(x,t) : x_0 - 1 < x < x_0 + 1,\ \frac{t_0}{2} < t < \frac{3t_0}{2}\right\}, \quad t_0 > 0. \tag{57}$$

By this decay property, one can check that the integral

$$\int_{-\infty}^{\infty}\left[\left(\frac{\partial^{m+n}}{\partial t^m \partial x^n}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\right)(x,t,y)\right]dy, \quad (x,t)\ \text{is near}\ \ (x_0,t_0)$$

converges uniformly for all (x,t) in R. By standard theory in advanced calculus, the function (as a function of $(x,t) \in R$)

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy$$

is **differentiable** with respect to t up to m times and differentiable with respect to x up to n times, i.e. one can apply $\frac{\partial^{m+n}u}{\partial t^m \partial x^n}$ onto it and obtain the identity

$$\left(\frac{\partial^{m+n}u}{\partial t^m \partial x^n}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy\right)(x,t)$$
$$=\int_{-\infty}^{\infty}\left[\left(\frac{\partial^{m+n}u}{\partial t^m \partial x^n}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\right)(x,t,y)\right]dy, \quad \forall\ (x,t)\in R. \tag{58}$$

As the point $(x_0,t_0)\in(-\infty,\infty)\times(0,\infty)$ is arbitrary and the numbers $m,\ n\in\mathbb{N}\bigcup\{0\}$ are also arbitrary, the identity (52) is proved for all $(x_0,t_0)\in(-\infty,\infty)\times(0,\infty)$. Moreover, the function

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy$$

is a C^∞ function of $(x,t)\in(-\infty,\infty)\times(0,\infty)$, which implies $u(x,t)\in C^\infty((-\infty,\infty)\times(0,\infty))$. Finally, we have

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)u(x,t)=\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy\right)$$
$$=\int_{-\infty}^{\infty}\left(\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right)\phi(y)\,dy=\int_{-\infty}^{\infty}0\cdot\phi(y)\,dy=0,$$

which means that $u(x,t)$ satisfies the heat equation on $(-\infty,\infty)\times(0,\infty)$. □

3.4 Heat equation on the whole line with initial condition.

Motivated by the heat polynomials, to get unique solution (we hope so) of a heat equation $u_t=u_{xx}$, we focus on the following **initial value problem**:

$$\begin{cases} u_t(x,t)=u_{xx}(x,t), & (x,t)\in\mathbb{R}\times(0,\infty)\\ u(x,0)=f(x), & x\in\mathbb{R}.\end{cases} \tag{59}$$

Here $f(x)$ is a given **continuous function** on $\mathbb{R}$ and we want (hope) to find a "**unique**" solution $u(x,t)$ lying in the space $C^2(\mathbb{R}\times(0,\infty))\bigcap C^0(\mathbb{R}\times[0,\infty))$.

Remark 3.28 *In **physical reality**, most phenomena described by heat equation (and wave equation) has **initial-boundary** conditions (where the space domain for x is **bounded**). However, for $x\in(-\infty,\infty)$, the initial value problem (59) has a nice solution formula (this is similar to the wave equation $u_{tt}(x,t)=u_{xx}(x,t)$ with initial conditions $u(x,0)$ and $u_t(x,0)$) and it is easier to manipulate. Therefore, **for mathematical reason (not for physical reason)**, instead of looking at initial-boundary value problem for heat equation, we look at (59) first.*

Note that, unlike the wave equation, here we do not need the condition $u_t(x,0)=g(x)$ for the heat equation. This is due to **physical phenomenon** (heat equation is not a mechanical equation coming from Newton's law) and also due to the fact that if $u(x,t)$ is C^2 up to $t=0$ (with $f\in C^2(\mathbb{R})$), then we also have

$$u_t(x,0)=u_{xx}(x,0)=f''(x), \quad x\in\mathbb{R},$$

i.e. the condition $u_t(x,0)=g(x)$ is automatically a consequence of the condition $u(x,0)=f(x)$. On the other hand, for the case of wave equation, we can not determine $u_t(x,0)$ from the condition $u(x,0)$ (but we can determine $u_{tt}(x,0)$ from the condition $u(x,0)$ due to the identity $u_{tt}(x,0)=u_{xx}(x,0)$).

Unfortunately, the initial value problem (59) has infinitely many solutions (this is unlike the wave equation, which has a unique solution once we know $u(x,0)$ **and** $u_t(x,0)$**) unless we impose condition on the behavior of solution** $u(x,t)$ **for large** $|x|$. This is because the data is prescribed on the line $t=0$, which is a **characteristic line** of the heat equation $u_{xx}(x,t)-u_t(x,t)=0$.

In spite of this defect, when $f(x)$ is given and $x\in(-\infty,\infty)$, there is some "**special solution**" of (59), which is given by a **representation formula** (solution formula), which has **good properties** and **is close to the physical reality**.

Remark 3.29 *Recall that for a second order linear* ***parabolic*** *equation with constant coefficients for* $u(x,y)$ *(here we view* y *as time), given by*

$$au_{xx}+2bu_{xy}+cu_{yy}+(\textit{lower order terms})=0,\quad ac=b^2, \tag{60}$$

where $a,\ b,\ c$ *are constants with* $ac=b^2$*, the leading terms* $au_{xx}+2bu_{xy}+cu_{yy}$ *can be factored as*

$$au_{xx}+2bu_{xy}+cu_{yy}=\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)\left[\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)u\right]=0 \tag{61}$$

for some constants $A,\ B,\ C$*. The* ***1-parameter family of lines***

$$Bx-Ay=\lambda,\quad \lambda\in(-\infty,\infty) \tag{62}$$

are called the ***characteristic lines*** *of the parabolic equation (60). By this, for the standard heat equation* $u_{xx}+(-u_t)=0$*, the* ***1-parameter family*** *of characteristic lines are given by (we now have* $B=0,\ A=1,\ y=t$ *in (62))*

$$-t=\lambda,\quad \lambda\in(-\infty,\infty). \tag{63}$$

From it, we know that the line $t=0$ *(i.e.* x*-axis) is a* ***characteristic line*** *of the heat equation. This may explain the* ***nonuniqueness*** *of the initial value problem (59).*

We now consider the following initial value problem for heat equation defined on the whole line:

$$\begin{cases} u_t=u_{xx}, & x\in(-\infty,\infty),\quad t\in(0,\infty)\\ u(x,0)=\phi(x), & x\in(-\infty,\infty). \end{cases} \tag{64}$$

Here $\phi(x)$ is a given **continuous** function on $(-\infty,\infty)$ and we want to find a solution lying in the function space:

$$u(x,t)\in C^2((-\infty,\infty)\times(0,\infty))\bigcap C^0((-\infty,\infty)\times[0,\infty)), \tag{65}$$

where satisfies (64).

As a consequence of Lemma 3.24 and Lemma 3.26, we can obtain the following **solution formula** for the initial value problem (64):

Theorem 3.30 *Assume* $\phi(x)$ *is a* ***continuous bounded*** *function defined on* $(-\infty,\infty)$*. Then the function*

$$u(x,t)=\begin{cases} \displaystyle\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy, & x\in(-\infty,\infty),\quad t\in(0,\infty)\\ \phi(x), & t=0 \end{cases} \tag{66}$$

belongs to the space $C^\infty(\mathbb{R}\times(0,\infty))\bigcap C^0(\mathbb{R}\times[0,\infty))$ *(i.e.* ***continuous up to*** $t=0$*) and satisfies the initial value problem*

$$\begin{cases} u_t(x,t)=u_{xx}(x,t), & x\in(-\infty,\infty),\quad t\in(0,\infty)\\ u(x,0)=\phi(x), & x\in(-\infty,\infty). \end{cases} \tag{67}$$

Proof. This is a direct consequence of Lemma 3.24 and Lemma 3.26. □

Remark 3.31 *(**Important.**) As long as $t > 0$, $u(x,t)$ becomes a **smooth** function even if the initial data $\phi(x)$ is only a continuous function. We call this a **smoothing effect** of the heat equation. This is unlike the wave equation, which has no smoothing effect.*

Corollary 3.32 *(**The maximum principle.**) The solution $u(x,t)$ given by (66), where $\phi(x)$ is a **continuous bounded** function defined on $(-\infty,\infty)$, satisfies the maximum principle:*

$$\inf_{\mathbb{R}} \phi \le u(x,t) \le \sup_{\mathbb{R}} \phi \quad \text{for all} \quad x \in (-\infty,\infty), \quad t \in (0,\infty). \tag{68}$$

Proof. We have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \underbrace{\phi(y)} dy \le \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \underbrace{\sup_{\mathbb{R}} \phi} dy \\ &= \sup_{\mathbb{R}} \phi \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy = \sup_{\mathbb{R}} \phi \end{aligned}$$

and similarly $u(x,t) \ge \inf_{\mathbb{R}} \phi$. □

Corollary 3.33 *(**Infinite speed of propagation of the heat equation.**) Let $\phi(x)$ be a **continuous bounded** function defined on $(-\infty,\infty)$. Assume $\phi(x) \ge 0$ everywhere, has **compact support**, and $\phi \not\equiv 0$. Then the solution $u(x,t)$ given by (66) satisfies*

$$u(x,t) > 0, \quad \forall\, x \in (-\infty,\infty), \quad t \in (0,\infty), \tag{69}$$

*i.e., as long as time is positive, $u(x,t)$ is positive everywhere no matter how large $|x|$ is (that is why we say the equation has **infinite speed of propagation**).*

Remark 3.34 *This is different from the wave equation. The function $u(x,t) = \phi(x-t)$ satisfies the wave equation $u_{tt} = u_{xx}$ with $u(x,0) = \phi(x)$. However, for $t > 0$, $u(x,t) = 0$ if $|x| > 0$ is large enough.*

Proof. Since ϕ is not a zero function, we have $\phi(x_0) > 0$ for some $x_0 \in (-\infty,\infty)$. By continuity, $\phi > 0$ on $(x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$. Now at any $(x,t) \in (-\infty,\infty) \times (0,\infty)$, we have

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)\, dy \ge \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)\, dy > 0.$$

The proof is done. □

To go on, we need the following special case of Fubini Theorem from **advanced calculus** textbook:

Lemma 3.35 *(**Tonelli's theorem.**) Let $\phi(x,y)$ be a **continuous "nonnegative"** function defined on $\mathbb{R}^2 = (-\infty,\infty) \times (-\infty,\infty)$. Then the finiteness of any one of the following three integrals:*

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi(x,y)\, dx \right) dy, \quad \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi(x,y)\, dy \right) dx, \quad \iint_{\mathbb{R}^2} \phi(x,y)\, dxdy$$

implies that of the other two. Moreover, their values are all equal.

Remark 3.36 *The condition $\phi(x,y) \ge 0$ on $\mathbb{R}^2$ is essential.*

Proof. We omit it. □

Lemma 3.37 *(**Conservation of total energy.**) Let $\phi(x)$ be a **continuous bounded** function defined on $(-\infty,\infty)$ ($\phi(x)$ may not be nonnegative). Assume $\int_{-\infty}^{\infty}|\phi(x)|\,dx$ converges. Then the solution $u(x,t)$ given by (66) satisfies*

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty}\phi(x)\,dx, \quad \forall\, t\in(0,\infty). \tag{70}$$

*This means that **the total energy (heat) is conserved.***

Proof. For each $x\in(-\infty,\infty)$, let $\phi^+(x)=\max\{\phi(x),0\}\geq 0$ and $\phi^-(x)=-\min\{\phi(x),0\}\geq 0$. Then we have

$$\phi(x)=\phi^+(x)-\phi^-(x), \quad |\phi(x)|=\phi^+(x)+\phi^-(x), \quad \forall\, x\in(-\infty,\infty).$$

The convergence of $\int_{-\infty}^{\infty}|\phi(x)|\,dx$ implies that of $\int_{-\infty}^{\infty}\phi^+(x)\,dx$ and $\int_{-\infty}^{\infty}\phi^-(x)\,dx$. Also, since $\phi(x)$ is a bounded function, for each fixed $(x,t)\in(-\infty,\infty)\times(0,\infty)$, the three improper integrals

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy, \ \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi^+(y)\,dy, \ \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi^-(y)\,dy$$

all converge. Now we have

$$\begin{aligned}\int_{-\infty}^{\infty}u(x,t)\,dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy\right]dx\\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi^+(y)\,dy-\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi^-(y)\,dy\right]dx \end{aligned}\tag{71}$$

and by Lemma 3.35, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi^+(y)\,dy\right]dx\\ &= \int_{-\infty}^{\infty}\left[\phi^+(y)\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dx}\right]dy = \int_{-\infty}^{\infty}\phi^+(y)\,dy<\infty\end{aligned}$$

and similarly

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi^-(y)\,dy\right]dx = \int_{-\infty}^{\infty}\phi^-(y)\,dy<\infty.$$

Therefore, the two iterated integrals in (71) converge and we conclude

$$\int_{-\infty}^{\infty}u(x,t)\,dx=\int_{-\infty}^{\infty}\phi^+(y)\,dy-\int_{-\infty}^{\infty}\phi^-(y)\,dy=\int_{-\infty}^{\infty}\phi(x)\,dx, \quad \forall\, t\in(0,\infty).$$

The proof is done. □

3.5 The maximum principle.

Assume that $u(x,t)$ is a solution of the heat equation $u_t=u_{xx}$ on $(-\infty,\infty)\times(-\infty,\infty)$. The maximum principle of the diffusion equation says that (roughly speaking), for fixed time t_0, we have $u_t(x_0,t_0)\leq 0$ if $u(x_0,t_0)$ has a *local maximum* at $x=x_0$ (so the value of $u(x_0,t_0)$ will "decrease" at the moment $t=t_0$); and $u_t(x_0,t)\geq 0$ if $u(x_0,t)$ has a *local minimum* at $x=x_0$ (so the value of $u(x_0,t_0)$ will "increase" at the moment $t=t_0$) (draw a picture for this). This is called **the maximum principle** of the heat equation. It matches with the physical phenomenon that heat goes from hot points to cold points and vice versa.

The maximum principle on the unbounded domain $x\in(-\infty,\infty)$ is more difficult to describe. We will discuss the maximum principle on **bounded domains** only.

3.5.1 The maximum principle on bounded domains.

Let $U_T \subset \mathbb{R}^2$ be the set given by

$$U_T = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < \ell,\ 0 < t \le T\right\}, \quad \ell,\ T > 0 \tag{72}$$

and assume that $u = u(x,t) \in C^2(U_T) \bigcap C^0(\bar{U}_T)$ satisfies the heat equation $u_t = u_{xx}$ on U_T (note that the segment $(x,T),\ 0 < x < \ell$, is included). Note that since u is continuous on the compact set $\bar{U}_T$, it has global maximum and minimum on $\bar{U}_T$.

Remark 3.38 *Explain the meaning of* $u \in C^2(U_T)$.

The maximum principle says the following:

Lemma 3.39 *(**Weak maximum principle for heat equation.**) Assume* $u \in C^2(U_T) \bigcap C^0(\bar{U}_T)$ *satisfies the heat equation* $u_t = u_{xx}$ *on* U_T*. Then we have*

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u, \tag{73}$$

where $\Gamma_T := \bar{U}_T - U_T$*, which is called the **parabolic boundary** of* U_T*.*

Remark 3.40 *The above result is still true if we have* $u_t \le u_{xx}$ *on* U_T*.*

Proof. Assume $v \in C^2(U_T) \bigcap C^0(\bar{U}_T)$ is a function such that

$$v_{xx}(x,t) - v_t(x,t) > 0 \quad \text{in} \quad U_T. \tag{74}$$

Then since $v \in C^0(\bar{U}_T)$, there is a point $(x_0,t_0) \in \bar{U}_T$ such that $v(x_0,t_0) = \max_{\bar{U}_T} v$. If $(x_0,t_0) \in U_T$ with $t < T$, then from calculus we know that

$$v_x(x_0,t_0) = 0, \qquad v_{xx}(x_0,t_0) \le 0, \qquad v_t(x_0,t_0) = 0. \tag{75}$$

This contradicts $v_{xx} - v_t > 0$ in U_T.

If $(x,t) \in U_T$ with $t = T$, then we replace $v_t(x_0,t_0) = 0$ by $v_t(x_0,t_0) \ge 0$ in (75) and get the same contradiction. Thus the point (x_0,t_0) **must lie on the parabolic boundary** of U_T and cannot lie on U_T (for $v(x,t)$ satisfying the differential inequality (74)). In such a case we have

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v \text{ (call this value } M), \quad \text{where } v_{xx}(x,t) - v_t(x,t) > 0 \text{ in } U_T, \tag{76}$$

and moreover, $v(x,t)$ cannot attain the value M on U_T.

Now let $v(x,t) = u(x,t) + \varepsilon x^2$ ($\varepsilon > 0$ is a small constant), where $u \in C^2(U_T) \bigcap C^0(\bar{U}_T)$ satisfies the heat equation on U_T. We now have

$$v_{xx}(x,t) - v_t(x,t) = u_{xx}(x,t) + 2\varepsilon - u_t(x,t) = 2\varepsilon > 0 \quad \text{in} \quad U_T.$$

By the above discussion, we know that

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v = \max_{\Gamma_T}\left(u(x,t) + \varepsilon x^2\right) \le \left(\max_{\Gamma_T} u(x,t)\right) + \varepsilon\ell^2$$

and by $u(x,t) = v(x,t) - \varepsilon x^2 \le v(x,t)$, we get

$$\max_{\bar{U}_T} u \le \max_{\bar{U}_T} v \le \left(\max_{\Gamma_T} u(x,t)\right) + \varepsilon\ell^2. \tag{77}$$

As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0^+$ (note that here ℓ is finite), we obtain $\max_{\bar{U}_T} u \le \max_{\Gamma_T} u(x,t)$. On the other hand, we also have $\max_{\bar{U}_T} u \ge \max_{\Gamma_T} u(x,t)$. Hence $\max_{\bar{U}_T} u = \max_{\Gamma_T} u(x,t)$. □

Exercise 3.41 *Instead of using $v(x,t) = u(x,t) + \varepsilon x^2$, now use the function $v(x,t) = u(x,t) - \varepsilon t,\ \varepsilon > 0$, and repeat the same argument of proof. Can you obtain the same result?*

Remark 3.42 *(**Be careful.**) In the above proof, we do not exclude the possibility that the maximum of $u(x,t)$ (note that $u_t = u_{xx}$) can also be attained at some point in U_T. For example, when the solution $u(x,t)$ is a **constant**, then this can happen. However, this is the only case that can happen (this is the **strong maximum principle**).*

We also have the following **minimum principle**:

Corollary 3.43 *(**Weak minimum principle for heat equation.**) Assume $u \in C^2(U_T) \bigcap C^0(\bar{U}_T)$ satisfies the heat equation $u_t = u_{xx}$ on U_T. Then we have*

$$\min_{\bar{U}_T} u = \min_{\Gamma_T} u. \tag{78}$$

Remark 3.44 *The above result is still true if we have $u_t \geq u_{xx}$ on U_T.*

Remark 3.45 *Again, here we do not exclude the possibility that the minimum can be attained at some point in U_T.*

Proof. The proof for the minimum case is similar by looking at $-u$ (it also satisfies the heat equation) and the identity $\max_{\bar{U}_T}(-u) = \max_{\Gamma_T}(-u)$ becomes $-\min_{\bar{U}_T} u = -\min_{\Gamma_T} u$. □

Corollary 3.46 *Assume $u \in C^2(U_T) \bigcap C^0(\bar{U}_T)$ satisfies the heat equation $u_t = u_{xx}$ on U_T and $u \equiv 0$ on the parabolic boundary Γ_T, then $u \equiv 0$ on $\bar{U}_T$.*

Proof. This is a consequence of the maximum-minimum principle. □

Example 3.47 *(**Give this as an homework problem ....**) Let $u(x,t)$ be one of the following functions:*

$$t + \frac{x^2}{2}, \quad e^{t+x}, \quad e^{t-x}, \quad e^{-t}\cos x, \quad e^{-t}\sin x, \quad e^t \cosh x, \quad e^t \sinh x, \quad (x,t) \in \mathbb{R}^2.$$

They all satisfy the heat equation $u_t = u_{xx}$ (note that $e^t \cosh x$ and $e^t \sinh x$ are linear combinations of e^{t+x} and e^{t-x}). Let $U_T = (0,1) \times (0,T]$. We have

$$\begin{cases} (1).\ \max_{\bar{U}_T}\left(t + \frac{x^2}{2}\right) = T + \frac{1}{2}, & \text{attained at } (1,T) \in \Gamma_T := \bar{U}_T - U_T \\ (2).\ \max_{\bar{U}_T}\left(e^{t+x}\right) = e^{T+1}, & \text{attained at } (1,T) \in \Gamma_T \\ (3).\ \max_{\bar{U}_T}\left(e^{t-x}\right) = e^{T-0}, & \text{attained at } (0,T) \in \Gamma_T \\ (4).\ \max_{\bar{U}_T}\left(e^{-t}\cos x\right) = e^{-0}\cos 0 = 1, & \text{attained at } (0,0) \in \Gamma_T \\ (5).\ \max_{\bar{U}_T}\left(e^{-t}\sin x\right) = e^{-0}\sin 1 = \sin 1, & \text{attained at } (1,0) \in \Gamma_T \\ (6).\ \max_{\bar{U}_T}\left(e^t \cosh x\right) = e^T \cosh 1, & \text{attained at } (1,T) \in \Gamma_T \\ (7).\ \max_{\bar{U}_T}\left(e^t \sinh x\right) = e^T \sinh 1, & \text{attained at } (1,T) \in \Gamma_T. \end{cases}$$

*From the above, we see that each solution attains its maximum point on the parabolic boundary Γ_T. Also note that the maximum can be attained at **any** corner point (there are four of them) of Γ_T.*

One can also use **Energy Method** (integral method) to prove the following (without using the maximum principle):

Lemma 3.48 *Assume $u \in C^2\left(\bar{U}_T\right)$ satisfies the heat equation $u_t = u_{xx}$ on U_T and $u \equiv 0$ on the parabolic boundary Γ_T, then $u \equiv 0$ on $\bar{U}_T$.*

Proof. Let $E(t)$, $0 \le t \le T$, be the quantity

$$E(t) = \frac{1}{2}\int_0^\ell u^2(x,t)\,dx \ge 0, \quad 0 \le t \le T.$$

Then $E(t)$ is a differentiable function on $[0,T]$, $E(0) = 0$, and we have

$$\begin{aligned}\frac{dE}{dt}(t) &= \int_0^\ell u(x,t)\,u_t(x,t)\,dx = \int_0^\ell u(x,t)\,u_{xx}(x,t)\,dx \\ &= \int_0^\ell \left[\left(\frac{d}{dx}\left[u(x,t)\,u_x(x,t)\right]\right) - \left(u_x(x,t)\right)^2\right]dx \\ &= \left[u(x,t)\,u_x(x,t)\right]\Big|_{x=0}^{x=\ell} - \int_0^\ell \left(u_x(x,t)\right)^2 dx = -\int_0^\ell \left(u_x(x,t)\right)^2 dx \le 0. \end{aligned} \tag{79}$$

Hence we have

$$0 \le E(t) \le E(0) = 0, \quad \forall\, t \in [0,T]. \tag{80}$$

Thus $E(t) = 0$ for all time $t \in [0,T]$ and so $u \equiv 0$ on $\bar{U}_T$. □

3.6 Discontinuous bounded initial data.

What happens if the initial condition $\phi(x)$ is a bounded function defined on $(-\infty,\infty)$ but discontinuous somewhere ? (here we assume that $\phi(x)$ is discontinuous only at a **finite number** of points and at each discontinuous point x_0 both $\lim_{x\to x_0^+}\phi(x)$ and $\lim_{x\to x_0^-}\phi(x)$ exist).

We have the following interesting result:

Lemma 3.49 *Let $\phi(x)$ be a* ***bounded*** *function defined on $(-\infty,\infty)$ and at $x = x_0$ it is* ***discontinuous*** *and satisfies*

$$\lim_{x\to x_0^+}\phi(x) = A, \quad \lim_{x\to x_0^-}\phi(x) = B, \quad \text{where} \quad A \ne B \tag{81}$$

Then the function

$$u(x,t) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy, \quad x \in (-\infty,\infty), \quad t \in (0,\infty)$$

lies in the space $u \in C^\infty((-\infty,\infty)\times(0,\infty))$ and satisfies the heat equation

$$u_t(x,t) = u_{xx}(x,t), \quad \forall\ (x,t) \in (-\infty,\infty)\times(0,\infty) \tag{82}$$

with

$$\lim_{t\to 0^+} u(x_0,t) = \lim_{t\to 0^+}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_0-y)^2}{4t}}\phi(y)\,dy = \frac{A+B}{2}. \tag{83}$$

Remark 3.50 *(**Be careful.**) In general, the limit*

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) \quad \textit{(note that this is not the same as } \lim_{t\to 0^+}u(x_0,t)\textit{)} \tag{84}$$

does not exist (see Example 3.51 below). On the other hand, if $\phi(x)$ is ***continuous*** *at $x = x_1$, then we have*

$$\lim_{(x,t)\to(x_1,0^+)} u(x,t) = \lim_{t\to 0^+} u(x_1,t) = \phi(x_1). \tag{85}$$

Proof. It suffices to verify (83). Let $M=\sup_{\mathbb{R}}|\phi|$ and let

$$u(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy,\quad (x,t)\in(-\infty,\infty)\times(0,\infty). \tag{86}$$

Then (let $y=x_0+\sqrt{4t}s$)

$$u(x_0,t)=\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-s^2}\phi\left(x_0+\sqrt{4t}s\right)ds+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{0}e^{-s^2}\phi\left(x_0+\sqrt{4t}s\right)ds. \tag{87}$$

For any $\varepsilon>0$, there exists $\delta>0$ such that if $x\in(x_0,x_0+\delta)$, then $|\phi(x)-A|<\varepsilon$. Hence the first integral in (87) satisfies

$$\begin{aligned}
&\left|\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-s^2}\phi\left(x_0+\sqrt{4t}s\right)ds-\frac{A}{2}\right|=\left|\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-s^2}\left[\phi\left(x_0+\sqrt{4t}s\right)-A\right]ds\right|\\
&\le\frac{1}{\sqrt{\pi}}\int_0^{\delta/\sqrt{4t}}e^{-s^2}\left|\phi\left(x_0+\sqrt{4t}s\right)-A\right|ds+\frac{1}{\sqrt{\pi}}\int_{\delta/\sqrt{4t}}^{\infty}e^{-s^2}\left|\phi\left(x_0+\sqrt{4t}s\right)-A\right|ds\\
&\le\frac{\varepsilon}{2}+2M\cdot\frac{1}{\sqrt{\pi}}\int_{\delta/\sqrt{4t}}^{\infty}e^{-s^2}ds
\end{aligned}$$

and so

$$\lim_{t\to0^+}\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-s^2}\phi\left(x_0+\sqrt{4t}s\right)ds=\frac{A}{2}.$$

Similarly, we have

$$\lim_{t\to0^+}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{0}e^{-s^2}\phi\left(x_0+\sqrt{4t}s\right)ds=\frac{B}{2}.$$

The proof is done. □

Example 3.51 *Let*

$$\phi(x)=\begin{cases}1, & x>0\\ 0, & x<0,\end{cases}\quad ,\quad \phi(x)\ \text{is not continuous at } x=0.$$

It is a bounded function. Define the function

$$u(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\phi(y)\,dy=\int_0^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy$$

and let $y=x-\sqrt{4t}s$ to get (we will get the same result if we let $y=x+\sqrt{4t}s$)

$$\begin{aligned}
u(x,t)&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4t}}e^{-s^2}ds=\frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{0}+\int_0^{x/\sqrt{4t}}\right)e^{-s^2}ds\\
&=\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}}e^{-s^2}ds,\quad (x,t)\in(-\infty,\infty)\times(0,\infty).
\end{aligned}$$

We note that

$$\begin{cases}
u(x,t)\in C^{\infty}((-\infty,\infty)\times(0,\infty))\ \text{ and }\ u_t=u_{xx}\ \text{ on }\ (-\infty,\infty)\times(0,\infty)\\
\lim_{(x,t)\to(x_0,0^+)}u(x,t)=1,\quad \text{if}\quad x_0>0\\
\lim_{(x,t)\to(x_0,0^+)}u(x,t)=0,\quad \text{if}\quad x_0<0\\
\lim_{t\to0^+}u(0,t)=\frac{1}{2}=\frac{1+0}{2},\\
\lim_{(x,t)\to(0,0^+)}u(x,t)=\lim_{(x,t)\to(0,0^+)}\left(\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}}e^{-s^2}ds\right)\quad \text{does not exist.}
\end{cases} \tag{88}$$

The last limit in (88) does not exist is due to the fact that as $(x,t) \to (0,0^+)$, *the quantity* $x/\sqrt{4t}$ *can approach any possible number in* $(-\infty,\infty)$. *Hence the limit*

$$\lim_{(x,t)\to(0,0^+)} \int_0^{x/\sqrt{4t}} e^{-s^2} ds$$

does not exist.

Example 3.52 *Let the initial data* $\phi(x)$ *be*

$$\phi(x) = \begin{cases} e^{-x}, & x \in (0,\infty) \\ 0, & x \in (-\infty,0). \end{cases}$$

ϕ *is* bounded *but not continuous at* $x = 0$. *Now we have*

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)\, dy \\ &= \int_0^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2-2xy+y^2+4ty}{4t}} dy = \int_0^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{[y+(2t-x)]^2}{4t}} e^{t-x} dy \quad (\text{let } y = x - 2t + \sqrt{4t}s) \\ &= \frac{1}{\sqrt{\pi}} e^{t-x} \int_{\frac{2t-x}{\sqrt{4t}}}^{\infty} e^{-s^2} ds, \quad (x,t) \in (-\infty,\infty) \times (0,\infty). \end{aligned}$$

It satisfies $u_t(x,t) = u_{xx}(x,t)$ *on* $(-\infty,\infty) \times (0,\infty)$ *and*

$$\begin{cases} \lim\limits_{(x,t)\to(x_0,0^+)} \left(\frac{1}{\sqrt{\pi}} e^{t-x} \int_{\frac{2t-x}{\sqrt{4t}}}^{\infty} e^{-s^2} ds \right) = \frac{1}{\sqrt{\pi}} e^{-x_0} \int_{-\infty}^{\infty} e^{-s^2} ds = e^{-x_0}, & \text{if} \quad x_0 > 0 \\ \lim\limits_{(x,t)\to(x_0,0^+)} \left(\frac{1}{\sqrt{\pi}} e^{t-x} \int_{\frac{2t-x}{\sqrt{4t}}}^{\infty} e^{-s^2} ds \right) = \frac{1}{\sqrt{\pi}} e^{-x_0} \int_{\infty}^{\infty} e^{-s^2} ds = 0, & \text{if} \quad x_0 < 0 \\ \lim\limits_{t\to 0^+} u(0,t) = \lim\limits_{t\to 0^+} \left(\frac{1}{\sqrt{\pi}} e^{t} \int_{\frac{2t}{\sqrt{4t}}}^{\infty} e^{-s^2} ds \right) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2}. \end{cases}$$

<u>This is the end of parabolic equations.</u>